Mathematical Sciences And Applications E-Notes
Volume 1 No. 1 pp. 18–35 (2013) ©MSAEN

DISTINGUISHED NORMALIZATION ON NON-MINIMAL NULL HYPERSURFACES

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(Communicated by Bayram SAHIN)

Abstract. We show that on a non-minimal lightlike hypersurface with nullity degree 1, there exists a unique null transversal (normalizing) vector field with prescribed calibrated divergence, for which the induced connection and the Levi-Civita connection of the associate non-degenerate metric coincide.

1. Introduction

In pseudo-Riemannian manifolds, due to the causal character of three categories of vector fields (namely, space-like, time-like and null), the induced metric on a hypersurface is a non-degenerate metric tensor field or a degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On non-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) notion of the unit orthogonal vector field is known to lead to a canonical decomposition of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore, by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations,... along with the second fundamental form, sharp operator, induced connection, etc. The case the normal vector field is null (also called lightlike), the hypersurface is called lightlike. The geometry of lightlike submanifolds is different and rather difficult since (contrary to the non-degenerate conterpart) the normal vector bundle intersects (non trivially) with the tangent bundle. Thus, one can not find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects on a lightlike submanifold. This is basically the normalization problem.

Date: Received: October 8, 2012 and Accepted: November 12, 2012.

2000 *Mathematics Subject Classification.* 53C50, 53B30; 53B50.

Key words and phrases. Lightlike hypersurface; normalization, associate metric, invariant normalizing differential equation.

Several authors considered this problem in various ways (Akivis-Goldberg [1, 2], Duggal-Bejancu[12], Penrose[17], Katsuno[13], Carter[9], Taub[19], Larsen[15, 16], Pinl[18],etc.). For the most part, these studies are specifc to a given problem, often with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Duggal and Bejancu in [12] introduced a general geometric technique to deal with the above anomaly. Their approach is basically extrinsic (in contrast to the intrinsic one developed by Kupeli [14]), that is very close to the known theory of non-degenerate submanifolds. This approach introduces a non-degenerate screen distribution (or equivalently a null transversal line vector bundle as we may see below) so as to get three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, sharp operators, induced connections, curvature, etc. Unfortunately, the screen distribution is not unique and there is no preferred one in general. As a consequence, it is a systematic task in this approach to study the dependence of the discussed structures and the induced geometric objects with respect to (not only) the screen distribution but also to the choice of the normalizing pair of null vectors. The least we can say is that for the above approach to be complete and consistent, we still need to build a distinguished normalization to accompany it. Most of our recent work are indeed devoted to this normalizing problem [3, 4, 7, 5], including the present one.

The paper is organized as follows. In section 2 we make a general set up on lightlike hypersuerfaces. Section 3 introduces associate metric to a normalized lightlike hypersurface through pseudo-inversion of degenerate metrics and section 4 deals with the determinant of the associate metric relative to the induced volume element. In section 5 we present a technical lemma accounting on how induced geometric objects change under change in normalization followed by a compatibility result needed in the formulation of our normalization constraints. Thereafter, we consider in section 6 the invariant normalizing differential equation and introduce in section 7 the calibrated divergence of sections along the null hypersuerfaces. Finally, we present the main result in section 8 followed by a basic example on the null cone $\wedge_0^3 \subset \mathbb{R}_1^4$.

2. Basic facts on null (lightlike) hypersurfaces

Consider a hypersurface M of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant index $0 < \nu < n+2$. In the classical theory of non-degenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent bundle and plays an important role in the introduction of the main induced geometric objects on M . In a lightlike setting, it is well known that the normal bundle $TM^\perp$ of the lightlike hypersurface M^{n+1} is a rank 1 vector subbundle of the tangent bundle TM. A complementary bundle of $TM^\perp$ in TM is a rank n non-degenerate distribution over M, called a *screen distribution* of M, which we denote by $\mathscr{S}(N)$, such that

$$TM = \mathscr{S}(N) \oplus_{Orth} TM^\perp, \tag{2.1}$$

where $\oplus_{Orth}$ denotes the orthogonal direct sum. Existence of $\mathscr{S}(N)$ is secured provided M be paracompact. A lightlike hypersurface with a specific screen distribution is denoted by $(M, g, \mathscr{S}(N))$. We know [12] that for such a triplet, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall\, W \in \mathscr{S}(N)|_{\mathscr{U}}). \tag{2.2}$$

Then $T\overline{M}$ is decomposed as follows:

$$T\overline{M}|_M \quad = \quad TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{Orth} \mathscr{S}(N). \tag{2.3}$$

We call $tr(TM)$ a *(null) transversal vector bundle* along M. In fact, from (2.2) and (2.3) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the screen distribution $\mathscr{S}(N)$. A vector field N as in (2.2) is called a *null transversal vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution and a unique radical vector field, say ξ, satisfying (2.2). Whence, from now on, by a *normalized lightlike hypersurface* we mean a triplet (M, g, N) where g is the induced metric on M along with a null transversal vector field N. In fact, in case the ambient manifold $\overline{M}$ has Lorentzian signature, at an arbitrary point x in M, a real lightlike cone C_x is invariantly defined in the (ambient) tangent space $T_x\overline{M}$ and is tangent to M along a generator emanating from x. This generator is exactly the radical fiber $\Delta_x = T_xM^{\perp}$. Each null vector field N, $x \longmapsto N_x \in C_x \setminus \Delta_x$ determines a normalization of M. Let (M, g, N) be a normalized lightlike hypersurface. A null vector field $\widetilde{N}$ is a normalizing field for (M, g) if and only if $\widetilde{N} = \phi N + \zeta$, for some nowhere vanishing $\phi \in C^{\infty}(M)$ and $\zeta \in \Gamma(TM)$. A change in normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$ is called *isotropic scaling* (from N) if $\zeta = 0$ that is $\widetilde{N} = \phi N$. In such a change of normalization the screen distribution corresponding to the null transversal vector field N is preserved while there is an "homothetic" scaling in the radical vector field $\widetilde{\xi} = \frac{1}{\phi}\xi$. It is called *tangential scaling* (from N) if $\phi = 1$, that is $\widetilde{N} = N + \zeta$. Here a change in screen distribution occurs and the null vector fields N and $\widetilde{N}$ are dual to the same radical vector field $\xi \in TM^{\perp}$, i.e $\langle \widetilde{N}, \xi \rangle = \langle N, \xi \rangle = 1$ as in (2.2). The general case $\widetilde{N} = \phi N + \zeta$ is called *mixed scaling* (from N).

