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ON GENERALIZED LN-SURFACES IN $\mathbb{E}^4$

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Abstract. The envelopes of one- and two-parameter families of spheres are very important for applied geometry. A surface M in $\mathbb{E}^4$ which is considered as envelopes of its tangent planes are called LN-surface. These surfaces are quadratically parametrized in $\mathbb{E}^4$. In the present study we calculate the Gaussian, normal and mean curvatures of these surfaces. Further, we have pointed out the flat and minimal points of the surfaces.

1. **Introduction**

The envelopes of one- and two-parameter families of spheres are very important for applied geometry [8]. Especially, rational surfaces with rational offsets are more involved, since the techniques for the curve case cannot be applied directly to surfaces. Although an explicit representation of all rational surfaces with rational offsets has been given already in [9], it is not obvious how to decide the rationality for particular surface classes. It has been proved that rational pipe surfaces [5], rational ruled surfaces [11] and all regular quadrics [4] possess rational offsets. These statements can also be found in [10] as specializations of a more general result concerning envelopes of one-parameter families of cones of revolution. Later it has been proved in [2] and [3] that rational surfaces with linear normal vector fields, so called LN-surfaces in $\mathbb{E}^3$, possess rational offset surfaces. In [12] it has been shown that even the convolution surface of an LN-surface and any rational surface admits rational parametrization.

In [7] M. Peternell and B. Odehnal investigates a class of two-dimensional rational surfaces M in $\mathbb{E}^4$ whose tangent planes satisfy the following property: For any three-space S in $\mathbb{E}^4$ there exists a unique tangent plane T of M which is parallel to S. The most interesting families of surfaces are constructed explicitly and geometric properties of these surfaces are derived. Quadratically parameterized surfaces in $\mathbb{E}^4$ occur as special cases. This construction generalizes the concept of LN-surfaces in $\mathbb{E}^3$ to two-dimensional surfaces in $\mathbb{E}^4$. The same authors defined seven type of generalized LN-surfaces in $\mathbb{E}^4$.

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The paper is organized as follows: Section 2 explains some geometric properties of the surfaces in $\mathbb{E}^4$. Section 3 tells about the rational construction of the envelope surfaces of two-parameter families of spheres corresponding to LN- surfaces. Further, we calculated the Gaussian curvature, normal curvature and mean curvatures of generalized LN- surfaces of several types. We have pointed out the flat and minimal points of these surfaces.

2. Geometric Background

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M is spanned by $\{X_u, X_v\}$.In the chart (u,v), the first fundamental form of M is given by

$$I = \langle DX, DX\rangle = Edu^2 + 2Fdudv + Gdv^2 \tag{2.1}$$

with

$$E = \langle X_u, X_u\rangle, F = \langle X_u, X_v\rangle, G = \langle X_v, X_v\rangle \tag{2.2}$$

where $\langle\ ,\rangle$ is the Euclidean inner product. We assume that $EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular.

For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus N_pM$ where N_pM is the orthogonal complement of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given local vector fields X_1, X_2 on M . The induced connection on M is defined by $\nabla_{X_1}X_2 = \left(\widetilde{\nabla}_{X_1}X_2\right)^T$.

Let $\chi(M)$ and $N(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map:

$$h : \chi(M) \times \chi(M) \to N(M), \quad h(X_1, X_2) = \widetilde{\nabla}_{X_i}X_j - \nabla_{X_i}X_j, \quad 1 \leq i,j \leq 2. \tag{2.3}$$

This map is well defined, symmetric and bilinear. For an orthonormal normal frame field $\{N_1, N_2\}$ on M recall the shape operator

$$A_N : T_pM \to T_pM, \ A_{N_i}X = -\left(\widetilde{\nabla}_X N_i\right)^T \tag{2.4}$$

where T means the tangent component. This operator is bilinear, self-adjoint and for any $X_1, X_2 \in T_pM$ satisfies the following equation:

$$\langle A_v X_1, X_2\rangle = \langle h(X_1, X_2), N_i\rangle, \ 1 \leq i \leq 2.$$

The equation (2.3) is called Gauss formula [1]. It is well-known that the coefficients of the second fundamental form c_{ij}^k satisfy

$$c_{ij}^k = \langle h(X_i, X_j), N_k\rangle, \quad i,j,k = 1,2. \tag{2.5}$$

So, the second fundamental form h of $M \subset \mathbb{E}^4$ is given by

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k , \qquad 1 \leq i,j \leq 2 \tag{2.6}$$

