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ACTIONS AND COVERINGS OF TOPOLOGICAL GROUPOIDS

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Abstract. Let R be a topological group-groupoid. We define a category $TGGdCov(R)$ of coverings of R and a category $TGGdOp(R)$ of actions of R on topological groups and then prove the equivalence of these categories. Further, if R is topological ring-groupoid then we define a category $TRGdCov(R)$ of coverings of R and a category $TRGdOp(R)$ of actions of R on topological rings and then prove the equivalence of these categories.

1. Introduction

The theory of covering groupoids plays an important role in the application of groupoids [1]. There is a result that if R is a groupoid then the category $GdCov(R)$ of groupoid coverings of R is equivalent to the category $GdOp(R)$ of the groupoid actions of R on sets [6]. The topological version of this problem is proved in [3]. They defined a category $TGdCov(R)$ of topological groupoid coverings of R and a category $TGdOp(R)$ of topological actions of R on topological spaces for the topological groupoid R and they proved the equivalence of these categories.

Mucuk in [5] proved that if R is a group-groupoid then the category $GGdCov(R)$ of group-groupoid coverings of R is equivalent to the category $GGdOp(R)$ of the group-groupoid actions of R on groups. He also proved that if R is a ring object in the category of groupoids which is called ring-groupoid then the category $RGdCov(R)$ of ring-groupoid coverings of R is equivalent to the category $RGdOp(R)$ of the ring-groupoid actions of R on rings [13].

Notion of topological group-groupoid was firstly presented in [8] in the proof of the equivalence of categories of topological group-groupoids and topological crossed modules. The topological ring-groupoid is a topological ring object in the category of topological groupoids, which appeared earlier in [14] in the proof of the equivalence of the categories of $UTRCov(X)$ of coverings of topological ring X in which

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both X and $\tilde{X}$ have universal coverings, and $UTRGdCov(\pi_1 X)$ of coverings of topological ring-groupoid $\pi_1 X$ in which X and $\tilde{R_0} = \tilde{X}$ have universal coverings.

In this paper, we present similar results for topological group-groupoids and topological ring-groupoids. If R is a topological group-groupoid then the category $TGGdCov(R)$ of coverings of R is equivalent to the category $TGGdOp(R)$ of actions of R on topological groups. In addition to this, we prove that if R is a topological group-groupoid which is transitive on underlying groupoid with Hausdorff object space then there exists a topological group-groupoid H and a topological covering morphism of topological group-groupoids $p : H \to R$.

Further, here we prove that if R is a topological ring-groupoid then the category $TRGdCov(R)$ of coverings of R is equivalent to the category $TRGdOp(R)$ of actions of R on topological rings. We also prove that if R is a topological ring-groupoid which is transitive on underlying groupoid with Hausdorff object space then there exists a topological ring-groupoid H and a topological covering morphism of topological ring-groupoids $p : H \to R$.

2. Topological Groupoid Coverings

A *groupoid* consists of two sets R and R_0 called respectively the set of morphisms or elements and the set of objects of the groupoid together with two maps $\alpha, \beta : R \to R_0$, called source and target maps respectively, a map $1_{(\ )} : R_0 \to R$, $x \mapsto 1_x$ called the object map and a partial multiplication or composition $R_{\alpha}\times_{\beta}R \to R$, $(b, a) \mapsto b \circ a$ defined on the pullback

$$R_{\alpha}\times_{\beta}R = \{(b, a) : \alpha(b) = \beta(a)\}$$

These maps are subject to following conditions:

(1) $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for each $(b, a) \in R_{\alpha}\times_{\beta}R$,
(2) $c\circ(b\circ a) = (c\circ b)\circ a$ for all $c, b, a \in R$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$,
(3) $\alpha(1_x) = \beta(1_x) = x$ for each $x \in R_0$, where 1_x is the identity at x,
(4) $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$ for all $a \in R$, and
(5) each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $a^{-1} \circ a = 1_{\alpha(a)}$, $a \circ a^{-1} = 1_{\beta(a)}$ [4].

Definition 2.1. ([10])A topological groupoid is a groupoid R such that the sets R and R_0 are topological spaces, and source, target, object, inverse and composition maps are contiuous.

Let R be a topological groupoid. For each $x, y \in R_0$ we write $R(x, y)$ as a set of all morphisms $a \in R$ such that $\alpha(a) = x$ and $\beta(a) = y$. We will write $St_R x$ for the set $\alpha^{-1}(x)$, and $CoSt_R x$ for the set $\beta^{-1}(x)$ for $x \in R_0$. The *object* or *vertex group* at x is $R(x) = R(x, x) = St_R x \cap CoSt_R x$. We say R is *transitive* (resp. 1-*transitive*, *simply transitive*) if for each $x, y \in R_0$, $R(x, y)$ is non-empty (resp. a singleton, has no more than one element).

