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ON ABSOLUTE ALMOST MATRIX SUMMABILITY OF ORTHOGONAL SERIES

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Abstract. In this paper we present some results on absolute almost matrix summability of an orthogonal series. Precisely, some sufficient conditions under which an orthogonal series will be absolute almost matrix summable are obtained. The most important corollaries of the main results also are deduced.

1. Introduction

As is known the absolute summability is a generalization of the concept of the absolute convergence just as the summability is an extension of the concept of the convergence. Lorentz [5], for the first time in 1948, defined almost convergence of a bounded sequence and it is shown in [6] that every convergent sequence is almost convergent. The idea of almost convergence led up to the definition of almost generalized Nörlund summability introduced by Qureshi [22] which includes almost Nörlund, Riesz, harmonic and Cesàro summability as particular cases.

The absolute summability of an orthogonal series has been studied by many authors, and for such examples, one can see the papers of Tandori [11], Leindler [7]–[10], Okuyama and Tsuchikura [12], Okuyama [13]–[16], Szalay [17], Billard [18], Grepaqevskaya [19], Spevakov and Kudrajatsev [20], and also recently by the present author [24]–[28]. Here in this paper, we shall not consider simply the absolute almost matrix summability of an orthogonal series but its absolute almost matrix summability of order k, $1 \leq k \leq 2$, which is our main aim. Note that this notion has been introduced by present author [26] motivated from a definition introduced by T. M. Flett [4].

2. Notations and notions

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$. A sequence $s := \{s_n\}$ is said to be almost convergent to a limit ℓ if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{v=m}^{n+m} s_v = \ell,$$

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uniformly with respect to m.

Let $A := (a_{nv})$ be a lower triangular matrix of non-zero diagonal entries.

The matrix A defines the sequence-to-sequence transformation, mapping the sequence $s := \{s_n\}$ to $As := \{A_n(s)\}$, where

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v,\ n = 0, 1, 2, \dots$$

A series $\sum_{n=0}^{\infty} a_n$ is said to be almost matrix summable to ℓ (see [21]) provided that

$$A_{n,m}(s) = \sum_{v=0}^{n} a_{nv} s_{v,m} \to \ell$$

uniformly with respect to m, where

$$s_{v,m} = \frac{1}{v+1} \sum_{i=m}^{v+m} s_i.$$

The following definition has been introduced in [26]:

A series $\sum_{n=0}^{\infty} a_n$ is said to be absolute almost matrix summable, briefly $|A|_{m;k}$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\triangle} A_{n,m}(s)|^k$$

converges uniformly with respect to m, where

$$\bar{\triangle} A_{n,m}(s) = A_{n,m}(s) - A_{n-1,m}(s),$$

and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |A|_{m;k}.$$

In the special case when $k = 1$, $a_{nv} = 0$ for $v = 0, 1, 2, \dots, n-1$, and $a_{nn} = 1$, then the above definition reduces to the following one introduced in [1]:

A series $\sum_{n=0}^{\infty} a_n$ is said to be absolute almost convergent if

$$\sum_{n=1}^{\infty} |s_{n,m} - s_{n-1,m}|$$

converges uniformly in m. Note that it was proved in [2] that the convergence of $\sum_{n=1}^{\infty} |s_{n,m} - s_{n-1,m}|$ for only one m implies convergence for any other value of m. We denote the set of all absolutely convergent sequences and absolutely almost convergent sequences respectively by $\bar{\ell}$ and $\hat{\ell}$. It is shown in [1] (pages 38 and 46) that the following hold true

$$\bar{\ell} \subset \hat{\ell} \subset |C, 1| \quad \text{and} \quad \hat{\ell} \not\subseteq \bar{\ell},$$

where $(C, 1)$ is the set of sequences which are Cesàro summable. It is important to emphasize here that in [3] also was concluded as follows: if an infinite series is absolutely almost convergent then it is almost convergent, but the converse of this is false.

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a,b). We assume that $f(x)$ belongs to $L^2(a,b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{2.1}$$

where $c_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Before starting the main results first we introduce some other notations.