The Gaussian and normal curvatures of the immersed surface $M \subset \mathbb{E}^4$ are given by

$$K = \frac{1}{W^2}\sum_{k=1}^{2}(c_{11}^k c_{22}^k - (c_{12}^k)^2 \tag{2.7}$$

and
(2.8)

$$K_N = \frac{1}{W^2}\left(E\left(c^1_{12}c^2_{22} - c^2_{12}c^1_{22}\right) - F\left(c^1_{11}c^1_{22} - c^2_{11}c^1_{22}\right) + G\left(c^1_{11}c^2_{12} - c^2_{11}c^1_{12}\right)\right),$$

respectively, where $EG - F^2 = W^2$

Further, the k^{th} mean curvature of a regular patch is given by

$$H_k = \frac{1}{2W^2}(c^k_{11}G - 2c^k_{12}F + c^k_{22}E)\ , \qquad 1 \leq k \leq 2 \tag{2.9}$$

respectively (see, [6]). Recall that a surface $M \subset \mathbb{E}^4$ is said to be *minimal* if its mean curvature $H = \sqrt{H_1^2 + H_2^2}$ vanishes identically [1].

3. Generalized LN-Surfaces in $\mathbb{E}^4$

In this section we will consider LN-surfaces in four dimensional Euclidean space $\mathbb{E}^4$. A surface M^2 in $\mathbb{E}^4$ is called quadratically parametrizable if it admits a parametrization (i.e. a surface patch) $X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v), x_4(u,v))$, where x_i are quadratic polynomials. The tangent space of M^2 is spanned by the linear vector fields $x_u(u,v)$ and $x_v(u,v)$. In [7] the authors determined a class of surfaces in $\mathbb{E}^4$ which generalize quadratically parametrizable surfaces M^2 concerning the structure of their tangent planes. Consequently, the two parameter family of spheres in $\mathbb{E}^3$ have envelops which admits rational parametrization. Recently, Peternell and Odehnal extend LN-surfaces to 4-dimensional Euclidean space.

Definition 3.1. A rational two-dimensional surface M^2 in $\mathbb{E}^4$ is called generalized LN-surface if for all 3-spaces $S \subset \mathbb{E}^4$ the surface parameters u and v can be expressed in terms of rational functions depending on the coefficients e_i of S [7].

The tangent plane $T(u,v)$ is defined by

$$T(u,v) = \{T/\ T = p + \lambda X_u + \mu X_v\} \tag{3.1}$$

and similarly the normal plane $N_p(u,v)$ is defined by

$$N(u,v) = \{N/\ N = p + \lambda N_1 + \mu N_2\} \tag{3.2}$$

where $N_1 \perp X_u$, $N_1 \perp X_v$ and $N_2 \perp X_u$, $N_2 \perp X_v$.

Generalized LN-surfaces in $\mathbb{E}^4$ which generalize quadratically parameterizeable surfaces M^2 concerning the structure of their tangent planes. So, this surfaces are considered as envelope of its tangent planes. The tangent space of M is spanned by the linear vector fields $X_u(u,v)$ and $X_v(u,v)$ which are the intersections of 3-spaces

$$\begin{aligned} S(u,v) &\ :\ e_1x_1 + ... + e_4x_4 = N_1^TX = a(u;v); \\ L(u,v) &\ :\ f_1x_1 + ... + f_4x_4 = N_2^TX = b(u;v): \end{aligned} \tag{3.3}$$

where $a(u;v)$ and $b(u;v)$ are rational functions and

$$\begin{aligned} N_1 &= (e_1, ..., e_4) \\ N_2 &= (f_1, ..., f_4) \\ X &= (x_1, ..., x_4). \end{aligned}$$

Actually the parametrization $X(u,v)$ is solution of the system (3.3) is general rational representation of M. A possible generalization interprets a surface $M \subset \mathbb{E}^4$ as envelope of its two-parameter family of tangent planes. The tangent planes T of

