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GENERALIZED (f, g)-DERIVATIONS OF LATTICES

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Abstract. In this paper as a generalization of derivation and f-derivation on a lattice we introduce the notion of generalized (f, g)-derivation of a lattice. We give illustrative example. If the function g is equal to the function f then the generalized (f, g)-derivation is the f-derivation defined in [8]. Also if we choose the function f and g the identity functions both then the derivation we define coincides with the derivation defined in [22].

1. Introduction

Many kind of derivations on rings, prime rings and lattices are studied by many authors [3, 4, 5, 10, 11, 12, 13, 16]. The derivation on a lattice was defined by Szasz, G [21]. X.L. Xin studied the derivation on a lattice and got interesting results. After these studies the f-derivation, symmetric bi derivation and symmetric f bi-derivation of lattices were defined and studied respectively in [8], [9] and [18]. In [8] Ceven and Ozturk gave a generalization of derivation on a lattice which was defined in [22]. Ceven in [9] introduced the symmetric bi derivations on lattices. The author investigated some related properties. He characterized the distributive and modular lattices by the trace of symmetric bi derivations. Ozbal and Firat in [18] introduced the notion of symmetric f bi-derivation of a lattice. They characterized the distributive lattice by symmetric f bi-derivation.

The lattice algebra has an important role and has many applications in information theory, information retrieval, information access controls and cryptanalysis. For more information one can see [1, 2, 6, 7, 14, 17, 19, 20, 23].

In this paper we introduce the notion of (f, g)-derivation of a lattice. We give illustrative example. We also characterize the distributive and isotone lattices by generalized (f, g)-derivations. We generalize the derivations defined in [8] and [22].

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2. Preliminaries

Definition 2.1. [15] Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. If $(L, \wedge, \vee)$ satisfies the following conditions for all $x, y, z \in L$

(1) $x \wedge x = x$, $x \vee x = x$

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$

then L is called a lattice.

Definition 2.2. [15] A lattice L is distributive if the identity (5) or (6) holds for all $x, y, z \in L$:

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Definition 2.3. [19] A lattice L is called modular if it satisfies the following conditions for all $x, y, z \in L$:

(7) If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$

Definition 2.4. [15] Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 2.5. [15] Let L and M be two lattices. The function $g : L \to M$ is called the lattice homomorphism if it satisfies the following conditions for all $x, y \in L$.

(8) $g(x \wedge y) = g(x) \wedge g(y)$

(9) $g(x \vee y) = g(x) \vee g(y)$.

Lemma 2.1. *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as in the Definition 2.4. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the g.l.b. of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

Definition 2.6. [22] A function $D : L \to L$ on a lattice L is called a derivation on L if D satisfies the following condition

$$D\,(x \wedge y) = (Dx \wedge y) \vee (x \wedge Dy)$$

The abbreviation Dx is used for $D(x)$ in above definition.

Definition 2.7. [22] Let L be a lattice and D be a derivation on L

(i) If $x \leq y$ implies $Dx \leq Dy$ then D is called an isotone derivation,

(ii) If D is one to one then D is called monomorphic derivation,

(iii) If D is onto then D is called epimorphic derivation.

3. Generalized (f, g)-Derivations of Lattices

Definition 3.1. Let L be a lattice, a function $d : L \to L$ is called generalized (f, g)-derivation of L if there exist functions $f, g : L \to L$ such that

$$d(x \wedge y) = (dx \wedge fy) \vee (gx \wedge dy) \tag{3.1}$$

for all $x, y \in L$.

It is obvious that if the function g equal to the function f then the generalized (f, g)-derivation is the f-derivation defined in [8]. Also if we choose the functions f and g the identity functions then the derivation we define in (3.1) coincides with the derivation defined in [22].

Example 3.1. Let L be the lattice of Figure 1 and define a function d by $d0 = 0,\ da = a,\ db = a,\ dc = c,\ d1 = a$.