For the matrix $A := (a_{nv})$, we associate four lower matrices with entries as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \ n, i = 0, 1, 2, \dots$$

$$\tilde{a}_{nv} := \sum_{j=v}^{n} \frac{a_{nj}}{j+1}, \ n, j = 0, 1, 2, \dots$$

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \ \hat{\tilde{a}}_{nv} = \tilde{a}_{nv} - \tilde{a}_{n-1,v}, \ n = 1, 2, \dots$$

where we note that $\hat{a}_{00} = \bar{a}_{00} = a_{00}$.

The following lemma due to B. Levi (see, for example [31]) is often used in the theory of functions. It will help us to prove main results.

Lemma 2.1. *If $h_n(t) \in L(U)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_U h_n(t)dt < \infty, \tag{2.2}$$

then the series

$$\sum_{n=1}^{\infty} h_n(t)$$

converges (absolutely) almost everywhere on U to a function $h(t) \in L(U)$.

Throughout this paper K denotes a positive constant depending only on k, and it may be different in different relations.

3. Main Results

Theorem 3.1. *If the series*

$$\sum_{n=1}^{\infty} \left\{ n^{2\left(1-\frac{1}{k}\right)} \sum_{j=0}^{n} \left(\hat{a}_{nj} - j\hat{\tilde{a}}_{nj}\right)^2 |c_{m+j}|^2 \right\}^{\frac{k}{2}}$$

converges uniformly with respect to m for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|A|_{m;k}$–summable almost everywhere.

Proof. Let $s_v(x)=\sum_{j=0}^{v}c_j\varphi_j(x)$ be vth partial sums of the series (2.1), and $1<k<2$. A straightforward calculation shows that

$$\begin{aligned}
s_{v,m}(x) &= \frac{1}{v+1}\sum_{k=m}^{v+m}s_k(x)\\
&= \frac{1}{v+1}\sum_{k=0}^{v}s_{k+m}(x)\\
&= \frac{1}{v+1}\sum_{k=0}^{v}\sum_{j=0}^{k+m}c_j\varphi_j(x)\\
&= s_{m-1}(x)+\sum_{j=0}^{v}\left(1-\frac{j}{v+1}\right)c_{m+j}\varphi_{m+j}(x)
\end{aligned}$$

and thus

$$\begin{aligned}
A_{n,m}(s)(x) &= \sum_{v=0}^{n}a_{nv}s_{v,m}(x)\\
&= \sum_{v=0}^{n}a_{nv}\left(s_{m-1}(x)+\sum_{j=0}^{v}\left(1-\frac{j}{v+1}\right)c_{m+j}\varphi_{m+j}(x)\right)\\
&= s_{m-1}(x)+\sum_{v=0}^{n}a_{nv}\sum_{j=0}^{v}c_{m+j}\varphi_{m+j}(x)-\sum_{v=0}^{n}\frac{a_{nv}}{v+1}\sum_{j=0}^{v}jc_{m+j}\varphi_{m+j}(x)\\
&= s_{m-1}(x)+\sum_{j=0}^{n}c_{m+j}\varphi_{m+j}(x)\sum_{v=j}^{n}a_{nv}-\sum_{j=0}^{n}jc_{m+j}\varphi_{m+j}(x)\sum_{v=j}^{n}\frac{a_{nv}}{v+1}\\
&= s_{m-1}(x)+\sum_{j=0}^{n}\bar{a}_{nj}c_{m+j}\varphi_{m+j}(x)-\sum_{j=0}^{n}j\tilde{a}_{nj}c_{m+j}\varphi_{m+j}(x)\\
&= s_{m-1}(x)+\sum_{j=0}^{n}\left(\bar{a}_{nj}-j\tilde{a}_{nj}\right)c_{m+j}\varphi_{m+j}(x).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\bar{\triangle}A_{n,m}(s)(x) &= A_{n,m}(s)(x)-A_{n-1,m}(s)(x)\\
&= \sum_{j=0}^{n}\left(\bar{a}_{nj}-j\tilde{a}_{nj}\right)c_{m+j}\varphi_{m+j}(x)-\sum_{j=0}^{n-1}\left(\bar{a}_{n-1,j}-j\tilde{a}_{n-1,j}\right)c_{m+j}\varphi_{m+j}(x)\\
&= \sum_{j=0}^{n}\left[\left(\bar{a}_{nj}-\bar{a}_{n-1,j}\right)-j\left(\tilde{a}_{nj}-\tilde{a}_{n-1,j}\right)\right]c_{m+j}\varphi_{m+j}(x)\\
&= \sum_{j=0}^{n}\left(\hat{a}_{nj}-j\tilde{\tilde{a}}_{nj}\right)c_{m+j}\varphi_{m+j}(x).
\end{aligned}$$

