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ON NADLER'S FIXED POINT THEOREM FOR PARTIAL METRIC SPACES

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Abstract. Recently, H. Aydi, M. Abbas and C. Vetro [Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, Topology Appl. 159 (2012), 3234-3242] have obtained a version of the well-known Nadler fixed point theorem for multi-valued maps on complete partial metric spaces. In this note we prove a new partial metric version of Nadler's theorem and derive some consequences of it.

1. Introduction and preliminaries

The notion of a partial metric space was introduced by Matthews ([11]) in the study of denotational semantics o programming languages. In this way, he modeled as partial metric spaces some distinguished examples of the theory of computation as the domain of words and the domain of the interval, and also proved a partial metric version of the celebrated Banach fixed point theorem ([11, Theorem 5.3]). Since then, many authors have obtained fixed point theorems for partial metric spaces that extend and generalize in several directions the one given by Matthews (see e.g. [1, 2, 3, 5, 8, 9, 13]). In particular, Aydi, Abbas and Vetro ([6]) started the fixed point theory for multi-valued maps on partial metric spaces, obtaining, among other results, a generalization of the well-known Nadler fixed point theorem ([12]). In this note we prove a new partial metric version of Nadler's result which is different to the one presented in [6]. Our contraction condition is based upon contraction conditions for single-valued self maps as used in [10], and conditions of Berinde's type ([7]) for partial metric spaces, recently explored by Altun and Acar in [4].

Next we recall some pertinent concepts and results of the basic theory of partial metric spaces, given in [11], which will be useful later on.

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The letters $\mathbb{R}^+$ and ω will denote the set of all non-negative real numbers and of all non-negative integer numbers, respectively.

Definition 1.1. A partial metric on a (non-empty) set X is a function $p : X \times X \to \mathbb{R}^+$ satisfying the following conditions for all $x, y, z \in X$:
(P1) $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then, the pair (X, p) is called a partial metric space.

Example 1.1. Let $X = \mathbb{R}^+$ and p defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space.

Each partial metric p on a set X induces a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $(x_n)_{n\in\omega}$ in a partial metric space (X, p) converges to $x \in X$ for τ_p if and only if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.

Given a partial metric space (X, p), the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$, is a metric on X.

We also have the following useful equivalence:

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 1.2. Let (X, p) be a partial metric space.

(1) A sequence $(x_n)_{n\in\omega}$ in X is called a Cauchy sequence in (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(2) (X, p) is called complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges for τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.1. *Let (X, p) be a partial metric space. Then:*

(a) A sequence $(x_n)_{n\in\omega}$ in X is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) (X, p) is complete if and only if the metric space (X, p^s) is complete.

Let (X, p) be a partial metric space. Following [6], a subset A of X is called bounded if there is $x_0 \in X$ and $M > 0$ such that $a \in B_p(x_0, M)$ for all $a \in A$, i.e., $p(x_0, a) < p(x_0, x_0) + M$ for all $a \in A$.

The set of all non-empty τ_p-closed and bounded subsets of (X, p) is denoted by $CB^p(X)$.

Aydi, Abbas and Vetro ([6]) defined the so-called partial Hausdorff metric of (X, p) on $CB^p(X)$ as follows.

Given $x \in X$ and $A \in CB^p(X)$, let $p(x, A) = \inf_{a\in A} p(x, a)$.

Now let $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ given by

$$\delta_p(A, B) = \sup_{a\in A} p(a, B),$$

for all $A, B \in CB^p(X)$.

The function $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ given by

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

for all $A, B \in CB^p(X)$, is said to be the partial Hausdorff metric of (X, p) ([6]).

If (X, d) is a metric space, then the partial Hausdorff metric constructed above is exactly the Hausdorff metric H_d of (X, d) on the set $CB(X)$ of all nonempty closed and bounded subsets of X.

Next we collect some interesting properties of H_p obtained in [6].