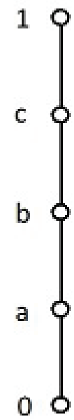

FIGURE 1

Then d is not a derivation on L since $a = d(b \wedge c) \neq (db \wedge c) \vee (b \wedge dc) = (a \wedge c) \vee (b \wedge c) = b$. If we define $f0 = 0,\ fa = a,\ fb = a,\ fc = 1,\ f1 = 1$ and $g0 = 0,\ ga = a,\ gb = a,\ gc = c,\ g1 = 1$, then d is a generalized (f,g)-derivation on L.

Proposition 3.1. *Let L be a lattice and d be a generalized (f,g)-derivation on L. Then*

$$dx \leq fx \vee gx$$

for all $x \in L$.

Proof. Since $dx \wedge fx \leq fx$ and $gx \wedge dx \leq gx$ then,

$$\begin{aligned} dx &= d(x \wedge x) \\ &= (dx \wedge fx) \vee (gx \wedge dx) \\ &\leq fx \vee gx. \end{aligned}$$

□

Proposition 3.2. *Let d be a generalized (f,g)- derivation on a distributive lattice L then*

$$dx \wedge dy \leq d(x \wedge y)$$

for all $x, y \in L$.

Proof. From Proposition (3.1) we have $dx \leq fx \vee gx$. Since

$$d(x \wedge y) = (dx \wedge fy) \vee (gx \wedge dy)$$

then

$$(dx \wedge fy) \leq d(x \wedge y)$$

and

$$(gx \wedge dy) \leq d(x \wedge y). \tag{3.2}$$

Also since

$$d(x \wedge y) = d(y \wedge x) = (dy \wedge fx) \vee (gy \wedge dx)$$

then

$$(dy \wedge fx) \leq d(x \wedge y) \tag{3.3}$$

and

$$(gy \wedge dx) \leq d(x \wedge y).$$

Combining (3.2) and (3.3) we have

$$(dy \wedge fx) \vee (gx \wedge dy) \leq d(x \wedge y) \tag{3.4}$$

From (3.4), and since L is a distributive lattice then we get

$$\begin{aligned} dx \wedge dy &\leq (fx \vee gx) \wedge dy \\ &= (fx \wedge dy) \vee (gx \wedge dy) \\ &\leq d(x \wedge y) \end{aligned}$$

It completes the proof. □

Proposition 3.3. *Let d be a generalized (f, g)-derivation on a lattice L. Then*

$$d(x \wedge y) \leq dx \vee dy$$

for all $x, y \in L$.

Proof. Since $dx \wedge fy \leq dx$ and $gx \wedge dy \leq dy$ then

$$\begin{aligned} d(x \wedge y) &= (dx \wedge fy) \vee (gx \wedge dy) \\ &\leq dx \vee dy \end{aligned}$$

□

Proposition 3.4. *Let d be a generalized (f, g)-derivation on a lattice L. If L has a least element 0,such that $f0 = 0$ and $g0 = 0$ then $d0 = 0$.*

Proof. We know that $dx \leq fx \vee gx$ for all $x \in L$ from Proposition (3.1). Since 0 is the least element of the lattice then we get $0 \leq d0 \leq f0 \vee g0 = 0$ means $d0 = 0$. □

Proposition 3.5. *Let L be a lattice with a greatest element 1, d be a generalized (f, g)-derivation on L and $f1 = g1 = 1$. Then the following identities hold;*
(i) If $fx \leq d1$ and $gx \leq d1$ then $dx = fx \vee gx$
(ii) If $fx \geq d1$ and $gx \geq d1$ then $dx \geq d1$

Proof. (i) Since $dx = d(x \wedge 1) = (dx \wedge f1) \vee (gx \wedge d1) = dx \vee gx$ then

$$gx \leq dx. \tag{3.5}$$

Similarly since $dx = d(1 \wedge x) = (d1 \wedge fx) \vee (g1 \wedge dx) = fx \vee dx$ we have

$$fx \leq dx. \tag{3.6}$$

Combining (3.5) and (3.6) then we get

$$gx \vee fx \leq dx. \tag{3.7}$$

We know $dx \leq fx \vee gx$ from Proposition (3.1). Finally

$$gx \vee fx \leq dx \leq fx \vee gx.$$

This means $dx = fx \vee gx$.

