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Δ-PRIMITIVE ELEMENTS OF FREE METABELIAN LIE ALGEBRAS

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Abstract. Let L_2 be a free metabelian Lie algebra of rank two. We obtain a characterization of Δ-primitive elements of L_2. In particular, we prove that every endomorphism Φ of L_2 is an automorhism if and only if Φ takes a Δ-primitive element of L_2 to another Δ-primitive element. Also we show that in a free metabelian Lie algebra of odd rank there are no Δ-primitive elements.

1. Introduction

Let F_n be the free Lie algebra of a finite rank $n \geq 2$ with a free generating set $X = \{x_1, ..., x_n\}$ over a field K. A primitive element of F_n is an element of some free generating set of F_n. Denote the universal enveloping algebra for F_n by $U(F_n)$. Algorithms to recognize primitive elements of free associative algebras of rank two and of free Lie algebras were obtained by V. Shpilrain, J. T. Yu, A. A. Mikhalev and A. A. Zolotykh in [3, 4, 8]. In the case free groups there is a matrix characterization of primitive elements given by Umirbaev [10]. He has also proved that the automorphism group of a free group acts transitively on the set of all primitive elements . An element $u \in F_n$ is called Δ-primitive if Fox derivatives [2] of the element u generate Δ as a left ideal of $U(F_n)$, where Δ is the augmentation ideal of $U(F_n)$. In this paper we focus on Δ-primitive elements of a free metabelian Lie algebra. In [9] V.Shpilrain considered Δ-primitive elements of free groups and free metabelian groups. An analogue of Shpilrain's result for free Lie algebras obtained by A. A. Mikhalev and J. T. Yu [5]. One can raise the result of Shpilrain also for relatively free Lie algebras. In this paper we consider the case of free metabelian Lie algebras. We prove that in a free metabelian Lie algebra L with free generators x_1, x_2 any Δ-primitive element is an automorphic image of the commutator $[x_1, x_2]$. This yields in particular a simple criterion for recognizing

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automorphisms of L: An endomorphism Φ of L is an automorphism if and only if Φ takes a Δ-primitive element of L to another Δ-primitive element.

2. Preliminaries

Let K be a field of characteristic zero. For any element u of the free associative algebra $U(F_n)$ we have the unique representation

$$u = a_1x_1 + a_2x_2 + ... + a_nx_n + c.1,$$

where $a_i \in U(F_n), c \in K$. We call the element a_i the left Fox derivative (see [2]) of the element u by x_i and we use the notation $a_i = \frac{\partial u}{\partial x_i}$. The operators $\frac{\partial}{\partial x_i}$: $U(F_n) \to U(F_n)$ are linear mappings such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$(Kronecker delta), $\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} \cdot \sigma(v) + u \cdot \frac{\partial v}{\partial x_i}$, where $\sigma : U(F_n) \to K$ is the homomorphism defined by $\sigma(x_i) = 0$ for all $x_i \in X$. By Δ we denote the augmentation ideal of $U(F_n)$ that is; the kernel of the augmentation homomorphism $\sigma : U(F_n) \to K$. The ideal Δ is a free left $U(F_n)$ module with free basis X. We denote by $F_n^{'}$ and $F_n^{''}$ the derived subalgebra of F_n and $F_n^{'}$ respectively. Let L_n be the free metabelian Lie algebra of rank n. We identify L_n with $F_n/F_n^{''}$ in the usual way. The adjoint representation of L_n induces a representation of L_n/L_n' on L_n'. Thus L_n' is furnished with the structure of a left $U(L_n/L_n')-$ module. We denote the action by a dot. Recall that if $f \in L_n'$ and $v_1, ..., v_k \in L_n/L_n'$ then $v_1...v_k.f = [v_1, [...[v_k, f]...]]$. In this way L_n' is a free left $U(L_n/L_n')-$module generated by the elements $c_{ij} = [x_i, x_j]$, $1 \le i < j \le n$.

The next lemma is an immediate consequence of the definitions of Fox derivatives and right ideals.

Lemma 2.1. *Let J be an arbitrary ideal of $U(F_n)$ and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in J$ for each $i, 1 \le i \le n$.*

The next lemma can be found in [11].