Using the Hölder's inequality, orthogonality, and the above equality we have that

$$
\begin{aligned}
\int_a^b |\bar{\triangle} A_{n,m}(s)(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}} \left(\int_a^b |A_{n,m}(s)(x) - A_{n-1,m}(s)(x)|^2 dx \right)^{\frac{k}{2}} \\
&= K \left(\int_a^b \left| \sum_{j=0}^{n} \left(\hat{a}_{nj} - j\tilde{\hat{a}}_{nj} \right) c_{m+j} \varphi_{m+j}(x) \right|^2 dx \right)^{\frac{k}{2}} \\
&= K \left\{ \sum_{j=0}^{n} \left(\hat{a}_{nj} - j\tilde{\hat{a}}_{nj} \right)^2 |c_{m+j}|^2 \right\}^{\frac{k}{2}} .
\end{aligned}
$$

Furthermore, the series

(3.1)

$$
\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\bar{\triangle} A_{n,m}(s)(x)|^k dx \leq K \sum_{n=1}^{\infty} n^{k-1} \left\{ \sum_{j=0}^{n} \left(\hat{a}_{nj} - j\tilde{\hat{a}}_{nj} \right)^2 |c_{m+j}|^2 \right\}^{\frac{k}{2}}
$$

converges, since the last one converges (by assumption) uniformly with respect to m. Since the functions $|\bar{\triangle} A_{n,m}(s)(x)|$ are non-negative, then by the Lemma 2.1 the series

$$
\sum_{n=1}^{\infty} n^{k-1} |\bar{\triangle} A_{n,m}(s)(x)|^k
$$

converges almost everywhere. For $k = 1$ we use the Schwartz's inequality, until for $k = 2$ we use just the orthogonality. This completes the proof of the theorem. □

We note that:

1. The absolute almost matrix summability of order k reduces to the absolute almost generalized Nörlund summability of order k ($|N, p, q|_{m;k}$−summability), if

$$
\begin{aligned}
a_{nv} &= \frac{p_{n-v} q_v}{R_n} \quad \text{for} \quad 0 \leq v \leq n, \\
a_{nv} &= 0 \quad \text{for} \quad v > n,
\end{aligned}
$$

   where for two given sequences of positive real constants $p = \{p_n\}$ and $q = \{q_n\}$, the convolution $R_n := (p * q)_n$ is defined by

$$
(p * q)_n = \sum_{v=0}^{n} p_v q_{n-v} = \sum_{v=0}^{n} p_{n-v} q_v .
$$

2. The absolute almost generalized Nörlund summability of order k reduces to the absolute almost Nörlund summability of order k ($|N, p|_{m;k}$−summability, $R_n \equiv P_n$), if $q_n = 1$ for all n.
3. The absolute almost generalized Nörlund summability of order k reduces to the absolute almost Riesz summability of order k ($|\overline{N}, q|_{m;k}$−summability, $R_n \equiv Q_n$), if $p_n = 1$ for all n.
4. In the special case when $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, the absolute almost Nörlund summability of order k reduces to the absolute almost generalized Cesàro summability of order k.

5. If $p_n = 1/(n+1)$ the absolute almost Nörlund summability of order k reduces to the absolute almost harmonic summability of order k.