Proposition 1.1. [6, Proposition 2.3] *Let (X, p) be a partial metric space. For each $A, B, C \in CB^p(X)$ the following hold:*
(a) $H_p(A, A) \le H_p(A, B)$;
(b) $H_p(A, B) = H_p(B, A)$;
(c) $H_p(A, B) \le H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

2. The results

Nadler proved in [12] the following multi-valued extension of the classical Banach fixed point theorem.

Theorem 2.1. [12, Theorem 5] *Let (X, d) be a complete metric space. If $T : X \to CB(X)$ is a multi-valued map such that for all $x, y \in X$, we have*

$$H_d(Tx, Ty) \le kd(x, y),$$

where $k \in (0, 1)$, then T has a fixed point, i.e., there exists $z \in X$ such that $z \in Tz$.

The main result of [6] is the following generalization of Nadler's fixed point theorem to the realm of partial metric spaces.

Theorem 2.2. [6, Theorem 3.2] *Let (X, p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multi-valued map such that for all $x, y \in X$, we have*

$$H_p(Tx, Ty) \le kp(x, y),$$

where $k \in (0, 1)$, then T has a fixed point.

In our main result (Theorem 2.3 below) we consider multi-valued maps from X into $X \cup CB^p(X)$. This approach is motivated, in part, by the fact that $CB^p(X) = \emptyset$ when (X, p) is the (complete) partial metric space of Example 1. Indeed, nonempty τ_p-closed sets are of the form $[r, +\infty[$, $r \in \mathbb{R}^+$. Hence, given $A = [r, +\infty[$, $r \in \mathbb{R}^+$, then for each $x_0 \in X$ and each $M > 0$, we have that $p(x_0, a) \ge p(x_0, x_0) + M$, where $a = \max\{r, x_0 + M\}$. Consequently $CB^p(X) = \emptyset$.

Moreover, our approach has also the advantage that fixed point results for self (single-valued) maps can be derived from the corresponding fixed point results for multi-valued maps (recall that if (X, p) is a partial metric space and $x \in X$, then $\{x\}$ does not necessarily belongs to $CB^p(X)$, as Example 1.1 shows).

Given a partial metric space (X, p), we shall write $T : X \to X \cup CB^p(X)$, whenever that T is a multi-valued map on X such that for each $x \in X$, $|Tx| = 1$ (i.e., $Tx = \{y\}$ for some $y \in X$), or $Tx \in CB^p(X)$. Then T will be called a mixed multi-valued map.

Note that, in particular, both a self map $T : X \to X$ and a multi-valued map $T : X \to CB^p(X)$, are mixed multi-valued maps. Of course, if τ_p is a T_1 topology on X and $Tx = \{y\}$, we also have $Tx \in CB^p(X)$.

If $Tx = \{y\}$ for some $y \in X$, we simply write $Tx = y$, if no confusion arises.

A mixed multi-valued map $T : X \to X \cup CB^p(X)$ will be called $T\mid_X$-orbitally continuous if whenever $(x_n)_{n\in\omega}$ is as sequence in X such that $x_{n+1} \in Tx_n$ for all $n \in \omega$, and $\lim_{n\to\infty} p(x, x_n) = p(x, x)$ for some $x \in X$ with $|Tx| = 1$, then $\lim_{n\to\infty} p(Tx, x_n) = p(Tx, Tx)$. When $T : X \to X$ we simply say that T is orbitally continuous.

Lemma 2.1. [10] *Let (X, p) be a partial metric space. A sequence $(x_n)_{n\in\omega}$ in X is a Cauchy sequence in (X, p) if and only if it satisfies the following condition:*

For each $\varepsilon > 0$ there is $n_0 \in \omega$ such that $p(x_n, x_m) - p(x_n, x_n) < \varepsilon$ whenever $n_0 \le n \le m$.