Let R and H be two topological groupoids. A morphism of topological groupoids is pair of maps $f : H \to R$ and $f_0 : H_0 \to R_0$ such that f and f_0 are continuous and $\alpha_R \circ f = f_0 \circ \alpha_H$, $\beta_R \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_{\alpha}\times_{\beta}H$.

We refer to [1] and [10] for more details concerning the basic concepts.

Definition 2.2. A morphism of topological groupoids $p : H \to R$ is called a topological covering morphism if the map

$$(p, \alpha) : H \to R_{\alpha}\times_{p_0}H_0$$

is a homeomorphism. In such a case the inverse to (p, α) is written $s_p : R_{\alpha\times p_0} H_0 \to H$ and called the lifting map.

Let R be a topological groupoid. Then we have a category of coverings of topological groupoid R $TGdCov(R)$ whose objects are the covering morphisms of topological groupoids $p : \tilde{R} \to R$ and a morphism from $p : \tilde{R} \to R$ to $q : \tilde{H} \to R$ is a morphism of topological groupoids $r : \tilde{R} \to \tilde{H}$ such that $p = qr$. Note that r is also covering morphism of topological groupoids.

We call a subset U of X *liftable* if it is open, path connected and the inclusion $U \to X$ maps each fundamental group $\pi_1(U, x)$, $x \in X$, to the trivial subgroup of $\pi_1(X, x)$. Remark that if X has a universal covering then each point $x \in X$ has a liftable neighbourhood [11].

Let X be a topological space which has a universal covering then the fundamental groupoid $\pi_1 X$ is a topological groupoid [2]. The following example was proved in [3].

Example 2.1. Let $p : \tilde{X} \to X$ be a covering map of topological spaces which both spaces $\tilde{X}$ and X have universal coverings. Then the induced morphism $\pi_1 p : \pi_1 \tilde{X} \to \pi_1 X$ is a covering morphism of topological groupoids.

Let X be a topological space which has a universal covering. Let $TCov(X)$ be a category whose objects are the covering maps $p : \tilde{X} \to X$ and a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a map $r : \tilde{X} \to \tilde{Y}$ such that $p = qr$. So r is also a covering map.

Let $UTCov(X)$ be the full subcategory of $TCov(X)$ on those objects $p : \tilde{X} \to X$ which both $\tilde{X}$ and X have universal coverings. Let $UTGdCov(\pi_1 X)$ be the full subcategory of $TGdCov(\pi_1 X)$ on those objects $p : \tilde{R} \to \pi_1 X$ such that X and $\tilde{R}_0 = \tilde{X}$ have universal coverings. Then we can give the following proposition which is proved in [13].

Proposition 2.1. *The categories $UTCov(X)$ and $UTGdCov(\pi_1 X)$ are equivalent.*

Definition 2.3. [13]Let R be a topological groupoid and let X be a topological space. Let $w : X \to R_0$ be a continuous map. We say that R acts topologically on X if there is a continuous map $\phi : R_{\alpha\times w} X \to X$, $(a, x) \mapsto {}^a x$ where $R_{\alpha\times w} X = \{(a, x) : a \in R,\ x \in X,\ \alpha(a) = w(x)\}$ such that

(1) $w({}^a x) = \beta(a)$,
(2) ${}^b({}^a x) = {}^{b\circ a} x$,
(3) ${}^{1_{w(x)}} x = x$,

whenever these expressions are defined. We also say that X is a left R-space via p.

As an example, if $p : H \to R$ is a covering morphism then R acts on H_0 via $p_0 : H_0 \to R_0$.

Example 2.2. ([3]) Let R be a topological groupoid which acts on a topological space X via $w : X \to R_0$. Then a topological groupoid $R \bowtie X$, called topological action groupoid, with object space X is defined as follows: The morphisms are the pairs (a, x) of $R_{\alpha\times w} X$, source and target maps are defined by $\alpha(a, x) = x$ and $\beta(a, x) = {}^a x$ and the groupoid multiplication is defined by $(b, y) \circ (a, x) = (b \circ a, x)$. The groupoid $R \bowtie X$ has product topology. Clearly source, target, object, inverse and composition maps of groupoid $R \bowtie X$ are continuous. Thus $R \bowtie X$ becomes a topological groupoid and the projection $p : R \bowtie X \to R$, $(a, x) \mapsto a$ is a covering morphism of topological groupoids.

Let R be a topological groupoid. We write (X, w) for an topological action of R on a space X via $w : X \to R_0$. A morphism from (X, w) to (X', w') is a map $f : X \to X'$ such that $w'f = w$ and $f(^a x) = {}^a f(x)$. So we have a category $GdOp(R)$ of topological actions of R on spaces.