In the following we shall use the notations

$$R_n^j := \sum_{v=j}^{n} p_{n-v} q_v, \quad R_{n-1}^n = 0, \quad R_n^0 = R_n$$

$$\widehat{R}_n^j := \sum_{v=j}^{n} \frac{p_{n-v} q_v}{v+1}, \quad \widehat{R}_{n-1}^n = 0.$$

Since

$$\begin{aligned}
\hat{a}_{nj} - j\tilde{\tilde{a}}_{nj} &= \bar{a}_{nj} - \bar{a}_{n-1,j} - j\left(\tilde{a}_{nj} - \tilde{a}_{n-1,j}\right) \\
&= \sum_{i=j}^{n} a_{ni} - \sum_{i=j}^{n-1} a_{n-1,i} - j\left(\sum_{i=j}^{n} \frac{a_{ni}}{i+1} - \sum_{i=j}^{n-1} \frac{a_{n-1,i}}{i+1}\right) \\
&= \frac{1}{R_n} \sum_{i=j}^{n} p_{n-i} q_i - \frac{1}{R_{n-1}} \sum_{i=j}^{n-1} p_{n-i} q_i \\
&\quad -j\left(\frac{1}{R_n} \sum_{i=j}^{n} \frac{p_{n-i} q_i}{i+1} - \frac{1}{R_{n-1}} \sum_{i=j}^{n-1} \frac{p_{n-i} q_i}{i+1}\right) \\
&= \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}}\right),
\end{aligned}$$

then we obtain

Corollary 3.1 ([28]). *If the series*

$$\sum_{n=1}^{\infty} \left\{ n^{2\left(1-\frac{1}{k}\right)} \sum_{j=0}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2 |c_{m+j}|^2 \right\}^{\frac{k}{2}}$$

converges uniformly with respect to m for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|N,p,q|_{m;k}-$summable almost everywhere.

Also the following corollaries can be obtain from the above theorem:

Corollary 3.2 ([28]). *If for $1 \le k \le 2$ the series*

$$\sum_{n=1}^{\infty} \left(\frac{n^{\left(1-\frac{1}{k}\right)} p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} p_{n-j}^2 \left[1 - \frac{P_{n-1-j}}{p_{n-j}} + j \sum_{v=0}^{n-j} \frac{P_n - (n+1-v)p_n}{(n-v)(n+1-v)p_n p_{n-j}} p_v\right]^2 |c_{m+j}|^2\right\}^{\frac{k}{2}}$$

converges uniformly with respect to m, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|N,p|_{m;k}-$summable almost everywhere.

Corollary 3.3 ([28]). *If for $1 \le k \le 2$ the series*

$$\sum_{n=1}^{\infty}\left(\frac{n^{\left(1-\frac{1}{k}\right)}q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{j=1}^{n}\left[Q_{j-1}+j\left(\frac{Q_n}{n+1}-\sum_{v=j}^{n}\frac{q_v}{v+1}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}$$

converges uniformly with respect to m, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)$$

is $|\overline{N},q|_{m;k}-$summable almost everywhere.

Now we shall prove a very general theorem on $|A|_{m;k}-$summability almost everywhere of an orthogonal series. It involves a positive sequence that satisfies certain conditions. We prove this theorem by Okuyama ([13]) and Ul'yanov's ([23]) scheme modifying it accordingly.

Indeed, if we put

$$\mathcal{B}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}n^{\frac{2}{k}}\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2 \tag{3.2}$$

then the following theorem holds true.

Theorem 3.2. *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty}|c_{m+n}|^2\Omega^{\frac{2}{k}-1}(n)\mathcal{B}^{(k)}(n)$ converges uniformly with respect to m, then the orthogonal series $\sum_{n=0}^{\infty}c_n\varphi_n(x)\in|A|_{m;k}$ almost everywhere, where $\mathcal{B}^{(k)}(n)$ is defined by (3.2).*