Now, on a normalized lightlike hypersurface (M, g, N), the local Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y \quad = \quad \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\overline{\nabla}_X N \quad = \quad -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY \quad = \quad \overset{\star}{\nabla}_X\, PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi \quad = \quad -\overset{\star}{A}_{\xi} X - \tau(X)\xi, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(N)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ on $\Gamma\big(\mathscr{S}(N)\big)$ with respect to the decomposition

(2.1). Now the $(0,2)$ tensors B and C are the local second fundamental forms on TM and $\mathscr{S}(N)$ respectively, $\overset{\star}{A}_\xi$the local shape operator on $\mathscr{S}(N)$ and τ a $1-$form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

The subbundle $\mathscr{S}(N)$ is canonically isomorphic to the factor vector bundle $TM/TM^\perp$ and the second fundamental form B satisfies

$$B(X,\xi) = 0, \quad \text{and} \quad B(X,Y) = g(\overset{\star}{A}_\xi X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.8}$$

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations [12, p. 93]

$$\begin{aligned} \langle \overline{R}(X,Y)Z, \xi\rangle &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \langle \overline{R}(X,Y)Z, PW\rangle &= \langle R(X,Y)Z, PW\rangle + B(X,Z)C(Y,PW) \\ &\quad - B(Y,Z)C(X,PW), \end{aligned} \tag{2.10}$$

$$\begin{aligned} \langle \overline{R}(X,Y)\xi, N\rangle &= \langle R(X,Y)\xi, N\rangle = C(Y, \overset{\star}{A}_\xi X) - C(X, \overset{\star}{A}_\xi Y) \\ &\quad - 2d\tau(X,Y), \quad \forall X, Y, Z, W \in \Gamma(TM|_{\mathscr{U}}). \end{aligned} \tag{2.11}$$

3. Pseudo-inversion of degenerate metrics

We recall from [6] the following results. Consider a normalized null hypersurface (M, g, N) and define the $1-$form

$$\eta(\bullet) \ = \ \overline{g}(\ N\ ,\ \bullet\ ).$$

For all $X \in \Gamma(TM)$, $\quad X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(\mathscr{S}(N))$. Now, we define $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\longmapsto X^\flat = g(\ X\ ,\ \bullet\ ) + \eta(X)\eta(\ \bullet\ ). \end{aligned} \tag{3.1}$$

Clearly, such a $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and can be used to generalize the usual non-degenerate theory. In the latter case, $\Gamma(\mathscr{S}(N))$ coincides with $\Gamma(TM)$, and as a consequence the $1-$form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp$ denote the inverse of the isomorphism $\flat$ given by (3.1). For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\sharp$) is called the dual $1-$form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (3.1) that if ω is a $1-$form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) \ = \ g(\omega^\sharp, X) \ + \ \omega(\xi)\eta(X).$$

Define a $(0,2)-$tensor $\underline{g}$ by

$$\underline{g}(X,Y) \ = \ X^\flat(Y), \quad \forall X, Y \in \Gamma(TM).$$

Clearly, $\underline{g}$ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to degenerate metric g on lightlike hypersurfaces ([6] for details). It is called the associate metric to g on (M, g, N). Also, observe that $\underline{g}$ coincides with g if the latter is non-degenerate. The $(0,2)-$tensor $g^{[\,\cdot\,,\,\cdot\,]}$, inverse of $\underline{g}$ is called *the pseudo-inverse of* g. With respect to the quasi orthonormal local frame field $\{\partial_0 := \xi, \partial_1, \cdots, \partial_n, N\}$ adapted to the decompositions (2.1) and (2.3) we have

$$\underline{g}(\xi,\xi) = 1, \quad \underline{g}(\xi, X) = \eta(X), \tag{3.2}$$
$$\underline{g}(X,Y) = g(X,Y) \; \forall X, Y \in \Gamma(\mathscr{S}(N)),$$

and the following holds [6].

Proposition 3.1. (α) *For any smooth function* $f : \mathscr{U} \subset M \to \mathbb{R}$ *we have*

$$grad^g f = g^{[\alpha\beta]} f_\alpha \partial_\beta \qquad where \quad f_\alpha = \frac{\partial f}{\partial x^\alpha} \quad \partial_\beta = \frac{\partial}{\partial x^\beta} \quad \alpha, \beta = 0, \ldots n$$

(β) *For any vector field* X *on* $\mathscr{U} \subset M$

$$div^g X = \sum_{\alpha=0}^{n} \varepsilon_\alpha \underline{g}(\nabla_{X_\alpha} X, X_\alpha) \;\; ; \; \varepsilon_0 = 1$$

(γ) *for a smooth function* f *defined on* $\mathscr{U} \subset M$ *we have*

$$\Delta^g f = \sum_{\alpha=0}^{n} \varepsilon_\alpha \underline{g}(\nabla_{X_\alpha} grad^g f, X_\alpha)$$

In particular, ρ being an endomorphism (resp. a symmetric bilinear form) on (M, g, N), we have

$$tr\rho = trace_g \rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \underline{g}(\rho(\partial_\alpha), \partial_\beta)$$

$$(\text{resp. } trace_g \rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \rho_{\alpha\beta} \;).$$

4. The determinant of the associate metric $\underline{g}$ relative to the induced volume element

Let (M^{n+1}, g, N) be a normalized null hypersurface of of an $(n+2)$-dimensional oriented semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant index $0 < \nu < n+2$. We denote by $\widetilde{\omega}$ the unique volume element on $\overline{M}$ compatible with $\overline{g}$ and the orientation. The induced volume element θ^N is defined by

$$\theta^N(X_0, \ldots, X_n) = \widetilde{\omega}(X_0, \ldots, X_n, N).$$

A unimodular basis for θ^N consists of a basis $(X_0, \ldots, X_n)$ of the tangent space $T_x M$ for which $\theta^N(X_0, \ldots, X_n) = 1$. It is called adapted unimodular if $X_0 = \xi$ and $\langle X_i, N\rangle = 0$. In general, a frame field $(X_0, \ldots, X_n)$ of M will be called adapted if $\text{span}(X_0) = TM^\perp \subset TM$ and $(X_1, \ldots, X_n)$ spans the (associated) screen distribution (we denote) $\mathscr{S}(N)$. Now, let $(X_0, \ldots, X_n)$ be a unimodular basis for θ in a neighborhood of a point $x \in M$. For the non-degenerate $\underline{g}$, if we set $\underline{g}_{\alpha\beta} = \underline{g}(X_\alpha, X_\beta)$, then the determinant of the matrix $[\underline{g}_{\alpha\beta}]$ is independent of the choice of unimodular basis $(X_0, \ldots, X_n)$. This number is denoted $\det_{\theta^N} \underline{g}$ and called the determinant of the associate metric $\underline{g}$ relative to the induced volume element. The basic fact of this section is that this number is invariant under any change of normalization $N \longrightarrow \widetilde{N} = \varphi N + \zeta$.

Theorem 4.1. *Let (M^{n+1}, g) be a null hypersurface of a $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then $\det_{\theta^N} \underline{g}$ is invariant under any change of normalization.*

Proof. Let N and $\widetilde{N} = \phi N + \zeta$ be two normalizations of (M, g). The associate metric to g relative to N is given by $\underline{g} = g + \eta \otimes \eta$. Let $\underline{\widetilde{g}}$ denote the associate metric to g relative to $\widetilde{N}$. We have

$$\underline{\widetilde{g}} = \underline{g} + (\phi^2 - 1)\eta \otimes \eta + \phi\Big(\eta \otimes \zeta^{\overline{\flat}} + \zeta^{\overline{\flat}} \otimes \eta\Big) + \zeta^{\overline{\flat}} \otimes \zeta^{\overline{\flat}}, \tag{4.1}$$

where $\zeta^{\overline{\flat}}$ denotes $\overline{g}(\zeta, .)$.