Theorem 2.1. *([3]) Let R be a topological groupoid. Then the categories $TGdCov(R)$ and $TGdOp(R)$ are equivalent.*

In a similar way to left R-spaces we define a right R-space. Let R be a topological groupoid and let X be a topological space. Let $w : X \to R_0$ be a continuous map. We say that X is a right R-space if there is a continuous map $\phi : X_{w\times\beta}R \to X$, $(x, a) \mapsto x^a$, where $X_{w\times\beta}R = \{(x, a) : a \in R,\ x \in X,\ \beta(a) = w(x)\}$ such that

(1) $w(x^a) = \alpha(a)$,
(2) $x(^a)^b = x^{b \circ a}$,
(3) $x^{1_{w(x)}} = x$,

whenever these expressions are defined [7].

Let R, H be topological groupoids and let X be a topological space. We call that X is $R-H$-bispace via $w - w'$ if X is a left R-space via w and also a right H-space via w' such that $w'(^a x) = w'(x)$, $w(x^b) = w(x)$ and ${}^a(x^b) = (^a x)^b$ whenever $x \in X$, $a \in R$, $b \in H$ and ${}^a x$, x^b are defined [7].

A standard example of $R-R$-bispace is the groupoid R itself via $\beta - \alpha$ with left and right actions given by composition in R.

An important use of $R-H$-bispaces is in constructing left actions of R on spaces of H-orbits. Suppose that X is a $R-H$-bispace via $w-w'$. Although the left action of R on X defines a left action of R on X/H, there is difficulty in proving continuity of this action due to the fact that a pullback of identification maps need not be an identification map. This difficulty can be overcome in the useful special case given by the following theorem in [7].

Theorem 2.2. *Let H be a topological group and let R be a topological groupoid such that R_0 is Hausdorff. If X is a $R - H$-bispace via $w - w'$, then the action of R on X determines the structure of a left R-space on the orbit space X/H.*

Corollary 2.1. *If R is a topological groupoid which object space R_0 is Hausdorff and $N(x)$ is a subgroup of $R(x)$, $St_R(N(x))$ becomes a left R-space.*

Thus we can give following proposition in [7]

Proposition 2.2. *Let R be a transitive topological groupoid with Hausdorff object space, let $x \in R_0$ and let $N(x)$ be a subgroup of $R(x)$. Then there exists a transitive groupoid H and a topological covering morphism $p : H \to R$. Further there is $y \in H_0$ such that $p(H(y)) = N(x)$.*

3. Topological Group-Groupoid Coverings

A topological group is a group X with a topology on the underlying set such that the group multiplication and inverse map are continuous. A topological group morphism (topological homomorphism) of a topological group into another is an abstract group homomorphism which is also a continuous map. A connected topological group X is called simply connected if the fundamental group of its underlying topological space consists of only the identity. Let X and $\tilde{X}$ be topological groups. A map $p : \tilde{X} \to X$ is called a covering morphism of topological groups if p is a morphism of groups and p is a covering map on the underlying spaces.

Definition 3.1. [8]A topological group-groupoid R is a topological groupoid endowed with a topological group structure such that the following maps are morphism of topological groupoids:

(1) $m : R \times R \to R$, $(a, b) \mapsto a + b$, group multiplication,
(2) $u : R \to R$, $a \mapsto -a$, group inverse map,
(3) $* : (\star) \to R$, where $(\star)$ is a singleton groupoid.

We write $a + b$ for the group multiplication of a and b, and write $b \circ a$ for the composition in the topological groupoid R. The group inverse of an element a is written $-a$. Also by 3, if e is the identity element of R_0 then 1_e is that of R.

Note that we have the interchange law

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c)$$

whenever both $b \circ a$ and $d \circ c$ are defined.

Example 3.1. Let X be a topological group. We obtain a topological group-groupoid $X \times X$ with the object set X. The morphisms are the pairs (y, x), the source and target maps are defined by $\alpha(y, x) = x$ and $\beta(y, x) = y$, the groupoid composition is defined by $(z, y) \circ (y, x) = (z, x)$ and the group multiplication is defined by $(z, t) + (y, x) = (z + y, t + x)$. The group-groupoid $X \times X$ has product topology. So source, target, object, inverse, group inverse maps, group multiplication and composition of $X \times X$ are continuous. Then $X \times X$ becomes a topological group-groupoid.

Following two results are proved in [9].

Proposition 3.1. *Let R be a topological group-groupoid, e the identity of R_0. Then the transitive component $C(R)_e$ of e is a topological group-groupoid.*

Proposition 3.2. *Let R be a topological group-groupoid, e the identity of R_0. Then the star $St_R e = \{a \in R : \alpha(a) = e\}$ of e becomes a topological group.*

Let R and H be two topological group-groupoids. A morphism $f : H \to R$ from H to R is a morphism of underlying topological groupoids preserving the topological group structure, i.e., $f(a + b) = f(a) + f(b)$ for $a, b \in H$. A morphism $f : H \to R$ of topological group-groupoids is called a topological covering if it is a covering morphism on the underlying topological groupoids [9].