Theorem 2.3. *Let (X, p) be a complete partial metric space. If $T : X \to X \cup CB^p(X)$ is a $T\mid_X$-orbitally continuous mixed multi-valued map such that for each $x, y \in X$ we have*

$$H_p(Tx, Ty) \le k[p(x, y) - p(x, x)] + p(y, y) + L \min\{p^s(x, Ty), p^s(y, Tx)\}, \tag{2.1}$$

where $k \in (0, 1)$ and $L \in \mathbb{R}^+$, then T has a fixed point.

Proof. Fix $r \in (k, 1)$. We first show that there exists a sequence $(x_n)_{n\in\omega}$ in X such that for each $n \in \omega$, $x_{n+1} \in Tx_n$ and

$$p(x_{n+1}, x_{n+2}) \le r[p(x_n, x_{n+1}) - p(x_n, x_n)] + p(x_{n+1}, x_{n+1}).$$

To this end, choose an $x_0 \in X$, and take $x_1 \in Tx_0$. Since

$$p^s(x_1, Tx_0) = \inf_{y\in Tx_0} p^s(x_1, y) = 0,$$

we deduce from (2.1) that

$$H_p(Tx_0, Tx_1) \le k[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1). \tag{2.2}$$

Now we consider two cases.

- *Case 1:* $|Tx_1| = 1$. Then, there exists $x_2 \in X$ such that $Tx_1 = x_2$, and hence $p(x_1, x_2) \le H_p(Tx_0, Tx_1)$. It follows from (2.2) that

$$p(x_1, x_2) \le k[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1).$$

- *Case 2:* $|Tx_1| > 1$. Then $Tx_1 \in CB^p(X)$.

  If $p(x_0, x_1) = p(x_0, x_0)$, we deduce from (2.2) that $H_p(Tx_0, Tx_1) \le p(x_1, x_1)$, so, in particular,

$$\inf_{z\in Tx_1} p(x_1, z) \le p(x_1, x_1),$$

  i.e.,

$$\inf_{z\in Tx_1} p(x_1, z) = p(x_1, x_1). \tag{2.3}$$

  By (2.3), there is a sequence $(z_n)_{n\in\omega}$ in Tx_1 such that

$$\lim_{n\to\infty} p(x_1, z_n) = p(x_1, x_1),$$

  i.e., $(z_n)_{n\in\omega}$ converges to x_1 for τ_p, and thus $x_1 \in Tx_1$. Therefore, putting $x_2 = x_1$, we trivially deduce that

$$p(x_1, x_2) = r[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1).$$

If $p(x_0, x_1) > p(x_0, x_0)$, we have

$$\begin{array}{rcl} H_p(Tx_0, Tx_1) & \leq & k[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1) \\ & < & r[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1). \end{array}$$

So, in particular,

$$\inf_{z \in Tx_1} p(x_1, z) < r[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1).$$

Therefore, there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) < r[p(x_0, x_1) - p(x_0, x_0)] + p(x_1, x_1).$$

Now, repeating the above arguments, there exists $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \leq r[p(x_1, x_2 - p(x_1, x_1)] + p(x_2, x_2],$$

and following this process we find a sequence a sequence $(x_n)_{n \in \omega}$ in X such that for each $n \in \omega$, $x_{n+1} \in Tx_n$ and

$$p(x_{n+1}, x_{n+2}) \leq r[p(x_n, x_{n+1}) - p(x_n, x_n)] + p(x_{n+1}, x_{n+1}).$$

Consequently

$$p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) \leq r^{n+1}[p(x_0, x_1) - p(x_0, x_0)],$$

for all $n \in \omega$. It immediately follows from the triangle inequality (P4) and standard techniques that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, x_m) - p(x_n, x_n) < \varepsilon,$$

whenever $n_0 \leq n \leq m$. Thus $(x_n)_{n \in \omega}$ is a Cauchy sequence in (X, p) by Lemma 2.1. Let $z \in X$ be such that

$$\lim_{n \to \infty} p(z, x_n) = \lim_{n \to \infty} p(x_n, x_n) = p(z, z), \tag{2.4}$$

or, equivalently, $\lim_{n \to \infty} p^s(z, x_n) = 0$.