(ii) Since

$$\begin{aligned} dx &= d(x \wedge 1) \\ &= (dx \wedge f1) \vee (gx \wedge d1) \\ &= dx \vee d1 \end{aligned}$$

then

$$dx \geq d1.$$

□

Definition 3.2. Let L be a lattice and d be a generalized (f,g)-derivation. Define a set $F = \{x \in L : dx = fx \vee gx\}$.

Proposition 3.6. *Let L be a lattice, d be an isotone generalized (f,g)-derivation. If $x, y \in F$ and f, g are decreasing functions, then $x \vee y \in F$.*

Proof. Since $x \leq x \vee y$ then $f(x \vee y) \leq fx$ and since $y \leq x \vee y$ then $g(x \vee y) \leq gy$. Then we have

$$\begin{aligned} f(x \vee y) &\leq fx \vee gx \\ g(x \vee y) &\leq fy \vee gy \end{aligned}$$

Since d is an isotone generalized (f,g)-derivation then

$$\begin{aligned} f(x \vee y) \vee g(x \vee y) &\leq (fx \vee gx) \vee (fy \vee gy) \\ &= dx \vee dy \\ &\leq d(x \vee y). \end{aligned}$$

We know that $d(x \vee y) \leq f(x \vee y) \vee g(x \vee y)$. As a result

$$d(x \vee y) = f(x \vee y) \vee g(x \vee y)$$

and $x \vee y \in F$. □

Proposition 3.7. *Let L be a lattice and d be a generalized (f,g)-derivation of L Then the following conditions are equivalent;*
(i) d is an isotone generalized (f,g)-derivation
(ii) $dx \vee dy \leq d(x \vee y)$

Proof. (1)⇒(2) Suppose that d is an isotone generalized (f,g)-derivation. We know that $x \leq x \vee y$ and $y \leq x \vee y$. Since d is isotone then $dx \leq d(x \vee y)$ and $dy \leq d(x \vee y)$. Hence we get $dx \vee dy \leq d(x \vee y)$.

(2)⇒(1) Suppose that $dx \vee dy \leq d(x \vee y)$ and $x \leq y$. Then we get $dx \leq dx \vee dy \leq d(x \vee y) = dy$. This means that d is an isotone derivation. □

Theorem 3.1. *Let L be a lattice with greatest element 1 and d be an isotone generalized (f,g)-derivation on L. Let $f1 = g1 = 1$ and either $fx \geq gx$ or $fx \leq gx$ for all $x \in L$. Then*

$$dx = (fx \vee gx) \wedge d1$$

for all $x \in L$.

Proof. If d is an isotone generalized (f,g)-derivation then $dx \leq d1$.

Let $fx \geq gx$. Then we have $dx \leq fx \vee gx = fx$. From this we get $dx \leq fx \wedge d1$. Also

$$\begin{aligned} dx &= d\left((x \vee 1) \wedge x\right) \\ &= [d(x \vee 1) \wedge fx] \vee [g(x \vee 1) \wedge dx] \\ &= (d1 \wedge fx) \vee (g1 \wedge dx) \\ &= (d1 \wedge fx) \vee (1 \wedge dx) \\ &= (d1 \wedge fx) \vee dx \\ &= d1 \wedge fx. \end{aligned}$$

Since $fx \vee gx = fx$ then we can get

$$dx = (fx \vee gx) \wedge d1.$$

Now suppose that $gx \geq fx$. Then $dx \leq fx \vee gx = gx$. From this we get $dx \leq gx \wedge d1$. Also

$$\begin{aligned} dx &= d\left(x \wedge (x \vee 1)\right) \\ &= [dx \wedge f(x \vee 1)] \vee [gx \wedge d(x \vee 1)] \\ &= (dx \wedge f1) \vee (gx \wedge d1) \\ &= dx \vee (gx \wedge d1) \\ &= gx \wedge d1. \end{aligned}$$

Since $gx \geq fx$ then $fx \vee gx = gx$ and

$$dx = (fx \vee gx) \wedge d1.$$

This completes the proof. □

4. Conclusion

In this study we introduced the notion of generalized (f,g)-derivation of a lattice. If the function g is equal to the function f then the (f,g)-derivation is the f-derivation defined in [8]. Also if we choose the function f and g the identity functions both, then the derivation we define coincides with the derivation defined in [22].

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