We know from [8] that if X is a topological group whose underlying space X has a universal covering, then the fundamental groupoid $\pi_1 X$ becomes a topological group-groupoid. So we can give the following example.

Example 3.2. Let X, $\tilde{X}$ be a topological groups whose underlying spaces X, $\tilde{X}$ have universal covering and let $p : X \to \tilde{X}$ be a covering map of topological groups. Then the induced morphism $\pi_1 p : \pi_1 X \to \tilde{X}$ becomes a covering morphism of topological group-groupoid.

Let X be a topological space. Then we have a category of coverings of topological space X denoted by $TCov(X)$ whose objects are covering maps $p : \tilde{X} \to X$ and a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a map $f : \tilde{X} \to \tilde{Y}$ (hence f is a covering map) such that $p = qf$. Further, we have a groupoid $\pi_1 X$ called a fundamental groupoid [1] and have a category of coverings of fundamental groupoid $\pi_1 X$ denoted by $GdCov(\pi_1 X)$ whose objects are the groupoid coverings $p : \tilde{R} \to \pi_1 X$ of $\pi_1 X$ and a morphism from $p : \tilde{R} \to \pi_1 X$ to $q : \tilde{H} \to \pi_1 X$ is a morphism $f : \tilde{R} \to \tilde{H}$ of groupoids (hence f is a covering morphism) such that $p = qf$.

We recall the following result from Brown [1].

Proposition 3.3. *Let X be a topological space which has a universal covering. Then the category $TCov(X)$ of topological coverings of X and the category $GdCov(\pi_1 X)$ of covering groupoids of fundamental groupoid $\pi_1 X$ are equivalent.*

We have a category of coverings of topological group X denoted by $TGCov(X)$ whose objects are topological group coverings $p : \tilde{X} \to X$ and a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a map $f : \tilde{X} \to \tilde{Y}$ (hence f is a covering map) such that $p = qf$. For a topological group X, the fundamental groupoid $\pi_1 X$ is a group-groupoid and so we have a category denoted by $GGdCov(\pi_1 X)$ whose objects are the group-groupoid coverings $p : \tilde{R} \to \pi_1 X$ of $\pi_1 X$ and a morphism from $p : \tilde{R} \to \pi_1 X$ to $q : \tilde{H} \to \pi_1 X$ is a morphism $f : \tilde{R} \to \tilde{H}$ of group-groupoids (hence f is a covering morphism) such that $p = qf$.

Thus the following result is given in [5].

Proposition 3.4. *Let X be a topological group whose underlying space has a universal covering. Then the category $TGCov(X)$ of topological group coverings of X is equivalent to the category $GGdCov(\pi_1 X)$ of group-groupoid coverings of the group-groupoid $\pi_1 X$.*

In addition to these results, we can give Theorem 3.1 in [9].

Let $UTGCov(X)$ be the full subcategory of $TGCov(X)$ on those objects $p : \tilde{X} \to X$ in which both $\tilde{X}$ and X have universal coverings. Let $UTGGdCov(\pi_1 X)$ be the full subcategory of $TGGdCov(\pi_1 X)$ on those objects $p : \tilde{R} \to \pi_1 X$ in which X and $\tilde{R}_0 = \tilde{X}$ have universal coverings [13].

Theorem 3.1. *The categories $UTGCov(X)$ and $UTGGdCov(\pi_1 X)$ are equivalent.*

Definition 3.2. Let $p : \tilde{R} \to R$ be a covering morphism of groupoids and $q : H \to R$ a morphism of groupoids. If there exists a unique morphism $\tilde{q} : H \to \tilde{R}$ such that $p\tilde{q} = q$ then we say that q lifts to $\tilde{q}$ by p.

We recall the following theorem from [1], which is an important result to have the lifting maps on covering groupoids.

Theorem 3.2. *Let $p : \tilde{R} \to R$ be a covering morphism of groupoids, $x \in R_0$ and $\tilde{x} \in \tilde{R}_0$ such that $p(\tilde{x}) = x$. Let $q : H \to R$ be a morphism of groupoids such that H is transitive and $\tilde{y} \in H_0$ such that $q(\tilde{y}) = x$. Then the morphism $q : H \to R$ uniquely lifts to a morphism $\tilde{q} : H \to \tilde{R}$ such that $\tilde{q}(\tilde{y}) = \tilde{x}$ if and only if $q[H(\tilde{y})] \subseteq p[\tilde{R}(\tilde{x})]$, where $H(\tilde{y})$ and $\tilde{R}(\tilde{x})$ are the object groups.*

Let R be a topological group-groupoid, e the identity of R_0. Let $\tilde{R}$ be just a topological groupoid, and let $p : \tilde{R} \to R$ be a covering morphism of topological groupoids such that $p(\tilde{e}) = e$ for the identity $\tilde{e} \in \tilde{R}_0$. We say the topological group structure of R lifts to $\tilde{R}$ if there exists a topological group structure on $\tilde{R}$ with the identity element $\tilde{e} \in \tilde{R}_0$ such that $\tilde{R}$ is a topological group-groupoid and $p : \tilde{R} \to R$ is a morphism of topological group-groupoids.