Let $(X_0, \ldots, X_n)$ be an adapted unimodular basis for θ^N, i.e with

$$\langle X_0, N \rangle = 1 \quad \text{and} \quad X_i \in \mathscr{S}(N)\ \forall i = 1, \ldots, n,$$

where $\langle , \rangle$ stands for both $\overline{g}$ or g (accordingly) and we use the following range of indices. Letters $i, j, k, \cdots = 1, \ldots n$; $\alpha, \beta, \gamma, \cdots = 0, \ldots n$ and $A, B, \cdots = 0, \ldots n+1$. Then using (4.1), we have

$$\underline{\widetilde{g}}_{\alpha 0} = \underline{\widetilde{g}}_{0\alpha} = \begin{cases} \phi^2 & \text{si} \quad \alpha = 0 \\ \phi\langle \zeta, X_i \rangle & \text{si} \quad \alpha = i. \end{cases}$$

and

$$\underline{\widetilde{g}}_{ij} = \underline{g}_{ij} + \langle \zeta, X_i \rangle \langle \zeta, X_j \rangle.$$

So,

$$\det_{\theta^N} \underline{\widetilde{g}} = \begin{vmatrix} \phi^2 & \phi\langle \zeta, X_1 \rangle & \ldots\ldots & \phi\langle \zeta, X_n \rangle \\ \phi\langle \zeta, X_1 \rangle & & & \\ \vdots & & & \\ \vdots & & \underline{g}_{ij} + \langle \zeta, X_i \rangle \langle \zeta, X_j \rangle & \\ \phi\langle \zeta, X_n \rangle & & & \end{vmatrix}.$$

We distinguish two cases: $\zeta \neq 0$ and $\zeta = 0$.

Case $\zeta \neq 0$. This case implies $P\zeta \neq 0$ (as $\langle \widetilde{N}, \widetilde{N} \rangle = 0 = 2\phi\eta(\zeta) + \|P\zeta\|^2$ with $\phi \neq 0$ and $\zeta = \eta(\zeta) + P(\zeta)$) and without loss of generality, we may consider an adapted unimodular basis for θ^N such that

$$X_0 = \xi, \quad X_1 = P\zeta, \quad \text{and } X_i \in \mathscr{S}(N) \cap (P\zeta)^{\perp_{\mathscr{S}(N)}} \quad i = 2, \ldots, n.$$

It follows that

$$
\begin{aligned}
\det_{\theta^N} \widetilde{\underline{g}} &= \phi^2 \begin{vmatrix} 1 & \|\zeta\|^2 & 0 & \cdots\cdots & 0 \\ \|\zeta\|^2 & \underline{g}_{11} + \|\zeta\|^4 & \underline{g}_{12} & \cdots & \underline{g}_{12} \\ 0 & \underline{g}_{12} & & & \\ \vdots & & & & \\ \vdots & \vdots & & \underline{g}_{ij} & \\ 0 & \underline{g}_{1n} & & & \end{vmatrix} \\
&= \phi^2 \begin{vmatrix} \underline{g}_{11} + \|\zeta\|^4 & \underline{g}_{12} & \cdots & \underline{g}_{12} \\ & \underline{g}_{12} & & \\ & & & \\ \vdots & & \underline{g}_{ij} & \\ \underline{g}_{1n} & & & \end{vmatrix} \\
& \quad -\phi^2 \|\zeta\|^2 \begin{vmatrix} \|\zeta\|^2 & \underline{g}_{12} & \cdots & \underline{g}_{12} \\ 0 & \vdots & & \\ \vdots & \vdots & & \\ \vdots & \vdots & \underline{g}_{ij}\ (i, j \geq 2) & \\ 0 & \underline{g}_{1n} & & \end{vmatrix}
\end{aligned}
$$

and direct development of above two determinants gives $\det_{\theta^N} \widetilde{\underline{g}} = \phi^2 \det_{\theta^N} \underline{g}$. But with $\widetilde{N} = \phi N + \zeta$ we have $\theta^{\widetilde{N}} = \phi \theta^N$. So, if $(X_0, \ldots, X_n)$ is an adapted unimodular basis for θ^N, then $(\frac{1}{\phi} X_0, \ldots, X_n)$ is an adapted unimodular basis for $\theta^{\widetilde{N}}$. Thus $\det_{\theta^{\widetilde{N}}} \widetilde{\underline{g}} = \phi^{-2} \det_{\theta^N} \widetilde{\underline{g}}$. Finally, we have

$$\det_{\theta^{\widetilde{N}}} \widetilde{\underline{g}} = \phi^{-2} \det_{\theta^N} \widetilde{\underline{g}} = \phi^{-2} \cdot \phi^2 \det_{\theta^N} \underline{g} = \det_{\theta^N} \underline{g}.$$

Case $\zeta = 0$. Relation (4.1) reduce to

$$\widetilde{\underline{g}} = \underline{g} + (\phi^2 - 1)\eta \otimes \eta.$$

Then,

$$\widetilde{\underline{g}}_{\alpha 0} = \widetilde{\underline{g}}_{0\alpha} = \begin{cases} \phi^2 & \text{if} \quad \alpha = 0 \\ \\ 0 & \text{if} \quad \alpha = i. \end{cases}$$

and

$$\widetilde{\underline{g}}_{ij} = \underline{g}_{ij}, \quad \forall\ i, j.$$

It follows that

$$
\begin{aligned}
\det_{\theta^N} \underline{\widetilde{g}} &= \begin{vmatrix} \phi^2 & 0 & \ldots\ldots & 0 \\ 0 & & & \\ \vdots & & & \\ \vdots & & \underline{g}_{ij} & \\ 0 & & & \end{vmatrix} \\
&= \phi^2 \begin{vmatrix} 1 & 0 & \ldots\ldots & 0 \\ 0 & & & \\ \vdots & & & \\ \vdots & & \underline{g}_{ij} & \\ 0 & & & \end{vmatrix} \\
&= \phi^2 \det_{\theta^N} \underline{g},
\end{aligned}
$$

and taking account of $\theta^{\widetilde{N}} = \phi\theta^N$, we get,

$$\det_{\theta^{\widetilde{N}}} \underline{\widetilde{g}} = \phi^{-2} \det_{\theta^N} \underline{\widetilde{g}} = \phi^{-2} \cdot \phi^2 \det_{\theta^N} \underline{g} = \det_{\theta^N} \underline{g}. \blacksquare$$

5. Some technical results

5.1. **A technical lemma.** The following lemma (we give a detailed proof in [3]) accounts for relationship between the induced geometric objects described in section 2 with respect to a change of normalization $N \longrightarrow \widetilde{N} = \phi N + \zeta$.