Proof. Applying Hölder's inequality to the inequality (3.1) we get that

$$\begin{aligned}
\sum_{n=1}^{\infty}n^{k-1}\int_a^b|\bar{\triangle}A_{n,m}(s)(x)|^kdx &\le K\sum_{n=1}^{\infty}n^{k-1}\left\{\sum_{j=0}^{n}\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2|c_{m+j}|^2\right\}^{\frac{k}{2}}\\
&= K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{2-k}{2}}}\left\{n\Omega^{\frac{2}{k}-1}(n)\sum_{j=0}^{n}\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2|c_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\le K\left(\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))}\right)^{\frac{2-k}{2}}\left\{\sum_{n=1}^{\infty}n\Omega^{\frac{2}{k}-1}(n)\sum_{j=0}^{n}\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2|c_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\le K\left\{\sum_{j=1}^{\infty}|c_{m+j}|^2\sum_{n=j}^{\infty}n\Omega^{\frac{2}{k}-1}(n)\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2\right\}^{\frac{k}{2}}\\
&\le K\left\{\sum_{j=1}^{\infty}|c_{m+j}|^2\left(\frac{\Omega(j)}{j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}n^{\frac{2}{k}}\left(\hat{a}_{nj}-j\tilde{\hat{a}}_{nj}\right)^2\right\}^{\frac{k}{2}}\\
&= K\left\{\sum_{j=1}^{\infty}|c_{m+j}|^2\Omega^{\frac{2}{k}-1}(j)\mathcal{B}^{(k)}(j)\right\}^{\frac{k}{2}}
\end{aligned}$$

which by assumption is finite uniformly with respect to m. For the proof now one can do the same reasoning as in the proof of Theorem 3.1. The proof is completed. $\square$

The following corollaries follow from Theorem 3.2.

Corollary 3.4 ([28]). *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the series*

$$\sum_{n=1}^{\infty} |c_{m+n}|^2 \Omega^{\frac{2}{k}-1}(n) \mathcal{N}^{(k)}(n)$$

converges uniformly with respect to m, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p, q|_{m;k}$$

almost everywhere, where $\mathcal{N}^{(k)}(n)$ is defined by

$$\mathcal{N}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2.$$

In the special case, when $p_v = 1$ for all v, we obtain the equality (see for details [28], page 285 and 287)

$$\begin{aligned} \mathcal{D}_n^j &:= \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \\ &= -\frac{q_n}{Q_n Q_{n-1}} \left[Q_{j-1} + j\left(\frac{Q_n}{n+1} - \sum_{v=j}^{n} \frac{q_v}{v+1} \right) \right], \end{aligned}$$

and also for $q_v = 1$ and all v

$$\mathcal{D}_n^j = \frac{p_n p_{n-j}}{P_n P_{n-1}} \left[1 - \frac{P_{n-1-j}}{p_{n-j}} + j \sum_{v=0}^{n-j} \frac{P_n - (n+1-v)p_n}{(n-v)(n+1-v)p_n p_{n-j}} p_v \right].$$

Therefore we deduce the following.

Corollary 3.5. *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the series*

$$\sum_{n=1}^{\infty} |c_{m+n}|^2 \Omega^{\frac{2}{k}-1}(n) \mathcal{R}^{(k)}(n)$$

converges uniformly with respect to m, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |\overline{N}, q|_{m;k}$$

almost everywhere, where $\mathcal{R}^{(k)}(n)$ is defined by

$$\mathcal{R}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left\{ \frac{q_n}{Q_n Q_{n-1}} \left[Q_{j-1} + j\left(\frac{Q_n}{n+1} - \sum_{v=j}^{n} \frac{q_v}{v+1} \right) \right] \right\}^2.$$

Corollary 3.6. *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the series*

$$\sum_{n=1}^{\infty} |c_{m+n}|^2 \Omega^{\frac{2}{k}-1}(n) \mathcal{P}^{(k)}(n)$$

converges uniformly with respect to m, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p|_{m;k}$$

almost everywhere, where $\mathcal{P}^{(k)}(n)$ is defined by

$$\mathcal{P}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left\{ \frac{p_n p_{n-j}}{P_n P_{n-1}} \left[1 - \frac{P_{n-1-j}}{p_{n-j}} + j \sum_{v=0}^{n-j} \frac{P_n - (n+1-v)p_n}{(n-v)(n+1-v)p_n p_{n-j}} p_v \right] \right\}^2 .$$

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