Lemma 2.2. *Let R be an ideal of F_n and $u \in F_n$. Then $u \in \Delta_R\Delta$ if and only if $u \in R^{'}$, where Δ_R is the ideal of $U(F_n)$ generated by R.*

We have the following chain rule for Fox derivations [2].

Lemma 2.3. *Let Φ be an endomorphism of F_n (it can be linearly extended to $U(F_n)$) defined by $\Phi(x_k) = y_k$, $1 \le k \le n$ and let $v = \Phi(u)$ for some $u, v \in U(F_n)$. Then*

$$\frac{\partial v}{\partial x_j} = \sum_{i=1}^{n} \Phi(\frac{\partial u}{\partial x_k})\frac{\partial y_k}{\partial x_j} \ .$$

Lemma 2.4. *Let M be an arbitrary Lie algebra over a field and $U(M)$ its universal enveloping algebra. Let $v_1, ..., v_m, u$ be elements of M. Suppose u belongs to the left ideal of $U(M)$ generated by $v_1, ..., v_m$. Then u belongs to the subalgebra of M generated by $v_1, ..., v_m$.*

For the proof of this lemma see [7].

Now we need the following lemma:

Lemma 2.5. *Let J be a left ideal of $U(F_n)$ generated as a free module over $U(F_n)$ by $u_1, ..., u_m$. Then the following conditions are equivalent:*

a) A matrix $M = (a_{ij})_{1\le i,j\le m}$ *is invertible over* $U(F_n)$,

b)The elements $y_j = \sum_{k=1}^{m} a_{kj}u_k$, $1 \le j \le m$, *generate the ideal* J *as a left ideal of* $U(F_n)$.

The proof follows by using the analog of the Lemma 2.4 of [9] for free associative algebras.

The description of the automorphisms of L_2 is given in [1] and [6].

Lemma 2.6. [1] *Let* Φ *be an endomorphism of* L_2. *Then* Φ *is an automorphism if and only if* $[\Phi(x_1), \Phi(x_2)] = \alpha[x_1, x_2], 0 \neq \alpha \in K$.

3. $\Delta-$Primitive Elements

In this section we obtain a characterization of $\Delta-$primitive elements of free metabelian Lie algebras.

Definition 3.1. An element $u \in F_n$ is called $\Delta-$primitive if the elements $\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}$ generate Δ as a left ideal of $U(F_n)$.

In the case of the free metabelian Lie Algebra L_n we say that an element $v \in L_n$ is $\Delta-$primitive if Fox derivatives of v generate Δ as a left ideal of $U(L_n/L_n')$.

Recall that L_2' is a free left $U(L_2/L_2')-$module generated by $[x_1, x_2]$.

Theorem 3.1. *1.Any* $\Delta-$*primitive element of* L_2 *is an automorphic image of* $[x_1, x_2]$.

2.The group of automorphisms of L_2 *acts transitively on the set of all* $\Delta-$*primitive elements.*

Proof. If an element g of L_2 is $\Delta-$primitive, then $g \in L_2^{'}$; otherwise $\frac{\partial g}{\partial x_1}$ and $\frac{\partial g}{\partial x_2}$ wouldn't belong to Δ by Lemmas 1,2.

Now let $h = u.[x_1, x_2]$ be an element of $L_2^{'}$,where $u \in U(L_2/L_2')$. Suppose that the element h is $\Delta-$primitive.

Now evaluate Fox derivatives of the element h.

$$\frac{\partial h}{\partial x_1} = -u.x_2, \quad \frac{\partial h}{\partial x_2} = u.x_1 \ .$$

Hence $\frac{\partial h}{\partial x_i}$, $i = 1, 2$ generate the same ideal of $U(L_2/L_2')$ as $\frac{\partial}{\partial x_i}[x_1, x_2]$. This shows that the element u should be invertible in $U(L_2/L_2')$, which means it is the nonzero element of the field K.

Hence any $\Delta-$primitive element of L_2 has the form $\alpha[x_1, x_2]$ where, $0 \neq \alpha \in K$; in particular h is an automorphic image of $[x_1, x_2]$ by Lemma 6.