Theorem 3.3. *Let $\tilde{R}$ be a topological groupoid and let R be a topological group-groupoid. Let $p : \tilde{R} \to R$ be a universal covering on the underlying groupoids such that both groupoids R and $\tilde{R}$ are transitive. Let e be the identity element of R_0 and $\tilde{e} \in \tilde{R}_0$ such that $p(\tilde{e}) = e$. Then the topological group structure of R lifts to $\tilde{R}$ with identity $\tilde{e}$.*

Definition 3.3. Let R be a topological group-groupoid and let X be a topological group. An action of the topological group-groupoid R on X consists of a topological group morphism $w : X \to R_0$ and an action of underlying topological groupoid of R on the underlying space of X via $w : X \to R_0$ such that the following interchange law holds

$$({}^{b}y) + ({}^{a}x) =^{b+a} (x + y)$$

whenever both sides are defined. We write (X, w) for such an action.

Example 3.3. Let $p : H \to R$ be a covering morphism of topological group-groupoids. Then the topological group-groupoid R acts on $H_0 = X$ via $p_0 : H_0 \to R$. By the definition of topological covering morphism, we have a homeomorphism $s_p : R_{\alpha \times p_0} H_0 \to H$ and morphisms of topological groups $p, p_0 = w$. The composing s_p with target map $\tilde{\beta} : H \to H_0$ of H gives a continuous map $\phi : R_{\alpha \times p_0} H_0 \to H_0$, $(a, \tilde{x}) \mapsto {}^{a}\tilde{x} = \tilde{\beta}(\tilde{a})$. In addition to these, we can write $w({}^{a}\tilde{x}) = p_0({}^{a}\tilde{x}) = p_0(\tilde{\beta}(\tilde{a})) = \beta(a)$, ${}^{b}({}^{a}\tilde{x}) = {}^{b}\tilde{\beta}(\tilde{a}) = \tilde{\beta}(\tilde{b})$, ${}^{b\circ a}\tilde{x} = \tilde{\beta}(\tilde{b} \circ \tilde{a}) = \tilde{\beta}(\tilde{b})$ and ${}^{1_{p_0(\tilde{x})}}\tilde{x} = \tilde{\beta}(\tilde{c}) = \tilde{x}$,for $a, b \in R$, $\tilde{a}, \tilde{b}, \tilde{c} \in H$ and $\tilde{x} \in H_0$. Consequently R acts on H_0.

Example 3.4. Let R be a topological group-groupoid which acts on a topological group X. Then from [3], there are a topological action groupoid $R \bowtie X$ and a covering morphism of topological groupoids $p : R \bowtie X \to R$, $(a, x) \mapsto a$. Also there exists the action group-groupoid $R \bowtie X$ with group multiplication $(a, x) + (b, y) = (a + b, x + y)$ and projection $p : R \bowtie X \to R$ is a covering morphism of group-groupoids [13]. Since group multiplication of $R \bowtie X$ defined by group multiplication of topological group-groupoid R, it is continuous. In addition to these, $p((a, x) + (b, y)) = p(a + b, x + y) = a + b = p(a, x) + p(b, y)$, that is, p holds topological group structure. Thus $R \bowtie X$ becomes topological group-groupoid and p is covering morphism of topological group-groupoids.

Let R be a topological group-groupoid. We have a category of coverings of topological group-groupoid R denoted by $TGGdCov(R)$ whose objects are topological group-groupoids coverings $p : H \to R$ and morphism from $p : H \to R$ to $q : K \to R$ is a morphism $r : H \to K$ (hence r is a topological covering morphism) such that $p = qr$.

Let X, X' be two topological groups and let R be topological group-groupoid. Also R acts on X via $w : X \to R_0$ and let R be acts on X' via $w' : X' \to R_0$. We show these actions by (X, w) and (X', w') respectively. A morphism of topological group-groupoid actions $f : (X, w) \to (X', w')$ is a morphism $f : X \to X'$ of topological groups such that $w'f = w$ and $f({}^{a}x) = {}^{a}f(x)$ whenever ${}^{a}x$ is defined. Thus we have a category of actions of topological group-groupoid R denoted by $TGGdOp(R)$ whose objects are topological actions (X, w) and morphism from (X, w) to (X', w') is a morphism $f : X \to X'$ of topological groups such that $w'f = w$ and $f({}^{a}x) = {}^{a}f(x)$ whenever ${}^{a}x$ is defined. Now we can give following result.