We shall show that z is a fixed point of T. Indeed, since, by (2.1),

$$H_p(x_{n+1}, Tz) \leq k[p(x_n, z) - p(x_n, x_n)] + p(z, z) + L \min\{p^s(x_n, Tz), p^s(z, x_{n+1})\},$$

for all $n \in \omega$, we immediately deduce the existence of a subsequence $(x_{n_j})_{j \in \mathbb{N}}$ of $(x_n)_{n \in \omega}$ and of a sequence $(z_j)_{j \in \mathbb{N}}$ in Tz such that

$$p(x_{n_j+1}, z_j) < \frac{1}{j} + p(z, z), \tag{2.5}$$

for all $j \in \mathbb{N}$. Since

$$p(z, z) \leq p(z, z_j) \leq p(z, x_{n_j+1}) + p(x_{n_j+1}, z_j) - p(x_{n_j+1}, x_{n_j+1}),$$

for all $j \in \mathbb{N}$, it follows from (2.4) and (2.5) that

$$\lim_{j \to \infty} p(z, z_j) = p(z, z). \tag{2.6}$$

If $|Tz| > 1$, then $Tz \in CB^p(X)$. From (2.6) and the fact that $z_j \in Tz$ for all $j \in \mathbb{N}$, it follows that $z \in Tz$, i.e., z is a fixed point of T.

If $|Tz| = 1$, then $z_j = Tz$ for all $j \in \mathbb{N}$, and by (2.6), $p(z, Tz) = p(z, z)$. By (2.4) and our hypothesis that T is $T\,|_X$-orbitally continuous we deduce that

$$\lim_{n \to \infty} p(x_n, Tz) = p(Tz, Tz).$$

Since

$$p(Tz, Tz) \leq p(z, Tz) \leq p(z, x_n) + p(x_n, Tz) - p(x_n, x_n),$$

for all $n \in \omega$, we have taking limits when $n \to \infty$, $p(Tz, Tz) = p(z, Tz)$. Therefore $p(z, z) = p(Tz, Tz) = p(z, Tz)$, so $z = Tz$. This completes the proof. □

As a first consequence of Theorem 2.3 we have the following improvement of Nadler's fixed point theorem.

Corollary 2.1. *Let (X, d) be a complete metric space. If $T : X \to CB(X)$ is a multi-valued map such that for each $x, y \in X$ we have*

$$H_d(Tx, Ty) \leq kd(x, y) + L \min\{d(x, Ty), d(y, Tx)\},$$

where $k \in (0, 1)$ and $L \in \mathbb{R}^+$, then T has a fixed point.

Proof. We show that T is $T\mid_X$-orbitally continuous. Indeed, let $(x_n)_{n\in\omega}$ be a sequence in X such that $x_{n+1} \in Tx_n$ for all $n \in \omega$, and $\lim_{n\to\infty} d(x, x_n) = 0$ for some $x \in X$ with $|Tx| = 1$. Then

$$\begin{aligned} d(Tx, x_{n+1}) &\leq \sup_{y\in Tx_n} d(Tx, y) \leq H_d(Tx, Tx_n) \\ &\leq kd(x, x_n) + Ld(x, Tx_n) \leq kd(x, x_n) + Ld(x, x_{n+1}), \end{aligned}$$

for all $n \in \omega$. Consequently $\lim_{n\to\infty} d(Tx, x_{n+1}) = 0$. Theorem 2.3 concludes the proof. □

We also deduce the following fixed point result for single-valued self maps.