Lemma 5.1 ([3]). *Let $\{\xi, N\}$ be a normalizing pair as in (2.2) and consider the change of normalization $\widetilde{N} = \phi N + \zeta$ with corresponding radical vector field $\widetilde{\xi}$. Then,*

(a) $\widetilde{\xi} = \frac{1}{\phi}\xi$,

(b) $2\phi\eta(\zeta) + ||\zeta||^2 = 0$,

(c) $B^{\widetilde{N}}(X,Y) = \frac{1}{\phi} B^N(X,Y)$,

(d) $\widetilde{P} = P - \frac{1}{\phi} g(\zeta, \cdot)\xi$,

(e) $C^{\widetilde{N}}(X, \tilde{P}Y) = \phi C^N(X, PY) - g(\nabla_X \zeta, PY)$
$+ [\tau^N(X) + \frac{X \cdot \phi}{\phi} + \frac{1}{\phi} B^N(\zeta, X)] g(\zeta, Y)$,

(f) $\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{\phi} B^N(X,Y)\zeta$

(g) $\tau^{\widetilde{N}} = \tau^N + d\ln|\phi| + \frac{1}{\phi} B^N(\zeta, \cdot)$,

(h) $A_{\widetilde{N}} = \phi A_N - \nabla_{\cdot}\zeta + [\tau^N + d\ln|\phi| + \frac{1}{\phi} B^N(\zeta, \cdot)]\zeta$,

(i) $\overset{\star}{A}_{\widetilde{\xi}} = \frac{1}{\phi} \overset{\star}{A}_{\xi} - \frac{1}{\phi^2} B^N(\zeta, \cdot)\xi$, *for all tangent vector fields X and Y.*

5.2. **A compatibility result.** The induced metric g is not compatible with the induced connection ∇^N in general and this compatibility arises if and only if the lightlike hypersurface M is totally geodesic in $\overline{M}$. Let $\nabla^{\underline{g}}$ denote the Levi-Civita connection of the non-degenerate associate metric $\underline{g}$ on (M, g, N). We are now interested in characterizing the normalizations for which the Levi-Civita connection $\nabla^{\underline{g}}$ of $\underline{g}$ agrees with the induced connection ∇^N due to N, i.e $\nabla^{\underline{g}} = \nabla^N$. For this we recall the following.

Lemma 5.2 ([6]). *For all $X, Y, Z \in \Gamma(TM)$ we have,*

$$\begin{aligned} \big(\nabla_X \underline{g}\big)\big(Y, Z\big) &= \eta(Y)\big(B(X, PZ) - C(X, PZ)\big) \\ &\quad + \eta(Z)\big(B(X, PY) - C(X, PY)\big) \\ &\quad + 2\tau(X)\eta(Y)\eta(Z). \end{aligned} \tag{5.1}$$

We derive the following result on the compatibility condition.

Theorem 5.1. *Let (M, g, N) be a normalized lightlike hypersurface of a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. The induced connections ∇^N and the Levi-Civita connection $\nabla^{\underline{g}}$ of the associate metric $\underline{g}$ on (M, g, N) agree if and only if for all $X, Z \in \Gamma(TM)$,*

$$\left\{ \begin{array}{lcl} B(X, PZ) &=& C(X, PZ) \\ \\ \tau(X) &=& 0. \end{array} \right. \tag{5.2}$$

Proof. Let X, Y and Z be tangent vector fields on M. As $\widetilde{\nabla}$ is the Levi-Civita connection of $\underline{g}$ and ∇ is torsion-free, the compatibility condition read $\big(\nabla_X \underline{g}\,\big)\,\big(Y, Z\big) = 0$ for all $X,\ Y,\ Z \in \Gamma(TM)$. Put $Y = Z = \xi$ in (5.1) to get $\tau(X) = 0$ for all $X \in \Gamma(TM)$, and setting $Y = \xi$ yields $B(X, PZ) = C(X, PZ)$ for all $X,\ Z \in \Gamma(TM)$ ■

We derive from (5.1) the main covariant derivative formula

$$\begin{aligned} X \cdot \underline{g}(X, Z) &= \underline{g}(\nabla_X Y, Z) + \underline{g}(Y, \nabla_X Z) + \eta(Y)(B(X, PZ) - C(X, PZ)) \\ &\quad + \eta(Z)(B(X, PY) - C(X, PY)) + 2\tau(X)\eta(Y)\eta(Z). \end{aligned} \tag{5.3}$$

Recall that a normalization (M, g, N) is called *screen conformal* [7] if on any coordinate neighbourhood $\mathscr{U} \subseteq M$ there exists a non-vanishing smooth function φ on $\mathscr{U}$ such that $A_N = \varphi \overset{\star}{A}_\xi$. This is equivalent to saying $C(X, PY) = \varphi B(X, Y)$ for all tangent vector fields X and Y. The function φ is called the conformal factor. In order to avoid trivial ambiguities, the domain $\mathscr{U}$ is considered to be connected and maximal in the sense that there is no larger connected domain $\mathscr{U}' \supset \mathscr{U}$ on which the above equality holds. In case $\mathscr{U} = M$ the screen conformality is said to be global.

Théorem 5.1 asserts that the compatibility condition is fulfilled if and only if the normalization is screen conformal with constant conformal factor 1 and vanishing normalizing $1-$form τ.

6. The invariant normalizing differential equation

Let (M, g, N) be a normalized lightlike hypersurface with ξ the corresponding (radical) null vector and ν the (non-normalized) mean curvature function ν on (M, g, N) that is the trace of the endomorphism $\overset{\star}{A}_\xi$ which in terms of the (local) second fundamental form B^N is given by

$$\nu = \underline{g}^{\alpha\beta} B^N_{\alpha\beta}.$$

Assune $\nu \neq 0$ everywhere and consider the following partial differential linear equation with unknown ψ :

$$\xi \cdot \psi + \Big(\tau^N(\xi) + \frac{\xi \cdot \nu}{\nu}\Big)\psi = 0. \tag{6.1}$$

A special fact on this PDE is that it is invariant under any change of normalization. Indeed, let $\widetilde{N} = \phi N + \zeta$ (for nowhere vanishing function ϕ and $\zeta \in \Gamma(TM)$) be a change of normalization. Throughout, overtilded objects are related to $\widetilde{N}$. We show that ψ is solution of Eq. (6.1) if and only if

$$\widetilde{\xi} \cdot \psi + \Big(\tau^{\widetilde{N}}(\widetilde{\xi}) + \frac{\widetilde{\xi} \cdot \widetilde{\nu}}{\widetilde{\nu}}\Big)\psi = 0. \tag{6.2}$$