Theorem 3.4. *Let R be a topological group-groupoid. Then the categories $TGGdCov(R)$ and $TGGdOp(R)$ are equivalent.*

Proof. Define a functor

$$\Gamma : TGGdOp(R) \rightarrow TGGdCov(R)$$

as follows: Let R be a topological group-groupoid which acts on a topological group X. Then from Example 3.4 there are a topological action group-groupoid $R \bowtie X$ and a covering morphism of topological group-groupoids $p : R \bowtie X \rightarrow R$, $(a, x) \mapsto a$.

Conversely, we define a functor

$$\Phi : TGGdCov(R) \rightarrow TGGdOp(R)$$

as follows: Let $p : H \rightarrow R$ be a covering morphism of topological group-groupoids. Then from Example 3.3 the topological group-groupoid R acts on $H_0 = X$ via $p_0 : H_0 \rightarrow R$.

The natural equivalencies $\Gamma\Phi \simeq 1$ and $\Phi\Gamma \simeq 1$ follow. □

Proposition 3.5. *Let R be a topological group-groupoid which is transitive on underlying groupoid with Hausdorff object space. Let $e \in R_0$ be the identity element of R_0 and let $N(e)$ be a subgroup of $R(e)$. Then there exists topological group-groupoid H with the identity $\tilde{e} = N(e)$ of H_0 and a topological covering morphism of topological group-groupoids $p : H \rightarrow R$ such that $p(H(\tilde{e})) = N(e)$.*

Proof. Let X be the set of all cosets $a \circ N(e)$ for all $a \in St_R e$. If we define a group multiplication on X by

$$(a \circ N(e)) + (b \circ N(e)) = (a + b) \circ N(e)$$

then from [13] we have a action group-groupoid $R \bowtie X = H$ with $(R \bowtie X)_0 = X$ such that $p(H(\tilde{e})) = N(e)$ and covering morphism of group-groupoids $p : H \rightarrow R$. The multiplication on X is continuous for that it consists of the multiplication of topological group R and the composition of the topological groupoid. So X is a topological group. $R \bowtie X$ is also a topological group with group multiplication

$$(b, a' \circ N(e)) + (c, a \circ N(e)) = (b + c, (a' + a) \circ N(e))$$

and induced topology from $R \times X$. Since R is a transitive groupoid with Hausdorff object space, there exist a transitive topological groupoid H and a topological covering morphism $p : H \rightarrow R$ [7]. Hence H is a topological group-groupoid and p is a covering morphism of topological group-groupoids. □

4. Topological Ring-Groupoid Coverings

A topological ring is a ring R with a topology on the underlying set such that the ring structure maps (i.e., group multiplication, group inverse and ring multiplication) are continuous. A topological ring morphism (topological homomorphism) of a topological ring into another is an abstract ring homomorphism which is also a continuous map. Let X and $\tilde{X}$ be topological rings. A map $p : \tilde{X} \rightarrow X$ is called a covering morphism of topological rings if p is a morphism of rings and p is a covering map on the underlying spaces.

Definition 4.1. [14]A topological ring-groupoid R is a topological groupoid endowed with a topological ring structure such that the following ring structure maps are morphisms of topological groupoids:

(1) $m : R \times R \to R$, $(a, b) \mapsto a + b$, group multiplication,
(2) $u : R \to R$, $a \mapsto -a$, group inverse map,
(3) $* : (\star) \to R$, where $(\star)$ is a singleton,
(4) $n : R \times R \to R$, $(a, b) \mapsto ab$, ring multiplication.

We write $a + b$ for the group multiplication, and write ab for the ring multiplication of a and b, and write $b \circ a$ for the composition in the topological groupoid R. Also by 3, if 0 is the zero element of R_0 then 1_0 is that of R.

In a topological ring-groupoid R, we have the interchange laws

(1) $(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$ and
(2) $(c \circ a)(d \circ b) = (cd) \circ (ab)$

whenever both $(c \circ a)$ and $(d \circ b)$ are defined.

Example 4.1. Let R be a topological ring. Then a topological ring-groupoid $R \times R$ with object set R is defined as follows: The morphisms are the pairs (y, x), the source and target maps are defined by $\alpha(y, x) = x$ and $\beta(y, x) = y$, the groupoid composition is defined by $(z, y) \circ (y, x) = (z, x)$, the group multiplication is defined by $(z, t) + (y, x) = (z + y, t + x)$ and ring multiplication is defined by $(z, t)(y, x) = (zy, tx)$. $R \times R$ has product topology. So all structure maps of ring-groupoid $R \times R$ become continuous. Hence $R \times R$ is a topological ring-groupoid.

Let R and H be two topological ring-groupoids. A morphism $f : H \to R$ from H to R is a morphism of underlying topological groupoids preserving the topological ring structure, i.e., $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for $a, b \in H$. A morphism $f : H \to R$ of topological ring-groupoids is called a *topological covering morphism* if it is a covering morphism on the underlying topological groupoids [14].

We know from [14] that if X is a topological ring whose underlying space X has a universal covering, then the fundamental groupoid $\pi_1 X$ becomes a topological ring-groupoid. So we can give the following example.