Corollary 2.2. *Let (X, p) be a complete partial metric space. If $T : X \to X$ is an orbitally continuous map such that for each $x, y \in X$ we have*

$$p(Tx, Ty) \leq k[p(x, y) - p(x, x)] + p(y, y) + L \min\{p^s(x, Ty), p^s(y, Tx)\},$$

where $k \in (0, 1)$ and $L \in \mathbb{R}^+$, then T has a fixed point.

We finish the paper with two examples illustrating the obtained results.

Example 2.1. Let $X = \{a, b, c\}$ and let $p : X\times X \to \mathbb{R}^+$ given as $p(a, a) = p(c, c) = 0$, $p(b, b) = 1$, $p(a, b) = p(b, a) = 2$, $p(a, c) = p(c, a) = 4$, and $p(b, c) = p(c, b) = 5$. It is almost obvious that (X, p) is a complete partial metric space. Observe also that τ_p is the discrete topology on X. Now define $T : X \to CB(X)$ by $Ta = a$, $Tb = b$ and $Tc = \{a, b\}$. It is immediate to check that T is $T\mid_X$-orbitally continuous. We also obtain

$$\begin{aligned} H_p(Ta, Tb) &= 2 = \frac{1}{2}[p(a, b) - p(a, a)] + p(b, b) \\ &< 4 = \frac{1}{2}[p(a, b) - p(b, b)] + \min\{p^s(a, Tb), p^s(Ta, b)\}, \end{aligned}$$

$$H_p(Ta, Tc) = 2 = \frac{1}{2}[p(a, c) - p(a, a)] = \frac{1}{2}[p(a, c) - p(c, c)],$$

$$\begin{aligned} H_p(Tb, Tc) &= 2 = \frac{1}{2}[p(b, c) - p(b, b)] \\ &< \frac{7}{2} = \frac{1}{2}[p(b, c) - p(c, c)] + p(b, b), \end{aligned}$$

and hence condition (2.1) is satisfied for $k = 1/2$, $L = 0$. We have shown that all conditions of Theorem 2.3 hold. However, for every $k \in (0, 1)$, we have

$$H_p(Ta, Tb) = 2 > kp(a, b),$$

and thus Theorem 2.2 cannot be applied to this example.

Example 2.2. Let $X = [0, 1]$ and let p be the complete partial metric on X given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$ (compare Example 1.1). Now let $T : X \to X$ defined by $Tx = x^2$ for all $x \in X$. It is clear that T is orbitally continuous. Next we show that the contraction condition of Corollary 2.2 is satisfied for any $k \in (0, 1)$ and $L = 1$. Indeed, if $x = y$ we have

$$p(Tx, Tx) = x^2 \leq x = p(x, x).$$

If $x \neq y$, we suppose, without loss of generality that $x < y$, and consider two cases.

Case 1. $x \leq Ty$. Then we obtain

$$p(Tx, Ty) = y^2 \leq y = p(y, y),$$

and

$$\begin{aligned} p(Tx, Ty) &= y^2 = x + y^2 - x = p(x, x) + \min\{y^2 - x, y - x^2\} \\ &= k[p(x, y) - p(y, y)] + p(x, x) + \min\{p^s(x, Ty), p^s(y, Tx)\}. \end{aligned}$$

Case 2. $x > Ty$. Then we obtain

$$p(Tx, Ty) = y^2 \leq y = p(y, y),$$

and

$$p(Tx, Ty) = y^2 < x = p(x, x).$$

Therefore, we can apply Corollary 2.2. In fact T has two fixed points. However, we cannot apply this corollary when we consider the complete metric d on X given by $d(x, x) = 0$ for all $x \in X$, and $d(x, y) = p(x, y)$ whenever $x \neq y$. Indeed, given $k \in (0, 1)$ take $x \in (k, 1)$ and let $y = x^2$. Then $\min\{d(x, Ty), d(y, Tx)\} = 0$, and

$$d(Tx, Ty) = x^2 > kx = kd(x, y).$$

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