For this, observe that from Lemma 5.1, Eq. (6.2) is equivalent to

$$\frac{1}{\phi}\xi \cdot \psi + \left[\frac{1}{\phi}\Big(\tau^N(\xi) + \frac{\xi \cdot \phi}{\phi} + \frac{1}{\phi}B^N(\zeta, \xi)\Big) + \frac{\phi}{\nu}\frac{1}{\phi}\xi \cdot \Big(\frac{1}{\phi}\nu\Big)\right]\psi = 0,$$

where we also make use of $\widetilde{\nu} = \dfrac{1}{\phi}\nu$. As $\xi \in Ker B^N$, we get

$$\frac{1}{\phi}\xi \cdot \psi + \left[\frac{1}{\phi}\Big(\tau^N(\xi) + \frac{\xi \cdot \phi}{\phi}\Big) + \frac{1}{\nu}\Big(-\frac{\xi \cdot \phi}{\phi^2}\nu + \frac{1}{\phi}\xi \cdot \nu\Big)\right]\psi = 0,$$

and as $\phi \neq 0$ this is equivalent to

$$\xi \cdot \psi + \Big(\tau^N(\xi) + \frac{\xi \cdot \nu}{\nu}\Big)\psi = 0,$$

which is Eq. (6.1). From this invariance, without reference to any normalization, (it make sense and) we call Eq. (6.1) the invariant normalizing differential equation of the lightlike hypersurface (INDE in short).

For future use, we introduce the following:

Notation: Smooth (non zero) solutions of the INDE-equation (6.1) will be called normalizing functions (in short NF) of the null hypersurface.

Observe that for $\tau^N = 0$, the mean curvature function ν is a normalizing function (that is solution of Eq. (6.1)) if and only if it is constant along $\xi-$orbits.

7. The calibrated divergence of sections along normalized null hypersurfaces

Let (M, g, N) be a normalized null hypersurface of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$, K a vector field along M. The calibrated divergence of K on (M, g, N) we denote by $\Xi^N(K)$ is defined by

$$\Xi^N(K) := \operatorname{div}^{\overline{g}} K - \Big(\mathscr{L}_K \overline{g}\Big)(\xi, N), \tag{7.1}$$

where $\mathscr{L}_K$ stands for the Lie derivative with respect to K.

Let $\widetilde{N} = \phi N + \zeta$ be a change of normalization on M. Then,

$$\begin{aligned}
\Xi^{\widetilde{N}}(K) &= \operatorname{div}^{\overline{g}} K - \Big(\mathscr{L}_K \overline{g}\Big)(\widetilde{\xi}, \widetilde{N}) \\
&= \operatorname{div}^{\overline{g}} K - \Big(\mathscr{L}_K \overline{g}\Big)\Big(\frac{1}{\phi}\xi, \phi N + \zeta\Big) \\
&= \operatorname{div}^{\overline{g}} K - \Big(\mathscr{L}_K \overline{g}\Big)(\xi, N) - \frac{1}{\phi}\Big(\mathscr{L}_K \overline{g}\Big)(\xi, \zeta),
\end{aligned}$$

i.e

$$\Xi^{\widetilde{N}}(K) = \Xi^N(K) - \frac{1}{\phi}(\mathscr{L}_K \overline{g})(\xi, \zeta). \tag{7.2}$$

If we restrict on the Killing fields of $\overline{M}$ along M, we find that the calibrated divergence is invariant under change of normalization, due to $\mathscr{L}_K \overline{g} = 0$. In the sequel we shall be particularly interested to the case where $K = N$ is the null transversal normalization section.

Lemma 7.1. *Let N and $\widetilde{N} = \phi N + \zeta$ be two null transversal vector fields along M. Then,*

$$\Xi^{\widetilde{N}}(\widetilde{N}) = \phi \Xi^N(N) + \Xi^N(\zeta) - \left[\tau^N(\xi) + \frac{\zeta \cdot \phi}{\phi} + \frac{1}{\phi} B^N(\zeta, \zeta)\right] + \overline{g}\Big(N, \nabla^N_\xi \zeta\Big), \tag{7.3}$$

where ∇^N denotes the induced connection on M with respect to the null transversal vector field N.

Proof.

$$\begin{aligned}
\Xi^{\widetilde{N}}(\widetilde{N}) &\overset{(7.2)}{=} \Xi^N \widetilde{N} - \frac{1}{\phi}(\mathscr{L}_{\widetilde{N}} \overline{g})(\xi, \zeta) \\
&= \operatorname{div}^{\overline{g}} \widetilde{N} - \Big(\mathscr{L}_{\widetilde{N}} \overline{g}\Big)(\xi, N) - \frac{1}{\phi}(\mathscr{L}_{\widetilde{N}} \overline{g})(\xi, \zeta) \\
&= \operatorname{div}^{\overline{g}}(\phi N + \zeta) - \Big(\mathscr{L}_{\widetilde{N}} \overline{g}\Big)(\xi, N) - \frac{1}{\phi}(\mathscr{L}_{\widetilde{N}} \overline{g})(\xi, \zeta) \\
&= \overline{g}\Big(\overline{\nabla}^{\overline{g}} \phi, N\Big) + \phi \operatorname{div}^{\overline{g}} N + \operatorname{div}^{\overline{g}} \zeta - (\mathscr{L}_{\phi N} \overline{g})(\xi, N) \\
&\quad -(\mathscr{L}_\zeta \overline{g})(\xi, N) - \frac{1}{\phi}(\mathscr{L}_{\phi N} \overline{g})(\xi, \zeta) - \frac{1}{\phi}(\mathscr{L}_\zeta \overline{g})(\xi, \zeta),
\end{aligned}$$

where $\overline{\nabla}^{\overline{g}}\phi$ stands for the gradient of ϕ w.r.t $\overline{g}$. Now using elementary properties of Lie derivative and summing up lead to

$$\begin{aligned}
\Xi^{\widetilde{N}}(\widetilde{N}) &= \overline{g}\Big(\overline{\nabla}^{\overline{g}}\phi, N\Big) + \phi\Big(\mathrm{div}^{\overline{g}}N - (\mathscr{L}_N\overline{g})(\xi, N)\Big) + \Big(\mathrm{div}^{\overline{g}}\zeta - (\mathscr{L}_\zeta\overline{g})(\xi, N)\Big) \\
&\quad -\overline{g}\Big(\overline{\nabla}^{\overline{g}}\phi, N\Big) - \frac{1}{\phi}\Big((\xi\cdot\phi)\eta(\xi) + \zeta\cdot\phi\Big) - (\mathscr{L}_N\overline{g})(\xi,\zeta) - \frac{1}{\phi}(\mathscr{L}_\zeta\overline{g})(\xi,\zeta) \\
&= \phi\Xi^N(N) + \Xi^N(\zeta) - \frac{1}{\phi}\Big((\xi\cdot\phi)\eta(\xi) + \zeta\cdot\phi\Big) - \overline{g}(\overline{\nabla}_\xi N, \zeta) - \overline{g}(\xi, \overline{\nabla}_\xi N) \\
&\quad -\frac{1}{\phi}\Big(-\phi\xi\cdot(\eta(\zeta)) - (\xi\cdot\phi)\eta(\zeta) + B^N(\zeta,\zeta)\Big) \\
&= \phi\Xi^N(N) + \Xi^N(\zeta) - \frac{1}{\phi}\Big(\zeta\cdot\phi + B^N(\zeta,\zeta) - \phi\xi\cdot(\eta(\zeta))\Big) - \tau^N(\zeta) \\
&\quad -\xi\cdot(\eta(\zeta)) + \overline{g}(N, \overline{\nabla}_\xi\zeta) \\
&= \phi\Xi^N(N) + \Xi^N(\zeta) - \left[\tau^N(\xi) + \frac{\zeta\cdot\phi}{\phi} + \frac{1}{\phi}B^N(\zeta,\zeta)\right] + \overline{g}\Big(N, \nabla^N_\xi\zeta\Big)
\end{aligned}$$

where we use $\overline{g}(N, \overline{\nabla}_\xi\zeta) = \overline{g}\Big(N, \nabla^N_\xi\zeta\Big)$.■