Example 4.2. Let X, $\tilde{X}$ be two topological rings whose underlying spaces X, $\tilde{X}$ have universal covering and let $p : X \to \tilde{X}$ be covering map of topological rings. Thus the induced morphism $\pi_1 p : \pi_1 X \to \tilde{X}$ becomes a covering morphism of topological ring-groupoids.

Now from [14] we can give following propositions.

Proposition 4.1. *Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element of ring R_0. Then the transitive component $C_R(0)$ of 0 is a topological ring-groupoid.*

Proposition 4.2. *Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element in the ring R_0. Then the star $St_R 0 = \{a \in R : \alpha(a) = 0\}$ of 0 becomes a topological ring.*

The following result is given in [12].

Proposition 4.3. *Let X be a topological ring whose underlying space has a universal covering. Then the category $TRCov(X)$ of topological ring coverings of X is equivalent to the category $RGdCov(\pi_1 X)$ of ring-groupoid coverings of the ring-groupoid $\pi_1 X$.*

In addition to these results, in Theorem 4.1 we quote a similar result in [14].

Let $UTRCov(X)$ be the full subcategory of $TRCov(X)$ on those objects $p : \tilde{X} \to X$ in which both $\tilde{X}$ and X have universal coverings. Let $UTRGdCov(\pi_1 X)$ be the full subcategory of $TRGdCov(\pi_1 X)$ on those objects $p : \tilde{R} \to \pi_1 X$ in which X and $\tilde{R}_0 = \tilde{X}$ have universal coverings.

Theorem 4.1. *The categories $UTRCov(X)$ and $UTRGdCov(\pi_1 X)$ are equivalent.*

Let R be a topological ring-groupoid and let $0 \in R_0$ be the zero element of the ring R_0. Let $\tilde{R}$ be just a topological groupoid and let $p : \tilde{R} \to R$ be a covering morphism of topological groupoids, $\tilde{0} \in \tilde{R}_0$ such that $p(\tilde{0}) = 0$. We say the topological ring structure of R lifts to $\tilde{R}$ if there exists a topological ring structure on $\tilde{R}$ with the zero element $\tilde{0} \in \tilde{R}_0$ such that $\tilde{R}$ is a topological ring-groupoid and $p : \tilde{R} \to R$ is a morphism of topological ring-groupoids. Thus we can give following theorem which is proved in [14].

Theorem 4.2. *Let $\tilde{R}$ be a topological groupoid and let R be a topological ring-groupoid. Let $p : \tilde{R} \to R$ be a universal covering on the underlying groupoids such that both groupoids R and $\tilde{R}$ are transitive. Let 0 be the zero element in the ring R_0 and $\tilde{0} \in \tilde{R}_0$ such that $p(\tilde{0}) = 0$. Then the topological ring structure of R lifts to $\tilde{R}$ with zero element $\tilde{0}$.*

Definition 4.2. Let R be a topological ring-groupoid and let X be a topological ring. A topological action of the topological ring-groupoid R on X consists of a topological ring morphism $w : X \to R_0$ and a continuous action of the underlying topological groupoid of R on the underlying space of X via $w : X \to R_0$ such that the following interchange laws hold

(1) $({}^b y) + ({}^a x) = {}^{b+a}(y + x)$,

(2) $({}^b y)({}^a x) = {}^{ba}(yx)$,

whenever both sides are defined.

Example 4.3. Let R be a topological ring-groupoid which acts on a topological ring X via $w : X \to R_0$. In [3] it is proved that $R \bowtie X$ is a topological groupoid with object set $(R \bowtie X)_0 = X$ and morphism set $R \bowtie X = \{(a, x) \in R \times X : {}^a x = y\}$. Furthermore, the projection $p : R \bowtie X \to R$, $(a, x) \mapsto a$ becomes a covering morphism of topological groupoids. Also in [13] it is showed that if a ring-groupoid R acts on a ring X via $w : X \to R_0$ then $R \bowtie X$ becomes a ring-groupoid and the projection $p : R \bowtie X \to R$, $(a, x) \mapsto a$ is a covering morphism of ring-groupoids. Clearly, the ring operations

$$(a, x) + (b, y) = (a + b, x + y) \text{ and}$$
$$(a, x)(b, y) = (ab, xy)$$

are also continuous since they are defined by the operations of the topological rings R and X. Thus $R \bowtie X$ becomes a topological ring-groupoid and the projection $p : R \bowtie X \to R$, $(a, x) \mapsto a$ is a covering morphism of topological ring-groupoids.

Example 4.4. Let $p : H \to R$ be a covering morphism of topological ring-groupoids. Then the topological ring-groupoid R acts on $H_0 = X$ via $p_0 : H_0 \to R$. In Example 3.3, we showed that there exists the action of underlying topological groupoid of R on the underlying space of H_0. Because if $p : H \to R$ is a covering morphism of topological ring-groupoids then we have a homeomorphism

$s_p : R_{\alpha\times p_0}H_0 \to H$ and $p, p_0 = w$ are topological ring morphisms. Consequently, R acts on H_0.