M are represented as intersection of 3-spaces $S(u,v)$ and $L(u,v)$, i.e., $T = S \cap L$. We assume that the system of linear equations

$$\begin{aligned} S &: N_1^T X = a;\ S_u : (N_1)_u^T X = a_u;\ S_v : (N_1)_v^T X = a_v; \\ L &: N_2^T X = b;\ F_u : (N_2)_u^T X = b_u;\ L_v : (N_1)_v^T X = b_v; \end{aligned} \tag{3.4}$$

which has a (unique) solution $X(u,v)$. Differentiating $N_1^T X = a$ with respect to u and v and taking $(N_1)_u^T X = a_u$ and $(N_1)_v^T X = a_v$ into account leads to $N_1^T X_u = 0$, $N_1^T X_v = 0$ [7]. Similarly differentiating $N_2^T X = b$ with respect to u and v and using $(N_2)_u^T X = b_u$ with $(N_2)_v^T X = b_v$ we get $N_2^T X_u = 0$, $N_2^T X_v = 0$.

For the all suitable normal vector fields $N_1(u;v)$ and $N_2(u;v)$ M. Peternell, and B. Odehnal determined the generalized LN-surfaces in $\mathbb{E}^4$ which generalize quadratically parameterizable surfaces M concerning the structure of their tangent planes [7]. The same authors defined several type of generalized LN-surfaces in $\mathbb{E}^4$ (see, Table1).

Table 1. Tangents and normals of LN-surfaces

Type	X_i	N_i
Type 1	$X_1 = (-u, 0, 1, 0)$ $X_2 = (0, -v, 0, 1)$	$n_1 = (1, 0, u, 0)$ $n_2 = (0, 1, 0, v)$
Type 2	$X_1 = (-u, v, 1, 0)$ $X_2 = (0, u, 0, 1)$	$n_1 = (1, 0, u, 0)$ $n_2 = (\frac{-uv}{1+u^2}, -1, \frac{v}{1+u^2}, u)$
Type 3	$X_1 = (u, v, 1, 0)$ $X_2 = (-v, u, 0, 1)$	$n_1 = (1, 0, -u, v)$ $n_2 = (0, -1, v, u)$
Type 4	$X_1 = (u, 0, v, 1)$ $X_2 = (0, v, u, 0)$	$n_1 = (-1, 0, 0, u)$ $n_2 = (\frac{uv^2}{1+u^2}, u, -v, \frac{v^2}{1+u^2})$
Type 5	$X_1 = (u, 0, v, 1)$ $X_2 = (1, v, u, 0)$	$n_1 = (u, 0, -1, -u^2 + v)$ $n_2 = (-v - uA, 1, A, uv - (-u^2 + v)A)$
Type 6	$X_1 = (-u, 1, 0, 0)$ $X_2 = (-v, 0, 1, 0)$	$n_1 = (1, u, v, 0)$ $n_2 = (0, 0, 0, 1)$

where $N_i = \frac{n_i}{\|n_i\|}$ is the unit normal vector of the surface and

$$A(u,v) = \frac{uv(-u^2 + v - 1)}{1 + u^2 + (-u^2 + v)^2}. \tag{3.5}$$

These surfaces defined by the following surface patches;

Table 2. Surface patches of LN-surfaces

Type	$X(u,v)$
Type 1	$(a - ua_u, b - vb_v, a_u, b_v)$
Type 2	$(a - ua_u, va_u + ub_u - b, a_u, b_u)$
Type 3	$(a - ua_u - va_v, -b - va_u + ua_v, -a_u, a_v)$
Type 4	$(ua_u - a, b_u, 2va_u - b_v, a_u)$
Type 5	$(ua_v - b_v, b - vb_v, -a + b_u - ub_v, a_v)$
Type 6	$(a - ua_u - va_v, a_u, a_v, b)$

where $a(u,v)$ and $b(u,v)$ are rational functions defined by

$$\begin{aligned} a(u;v) &= N_1^T X, \\ b(u;v) &= N_2^T X. \end{aligned}$$

By the use of (3.4) with the tangent and normal vectors given in Table 1 and Table 2 we can find the rational functions $a(u, v)$ and $b(u, v)$ (see, Table 3).