Corollary 7.1. *Suppose $\widetilde{N} = \phi N + \zeta$ is such that*

$$\tau^{\widetilde{N}} = 0 \quad \textit{and} \quad B^{\widetilde{N}}(X, \widetilde{P}Y) = C^{\widetilde{N}}(X, \widetilde{P}Y)$$

for all X and Y tangent to M, then

$$\Xi^{\widetilde{N}}(\widetilde{N}) = \phi\Xi^N(N) + \Xi^N(\zeta). \tag{7.4}$$

Proof. From (7.3), we first have

$$\Xi^{\widetilde{N}}(\widetilde{N}) = \phi\Xi^N(N) + \Xi^N(\zeta) + \overline{g}\Big(N, \nabla_\xi\zeta\Big),$$

as from Lemma 5.1, $\tau^{\widetilde{N}} = \tau^N + d\ln|\phi| + \frac{1}{\phi}B^N(\zeta, .)$. It remains to show that the term $\overline{g}\Big(N, \nabla_\xi\zeta\Big)$ vanishes. But,

$$\overline{g}\Big(N, \nabla_\xi\zeta\Big) = \overline{g}(N, \overline{\nabla}_\xi\zeta) = \xi\cdot(\eta(\zeta)) + C^N(\xi, P\zeta) - \tau^N(\xi)\eta(\zeta),$$

and from $B^{\widetilde{N}}(X, \widetilde{P}Y) = C^{\widetilde{N}}(X, \widetilde{P}Y)$ we have setting $X = \xi$,

$$\begin{aligned}
0 &= \phi C^N(\xi, P\zeta) - g(\nabla_\xi\zeta, \zeta) \\
&= \phi C^N(\xi, P\zeta) - \overline{g}(\overline{\nabla}_\xi\zeta, \zeta) \\
&= \phi C^N(\xi, P\zeta) - \frac{1}{2}\xi\cdot\|\zeta\|^2 \\
&= \phi C^N(\xi, P\zeta) - \Big(- (\xi\cdot\phi)\eta(\zeta) - \phi\xi\cdot(\eta(\zeta))\Big),
\end{aligned}$$

by differentiating with respect to ξ the relation $\|\zeta\|^2 = -2\phi\eta(\zeta)$ from Lemma 5.1. Hence,

$$\phi\Big(C(\xi, P\zeta) + \xi\cdot(\eta(\zeta))\Big) + (\xi\cdot\phi)\eta(\zeta) = 0;$$

i.e

$$C(\xi, P\zeta) + \xi\cdot(\eta(\zeta)) = -\frac{1}{\phi}(\xi\cdot\phi)\eta(\zeta).$$

Thus,

$$\begin{aligned}
\overline{g}(N, \overline{\nabla}_\xi \zeta) &= -\frac{1}{\phi}(\xi\cdot\phi)\eta(\zeta) - \tau^N(\xi)\eta(\zeta)\\
&= -\Big[\tau^N(\xi) + \frac{\xi\cdot\phi}{\phi}\Big]\eta(\zeta)\\
&= -\Big[\tau^N(\xi) + \frac{\xi\cdot\phi}{\phi} + \frac{1}{\phi}B^N(\zeta,\xi)\Big]\eta(\zeta)\\
&= -\tau^{\widetilde{N}}(\xi)\eta(\zeta) = 0,
\end{aligned}$$

where we make use of $B^N(\zeta,\xi) = 0$ and $\tau^{\widetilde{N}} = 0$.■

8. A distinguished normalization

8.1. **Normalizing constraints and main results.** Let ψ be a smooth normalization function (NF) (i.e a solution of the INDE equation (6.1). Our purpose is to determinate appropriate normalization (M, g, N) so that the following holds

$$\text{(8.1)}\qquad \begin{cases} \nabla &= \nabla^{\underline{g}} \\ \\ \Xi^N(N) &= \psi, \end{cases}$$

or equivalently using Theorem 5.1,

$$\text{(8.2)}\qquad \begin{cases} B^N(X,PY) &= C^N(X,PY) \\ \\ \tau^N &= 0 \\ \\ \Xi^N(N) &= \psi, \end{cases}$$

where ∇ and $\nabla^{\underline{g}}$ are the induced connection on M by the normalization N and the Levi-Civita connection of the associate metric $\underline{g}$, respectively. Each of these systems will be called the normalization constraints (NC in short).

Now start with a tentative null transversal section N. If the normalization constraints are fulfield, there is nothing more to prove. If not, we consider a change of normalization $\widetilde{N} = \phi N + \zeta$ where $(\phi,\zeta) \in \mathscr{C}^\infty \times \Gamma(TM)$ is to be determinated in view of the (NC) conditions. Under the above change of normalization, using Lemma 5.1 and corollary 7.1 , we find out that the normalization constraints transform to

$$\text{(8.3)}\qquad \begin{cases} \phi^2 C^N(X,PY) - \phi g(\nabla_X\zeta, PY) - B^N(X,Y) &= 0 \\ \\ \tau^N(X) + \dfrac{X\cdot\phi}{\phi} + \dfrac{1}{\phi}B^N(\zeta, X) &= 0 \\ \\ \phi\Xi^N(N) + \Xi^N(\zeta) &= \psi, \end{cases}$$

for arbitrary tangent vector fields X, Y.