Example 4.5. Let R be a topological ring-groupoid which acts on a topological ring X. Then from Example 3.4, there are a topological action group-groupoid $R \bowtie X$ and a covering morphism of topological group-groupoids $p : R \bowtie X \to R$, $(a, x) \mapsto a$. Also we know that $R \bowtie X$ is a ring-groupoid with group multiplication $(a, x) + (b, y) = (a + b, x + y)$ and ring multiplication $(a, x)(b, y) = (ab, xy)$ [13]. The ring multiplication is continuous for that it consists of the multiplications of topological rings R and X. Thus $R \bowtie X$ becomes topological ring-groupoid and p is covering morphism of topological ring-groupoid.

Let R be a topological ring-groupoid. We have a category of coverings of topological ring-groupoid R denoted by $TRGdCov(R)$ whose objects are topological ring-groupoid coverings $p : H \to R$ and morphism from $p : H \to R$ to $q : K \to R$ is a morphism $r : H \to K$ (hence r is a topological covering morphism) such that $p = qr$.

Let X, X' be two topological rings and let R be topological ring-groupoid. Also R acts on X via $w : X \to R_0$ and let R be acts on X' via $w' : X' \to R_0$. We show these actions by (X, w) and (X', w') respectively. A morphism of topological ring-groupoid actions $f : (X, w) \to (X', w')$ is a morphism $f : X \to X'$ of topological rings such that $w'f = w$ and $f({}^a x) = {}^a f(x)$ whenever ${}^a x$ is defined. Thus we have a category of actions of topological ring-groupoid R denoted by $TRGdOp(R)$ whose objects are topological actions (X, w) and morphism from (X, w) to (X', w') is a morphism $f : X \to X'$ of topological rings such that $w'f = w$ and $f({}^a x) = {}^a f(x)$ whenever ${}^a x$ is defined. Now we can give following result.

Theorem 4.3. *Let R be a topological ring-groupoid. Then the categories $TRGdCov(R)$ and $TRGdOp(R)$ are equivalent.*

Proof. Define a functor

$$\Gamma : TRGdOp(R) \to TRGdCov(R)$$

as follows: Let R be a topological ring-groupoid which acts on a topological ring X. In Example 4.5, the action on a topological ring of the topological ring-groupoid R gives rise to a topological action ring-groupoid $R \bowtie X$ and a covering morphism of topological ring-groupoids $p : R \bowtie X \to R$, $(a, x) \mapsto a$.

Conversely, we define a functor

$$\Phi : TRGdCov(R) \to TRGdOp(R)$$

as follows: Let $p : H \to R$ be a covering morphism of topological ring-groupoids. Then the topological ring-groupoid R acts on $H_0 = X$ via $p_0 : H_0 \to R$ by Example 4.4.

The natural equivalencies $\Gamma\Phi \simeq 1$ and $\Phi\Gamma \simeq 1$ follow. □

Proposition 4.4. *Let R be a topological ring-groupoid which is transitive on underlying groupoid with Hausdorff object space. Let 0 be the zero element of R_0 and let $N(0)$ be a subgroup of $R(0)$. Then there exists a topological ring-groupoid H such that $\tilde{0} = N(0)$ is the zero element of H_0. There is also a topological covering morphism of topological ring-groupoids $p : H \to R$ such that $p(H(\tilde{0})) = N(0)$.*

Proof. Let X be the set of all cosets $a \circ N(0)$ for all $a \in St_R 0$. If we define group multiplication and ring multiplication on X by

$$(a \circ N(e)) + (b \circ N(e)) = (a + b) \circ N(e),$$
$$(a \circ N(e))(b \circ N(e)) = (ab) \circ N(e),$$

then from [13] we have a ring-groupoid $R \bowtie X = H$ with object set $(R \bowtie X)_0 = X$ such that $p(H(\tilde{0})) = N(0)$ and covering morphism of ring-groupoids $p : H \to R$. The group and ring multiplications on X are continuous, because they consist of the multiplications of topological ring R and the composition of the topological groupoid. So X is a topological ring. $R \bowtie X$ is also a topological ring with group and ring multiplications, respectively,

$$(b, a' \circ N(e)) + (c, a \circ N(e)) = (b + c, (a' + a) \circ N(e)),$$
$$(b, a' \circ N(e))(c, a \circ N(e)) = (bc, (a'a) \circ N(e)),$$

and induced topology from $R \times X$. Since R is a transitive groupoid with Hausdorff object space, there exist a transitive topological groupoid H and a topological covering morphism $p : H \to R$ [7]. Hence H is a topological ring-groupoid and p is a covering morphism of topological ring-groupoids. □

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