Table 3. Rational functions of LN-surfaces

Type	$a(u, v)$	$b(u, v)$
Type 1	$a = \frac{1}{2}u^2, \quad a_v = 0$	$b = \frac{1}{2}v^2, b_u = 0$
Type 2	$a = \frac{1}{2}u^2, a_v = 0$	$b = uv, a_u - b_v = 0$
Type 3	$a = \frac{1}{2}(v^2 - u^2), a_u + b_v = 0$	$b = uv, \ a_v = b_u$
Type 4	$a = \frac{1}{2}u^2, \quad a_v = 0$	$b = \frac{1}{2}uv^2, b = vb_v + ub_u - v^2 a_u$
Type 5	$a = uv - \frac{1}{3}u^3, \quad b_u = va_v$	$b = \frac{1}{2}u^2 v - \frac{1}{2}v^2, \ b_v = -a_u - ua_v$
Type 6	$a = \frac{1}{2}(u^2 + v^2)\ , a(u, v)$	$b = const$

Using (2.2) and (2.5), the normal vectors given in Table 1 and Table 3 we obtain the coefficients of the first and second fundamental form of LN-surfaces in $\mathbb{E}^4$ as follows:

Table 4. The coefficients of the first and second fundamental form

Type	first fund. form	second fund. form	
Type 1	$E = 1 + u^2$ $F = 0$ $G = 1 + v^2$	$c^1_{11} = -\frac{1}{\sqrt{1+u^2}}$ $c^1_{12} = 0$ $c^1_{22} = 0$	$c^2_{11} = 0$ $c^2_{12} = 0$ $c^2_{22} = -\frac{1}{\sqrt{1+v^2}}$
Type 2	$E = 1 + u^2 + v^2$ $F = uv$ $G = 1 + u^2$	$c^1_{11} = -\frac{1}{\sqrt{1+u^2}}$ $c^1_{12} = 0$ $c^1_{22} = 0$	$c^2_{11} = \frac{uv}{\sqrt{1+u^2}\sqrt{v^2+(1+u^2)^2}}$ $c^2_{12} = -\frac{\sqrt{1+u^2}}{\sqrt{v^2+(1+u^2)^2}}$ $c^2_{12} = 0$
Type 3	$E = 1 + u^2 + v^2$ $F = 0$ $G = 1 + u^2 + v^2$	$c^1_{11} = \frac{1}{\sqrt{1+u^2+v^2}}$ $c^1_{12} = 0$ $c^1_{22} = -\frac{1}{\sqrt{1+u^2+v^2}}$	$c^2_{11} = 0$ $c^2_{12} = -\frac{1}{\sqrt{1+u^2+v^2}}$ $c^2_{22} = 0$
Type 4	$E = 1 + u^2 + v^2$ $F = uv$ $G = u^2 + v^2$	$c^1_{11} = -\frac{1}{\sqrt{1+u^2}}$ $c^1_{12} = 0$ $c^1_{22} = 0$	$c^2_{11} = \frac{uv^2}{\sqrt{1+u^2}\sqrt{v^4+(u^2+v^2)(1+u^2)}}$ $c^2_{12} = -\frac{v\sqrt{1+u^2}}{\sqrt{v^4+(u^2+v^2)(1+u^2)}}$ $c^2_{22} = \frac{u\sqrt{1+u^2}}{\sqrt{v^4+(u^2+v^2)(1+u^2)}}$
Type 5	$E = 1 + u^2 + v^2$ $F = u + uv$ $G = 1 + u^2 + v^2$	$c^1_{11} = \frac{u}{\sqrt{1+u^2+(-u^2+v)^2}}$ $c^1_{12} = -\frac{1}{\sqrt{1+u^2+(-u^2+v)^2}}$ $c^1_{22} = 0$	$c^2_{11} = \frac{-v-uA}{\lVert n_2 \rVert}$ $c^2_{12} = \frac{A}{\lVert n_2 \rVert}$ $c^2_{22} = \frac{1}{\lVert n_2 \rVert}$
Type 6	$E = 1 + u^2$ $F = uv$ $G = 1 + v^2$	$c^1_{11} = -\frac{1}{\sqrt{1+u^2+v^2}}$ $c^1_{12} = 0$ $c^1_{22} = -\frac{1}{\sqrt{1+u^2+v^2}}$	$c^2_{11} = 0$ $c^2_{12} = 0$ $c^2_{22} = 0$

where the function $A(u, v)$ is defined in (3.5) and

$$\lVert n_2 \rVert = \sqrt{1 + A^2 + (Au + v)^2 + (uv - (-u^2 + v)A)^2}. \tag{3.6}$$