First, observe that from decompositions (2.1) and (2.3) we may consider frame fields $(\xi, X_1, \ldots, X_n, N)$ of $(\overline{M}^{n+2}, \overline{g})$ along M with

$$\eta(\xi) = 1 \quad \text{and} \quad \eta(X_i) = 0 \ \forall i = 1, \ldots, n;$$

i.e $\mathscr{S}(N) = \text{span}\{X_1, \ldots, X_n\}$. Then,

$$\begin{aligned}
\Xi^N(\zeta) &= \text{div}^{\overline{g}}\zeta - (\mathscr{L}_\zeta \overline{g})(\xi, N) \\
&= \overline{g}\Big(\overline{\nabla}_\xi \zeta, N\Big) + \overline{g}\Big(\xi, \overline{\nabla}_N \zeta\Big) + \sum_{i,j=1}^n \overline{g}^{ij}\overline{g}\Big(\overline{\nabla}_{X_i}\zeta, X_j\Big) - (\mathscr{L}_\zeta \overline{g})(\xi, N) \\
&= \sum_{i,j=1}^n \overline{g}^{ij}\overline{g}\Big(\nabla_{X_i}\zeta + B(X_i, \zeta)N, X_j\Big) \\
&= \sum_{i,j=1}^n \overline{g}^{ij}\overline{g}\Big(\nabla_{X_i}\zeta, X_j\Big) \quad \text{as} \quad \eta(X_j) = 0 \ \forall j.
\end{aligned}$$

Also,

$$\begin{aligned}
\Xi^N(N) &= \text{div}^{\overline{g}} N - (\mathscr{L}_N \overline{g})(\xi, N) \\
&= \sum_{i,j=1}^n \overline{g}^{ij}\overline{g}\Big(\overline{\nabla}_{X_i} N, X_j\Big) \\
&= \sum_{i,j=1}^n \overline{g}^{ij}\overline{g}\Big(-A_N X_i, X_j\Big) \quad \text{as} \quad \eta(X_j) = 0 \ \forall j. \\
&= -\sum_{i,j=1}^n \overline{g}^{ij} C^N(X_i, X_j).
\end{aligned}$$

Now, contracting the first equation in (8.3) with $\underline{g}^{ij} = \overline{g}^{ij}$ leads to

$$\phi^2 \underline{g}^{ij} C^N(X_i, X_j) - \phi \underline{g}^{ij} g\Big(\nabla_{X_i}\zeta, X_j\Big) - \underline{g}^{ij} B^N(X_i, X_j) = 0,$$

and as for $i = 1 \ldots, n$, $X_i \in \mathscr{S}(N)$ and $\underline{g}_{ij} = \overline{g}^{ij}$, we have

$$\phi^2 \overline{g}^{ij} C^N(X_i, X_j) - \phi \overline{g}^{ij} g\Big(\nabla_{X_i}\zeta, X_j\Big) - \overline{g}^{ij} B^N(X_i, X_j) = 0,$$

and as was said above $\underline{g}$ and g coincide on $\mathscr{S}(N)$ and noting that $B^N_{0\alpha} = B^N_{\alpha 0} = 0$, we infer using the previous expressions in coordinates of $\Xi^N(N)$ and $\Xi^N(\zeta)$,

$$-\phi^2 \Xi^N(N) - \phi \Xi^N(\zeta) - \nu = 0,$$

that is

$$-\phi\Bigg(\phi\Xi^N(N) + \Xi^N(\zeta)\Bigg) - \nu = 0,$$

and taking into account the third equation in (8.3) we find out that

$$-\phi\psi - \nu = 0. \tag{8.4}$$

As ψ is a (NF) smooth solution of the INDE-equation in which ν is non zero, we have from previous equation (8.4) $\psi \neq 0$ everywhere and

$$\phi = -\frac{\nu}{\psi}. \tag{8.5}$$

From now on, we assume the following condition:

Definition 8.1. *We say that the (non minimal) null hypersurface (M^{n+1}, g) has nullity degree 1 if the tensor $\overset{\star}{A}_\xi$ has minimal nullity degree 1 everywhere, or equivalently B^N has (constant) maximal rank n.*

By Lemma 5.1 it is obvious that the above degree is intrinsic that is invariant under normalization.

Consequently, the extracted matrix $B^N_{ij} := B^N(X_i, X_j)$ is invertible. Then consider the decomposition $\zeta = \zeta^0\xi + \zeta^i X_i$. From second equation in (8.3) we have for all k,

$$\tau^N(X_k) + \frac{X_k \cdot \phi}{\phi} + \frac{1}{\phi} B^N(\zeta^0\xi + \zeta^i X_i, X_k) = 0.$$

So

$$\forall i = 1\dots, n, \quad \zeta^i = -B^{ik}\left[\phi\tau^N(X_k) + X_k \cdot \phi\right] = B^{ik}\left[\frac{\nu}{\psi}\tau^N(X_k) + X_k \cdot \frac{\nu}{\psi}\right]. \tag{8.6}$$

The (characteristic) radical part $\zeta^0 = \eta(\zeta)$ of ζ is determined using item (b) in Lemma 5.1 and (8.5),

$$\zeta^0 = \eta(\zeta) = \frac{\psi}{2\nu}\left[g_{ij}B^{ik}B^{jl}\Big(\frac{\nu}{\psi}\tau^N(X_k) + X_k \cdot \frac{\nu}{\psi}\Big)\Big(\frac{\nu}{\psi}\tau^N(X_l) + X_l \cdot \frac{\nu}{\psi}\Big)\right] \tag{8.7}$$

and ζ is entirely determined.

Now, let us clarify the choice of the calibrated divergence ψ in the set of solutions of the INDE-equation (6.1). Note that setting $X = \xi$ in the second equation of (8.3) leads to the following equation in ϕ:

$$\phi\tau^N(\xi) + \xi \cdot \phi = 0, \tag{8.8}$$

which is equivalent to

$$\xi \cdot \psi + \Big(\tau^N(\xi) + \frac{\xi \cdot \nu}{\nu}\Big)\psi = 0.$$

Now, we prove the unicity of the section N.

Consider N and $\widetilde{N} = \phi N + \zeta$ to be two such sections. By the second equation in (8.3), the Lemma 7.1 and its Corollary 7.1, we have

$$\psi = \phi\psi + \Xi^N(\zeta),$$

i.e

$$(1-\phi)\psi = \Xi^N(\zeta). \tag{8.9}$$

Also, contracting with $\overline{g}^{ij}$ the first equation in (6.1) taking into account the first equality in (8.2) for the compatibility condition leads to

$$\Big(\phi^2 - 1\Big)\nu - \phi\Xi^N(\zeta) = 0,$$

which is equivalent to

$$(1-\phi^2)\psi = \Xi^N(\zeta), \tag{8.10}$$

using (8.5) and the non nullity of ϕ everywhere. It follows (8.9) and (8.10) that

$$\phi(1-\phi) = 0,$$

which gives $\phi = 1$, as $\phi \neq 0$.

Setting $\phi = 1$ in the second equation in (8.3) and taking into account $\tau^N = 0$ yields, $B^N(\zeta, \cdot) = 0$ i.e $\zeta \in KerB^N = \text{span}\{\xi\}$ as $\overset{\star}{A}_\xi$ has nullity degree 1. So, $\zeta = \eta(\zeta)\xi$. But, $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$. Hence, $\eta(\zeta) = 0$ as $\|\zeta\|^2 = 0$ and $\phi \neq 0$. Thus, we get $\phi = 1$ and $\zeta = 0$, i.e $\widetilde{N} = \phi N + \zeta = N$ and the unicity is proved.