2. It follows from (1) that the group of automorphisms of the algebra L_2 acts transitively on the set of all $\Delta-$primitive elements.

Corollary 3.1. *An endomorphism* Φ *of* L_2 *is an automorphism if and only if* Φ *takes a* $\Delta-$*primitive element of* L_2 *to another* $\Delta-$*primitive element.*

Proof. First suppose that Φ is an automorphism. If an element u of L_2 is $\Delta-$primitive element then by Theorem 8 it is of the form $u = \alpha[x_1, x_2]$,

where $0 \neq \alpha \in K$. If we now apply Φ to the element u we get

$$\begin{aligned} \Phi(u) &= \alpha[\Phi(x_1), \Phi(x_2)] \\ &= \alpha\beta[x_1, x_2] \end{aligned}$$

where $0 \neq \beta \in K$. Therefore $\Phi(u)$ is $\Delta-$primitive.

Conversely let $\Phi(v)$ be $\Delta-$primitive for any $\Delta-$primitive element v of L_2. By Theorem 8 the elements v and $\Phi(v)$ have the form

$$\begin{aligned} v &= \beta[x_1, x_2],\ 0 \neq \beta \in K\ , \\ \Phi(v) &= \alpha[x_1, x_2],\ 0 \neq \alpha \in K\ . \end{aligned}$$

Thus we have

$$[\Phi(x_1), \Phi(x_2)] = \beta^{-1}\alpha[x_1, x_2].$$

By Lemma 6 Φ is an automorphism.

Theorem 3.2. *If n is odd then there are no $\Delta-$primitive elements in L_n.*

Proof. Consider the elements $c_{ij} = [x_i, x_j]$, $1 \leq i < j \leq n$ in the algebra L_n. Each element u in the derived subalgebra $L_n^{'}$ can be written down as

$$u = \sum_{1 \leq i < j \leq n} \alpha_{ij}.c_{ij}$$

where $\alpha_{ij} \in U(L_n/L_n')$.

Evaluate Fox derivatives of the element u:

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= -\alpha_{12}x_2 - \alpha_{13}x_3 - \alpha_{14}x_4 - \ldots - \alpha_{1n}x_n, \\ \frac{\partial u}{\partial x_2} &= \alpha_{12}x_1 - \alpha_{23}x_3 - \alpha_{24}x_4 - \ldots - \alpha_{2n}x_n, \\ \frac{\partial u}{\partial x_3} &= \alpha_{13}x_1 + \alpha_{23}x_2 - \alpha_{34}x_4 - \ldots - \alpha_{3n}x_n, \\ &\vdots \\ \frac{\partial u}{\partial x_n} &= \alpha_{1n}x_1 + \alpha_{2n}x_2 + \alpha_{3n}x_3 + \ldots + \alpha_{n-1n}x_{n-1}. \end{aligned}$$

We are going to show that these derivatives are dependent over $U(L_n/L_n')$. The coefficient matrix

$$\begin{pmatrix} 0 & -\alpha_{12} & -\alpha_{13} & ... & -\alpha_{1n} \\ \alpha_{12} & 0 & -\alpha_{23} & ... & -\alpha_{2n} \\ \alpha_{13} & \alpha_{23} & 0 & ... & -\alpha_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \ldots & 0 \end{pmatrix}$$

is antisymmetric with zeroes on the diagonal. Denote this matrix by $B = (b_{ij})$. The determinant of B must be zero if n is odd. Indeed, consider the summands $b_{1j_1}.b_{2j_2}...b_{nj_n}$ and $b_{j_1 1}.b_{j_2 2}...b_{j_n n}$ in the decomposition of the determinant. If there is at least one diagonal element among these b_{kj_k} , then the product is zero . If all b_{kj_k} are off diagonal then these two summands go with different signs since $b_{ij} = -b_{ji}$ and n is odd. Therefore, they cancel out which proves that the determinant of B equals zero. This shows that the matrix B is not invertible over $U(L_n/L'_n)$. By Lemma 5 the derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, ..., n$ don't generate Δ as a free left $U(L_n/L'_n)-$module.

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