By the use of the coefficients of the first and second fundamental form of the surface we can calculate the Gaussian curvature, normal curvature and mean curvature of the surface. So, using the equations (2.7), (2.8) and (2.9) with Table 4, we can find the Gaussian and normal curvatures of LN-surfaces (see, Table 5);

Table 5. The Gaussian and normal curvatures of LN-surfaces

Type	K	K_N
Type 1	0	0
Type 2	$-\frac{1+u^2}{(v^2+(1+u^2)^2)^2}$	$\frac{1+u^2}{(v^2+(1+u^2)^2)^2}$
Type 3	$\frac{-2}{(u^2+v^2+1)^3}$	$\frac{-2}{(u^2+v^2+1)^3}$
Type 4	$\frac{v^2}{(v^4+(1+u^2)(v^2+u^2))^2}$	$\frac{v(2u^2+v^2)}{(v^4+(1+u^2)(v^2+u^2))^2}$
Type 5	$-\frac{1}{W^2}(\frac{1}{1+u^2+(v-u^2)^2}+\frac{v+Au+A^2}{\Vert n_2\Vert^2})$	$-\frac{1}{W^2}\frac{(1+v)(1+2u^2+v^2)}{\Vert n_2\Vert\sqrt{1+u^2+(v-u^2)^2}}$
Type 6	$\frac{1}{(u^2+v^2+1)^2}$	0

where $A(u,v)$ and $\Vert n_2\Vert$ are defined in (3.5) and (3.6) respectively, and

$$W^2=(1+u^2+v^2)^2-u^2(1+v)^2.$$

We get the following results;

Proposition 3.1. *Let M be a LN-surface of Type k $(1\leq k\leq 6)$. Given any point $p\in M$ the following statements are valid;*

i) At each point p, the surface of Type 1 has vanishing Gaussian curvature,

ii) At each point p, the surface of Type 2, Type 3 and Type 6 have non-vanishing Gaussian curvatures,

iii) On the u-parameter curve (i.e. $v=0$) of the LN-surface of Type 4 the Gaussian curvature vanishes identically.

Proposition 3.2. *Let M be a LN-surface of Type k $(1\leq k\leq 6)$. Given any point $p\in M$ the following statements are valid;*

i) At each point p, the surfaces of Type 1 and Type 6 have vanishing normal curvatures,

ii) At each points p, the surfaces of Type 2 and Type 3 have non-vanishing normal curvatures,

iii) On the u-parameter curve (i.e. $v=0$) of the LN-surface of Type 4 the normal curvature vanishes identically,

iv) On the u-parameter curve (with $v=-1$) of the LN-surface of Type 5 the normal curvature vanishes identically.

Using the equations in (2.9) with Table 4, we can find the k^{th} mean curvature of these surfaces (see, Table 6);

Table 6. The k^{th} Mean curvature of LN-surfaces

Type	H_1	H_2
Type 1	$\frac{-1}{2(1+u^2)^{3/2}}$	$\frac{-1}{2(1+v^2)^{3/2}}$
Type 2	$-\frac{\sqrt{1+u^2}}{2(v^2+(1+u^2)^2)}$	$\frac{3uv\sqrt{1+u^2}}{2(v^2+(1+u^2)^2)^{3/2}}$
Type 3	0	0
Type 4	$-\frac{u^2+v^2}{2\sqrt{1+u^2}(v^4+(1+u^2)(v^2+u^2))}$	$\frac{u((1+u^2)^2+v^2(4u^2+v^2+3))}{2\sqrt{1+u^2}(v^4+(1+u^2)(v^2+u^2))^{3/2}}$
Type 5	$\frac{u(3+2v+u^2+v^2)}{2W^2\sqrt{1+u^2+(v-u^2)^2}}$	$\frac{(1+u^2+v^2)(1-v-Au)-2Au(1+v)}{2W^2\Vert n_2\Vert}$
Type 6	$-\frac{u^2+v^2+2}{2(u^2+v^2+1)^{3/2}}$	0

Thus, we obtain the following result.

Proposition 3.3. *Let M be a LN-surface of Type k ($1 \leq k \leq 6$). Given any point $p \in M$ the following statements are valid;*

i)At each point p, the surfaces of Type 1 and Type 3 have vanishing mean curvatures,

ii)At each point p, the surfaces of Type 2, Type 4 and Type 6 have non vanishing mean curvatures.

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