To show that the constructed (normalization) null section is constant along the radical (or ξ)−orbits, use relation (8.3) and set $X = \xi$, $PY = X_i$, $i = 1, \ldots, n$. One gets

$$\phi C^N(\xi, X_i) - g(\nabla_\xi \zeta, X_i) = 0,$$

which is equivalent to

$$\overline{g}(\overline{\nabla}_\xi X_i, \phi N) - \overline{g}(\overline{\nabla}_\xi \zeta, X_i) = 0,$$

that is $\overline{g}(\overline{\nabla}_\xi \widetilde{N}, X_i) = 0$, $i = 1, \ldots, n$. Also,

$$\overline{g}\Big(\overline{\nabla}_\xi \widetilde{N}, \xi\Big) = \overline{g}\Big(-A_{\widetilde{N}}\xi + \tau^{\widetilde{N}}(\xi)\widetilde{N}, \xi\Big) = 0, \quad (\text{as } \tau^{\widetilde{N}} = 0),$$

and $\overline{g}\Big(\overline{\nabla}_\xi \widetilde{N}, \widetilde{N}\Big) = \dfrac{1}{2}\xi \cdot \overline{g}(\widetilde{N}, \widetilde{N}) = 0$. Finally, as the ambient metric $\overline{g}$ in non-degenerate, we get $\overline{\nabla}_\xi \widetilde{N} = 0$.■

The following is then proved:

Theorem 8.1. *Let (M, g) be a non minimal null hypersurface with nullity degree 1. Then there exist a unique normalization null section N along M with prescribed calibrated divergence a given smooth NF-function ψ and for which the induced connection and the Levi-Civita connection of $\underline{g}$ coincide.*

In particular, such a null section is constant along radical ξ−orbits.

8.2. **A basic example: the light-cone $\wedge_0^3 \subset \mathbb{R}_1^4$.** Let us consider the lightcone $\wedge_0^3$ as the immersion

$$\begin{aligned} f: M = \mathbb{R}^3 \setminus \{0\} &\longrightarrow \mathbb{R}_1^4 \\ (x, y, z) &\longmapsto \Big[x, y, z, \varepsilon(x^2 + y^2 + z^2)^{\frac{1}{2}}\Big], \varepsilon = \pm 1. \end{aligned}$$

Locally, $\wedge_0^3$ is the graph $t = \varepsilon(x^2 + y^2 + z^2)^{\frac{1}{2}}$ and it is an obvious fact that this is a lightlike hypersurface immersion. (We focus on the future directed connected component i.e $\varepsilon = 1$).

Start (the normalization) with the tentative null vector field

$$N = x\partial_x + y\partial_y + z\partial_z - t\partial_t, \tag{8.11}$$

and let P denote the morphism projection of the tangent bundle TM onto $\mathscr{S}(N)$. It is traightforward to check that the associate screen shape operator on $\wedge_0^3$ is given by

$$\overset{\star}{A}_\xi X = -\frac{1}{2t^2} PX, \ \forall X \in \Gamma(TM), \tag{8.12}$$

where

$$\xi = \frac{1}{2t^2}\Big(x\partial_x + y\partial_y + z\partial_z + t\partial_t\Big). \tag{8.13}$$

It follows that the mean curvature fonction ν on $\wedge_0^3 \subset \mathbb{R}_1^4$ and the connection $1-$form τ^N are given by

$$\nu = -\frac{1}{t^2} \quad \text{and} \quad \tau^N = -\frac{1}{2t^2}\eta + 2\frac{dT}{T}, \tag{8.14}$$

where $(x, y, z, t) \longmapsto T(x, y, z, t) = t$ is the global canonical time function on $\mathbb{R}_1^4$ restricted on $\wedge_0^3$ and we get the following INDE-equation:

$$\Bigg[x\partial_x + y\partial_y + z\partial_z + t\partial_t\Bigg]\psi - \psi = 0. \tag{8.15}$$

Solutions of this PDE are given by the following family of functions all of whose level sets are hyperplanes of $\mathbb{R}_1^4$:

$$\psi(x, y, z, t) = \alpha x + \beta y + \gamma z + \delta t, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \tag{8.16}$$

Observe that the global time function above is a solution $(\alpha = 0, \beta = 0, \gamma = 0, \delta = 1)$. Then, prescribe the calibrated divergence function to be the (global) time solution

$$\psi(x, y, z, t) = t. \tag{8.17}$$

It follows that

$$\phi(x, y, z, t) = -\frac{\nu}{\psi} = \frac{1}{t^3}. \tag{8.18}$$

The rank 2 distribution $\mathscr{S}(N)$ is spanned by

$$X_1 = y\partial_x - x\partial_y \quad \text{and} \quad X_2 = z\partial_x - x\partial_z.$$

It follows (Eq. 8.6) and (Eq. 8.7) and using (Eq. 8.14) that

$$\zeta^1 = \zeta^2 = \zeta^0 = 0, \quad \text{that is} \quad \zeta = 0.$$

Finally, we get the normalizing null vector field

$$\widetilde{N} = \frac{1}{t^3}\Big(x\partial_x + y\partial_y + z\partial_z - t\partial_t\Big). \tag{8.19}$$

The corresponding normalized radical (characteristic) null vector field is then given by

$$\widetilde{\xi} = t\Big(x\partial_x + y\partial_y + z\partial_z + t\partial_t\Big). \tag{8.20}$$

References

[1] Akivis, M. A. and Goldberg, V. V. *On Some Methods of Construction of Invariant Normalizations of Lightlike Hypersurfaces.*– Diff. Geom. Appl. **12** (2000), p. 121-143.

[2] Akivis, M. A. and Goldberg, V. V. *Lightlike Hypersurfaces on Manifolds Endowed with a Conformal Structur of Lorentzian Signature.*– Acta Appl. Math. **57** (1999), p.155-185.

[3] Atindogbe, C.*Normalization and prescribed extrinsic scalar curvature on lightlike hypersurfaces,* Journal of Geometry and Physics 60 (2010) 1762âĂŞ1770.

[4] Atindogbe, C. *Blaschke type normalization on light-Like Hypersurfaces,* Journal of Mathematical Physics, Analysis, Geometry 6 (2010) No. 4, pp. 362-382.

[5] Atindogbe, C. *Scalar curvature on lightlike hypersurfaces* Applied Sciences, Vol.11, 2009, pp. 9-18.

[6] Atindogbe, C. and Ezin, J.-P. and J. Tossa *Pseudo-inversion of degenerate metrics* Int. J. of Mathematics and Mathematical Sciences, 55 (2003), 3479-3501.

[7] Atindogbe, C. and K.L. Duggal, K. L. *Conformal screen on lightlike hypersurfaces.*, Int. J. of Pure and Applied Math., 11 (2004), 421-442.

[8] E. Bekkara, C. Frances and A. Zeghib *On lightlike geometry: isometric actions and rigidity aspects.* C. R. Acad. Sci. Paris, Ser. (2006), 1343, p. 317-321.

[9] Carter, B. Killing Horizons and Orthogonally Transitive Groups in Space-time. – J. Math. Phys. **10** (1969), 1, p. 70-81.

[10] Chrusciel, P. T. Delay, E. and Galloway, G. J. Howard, R. *Regularity of the horizon and the Area Theorem*, Annales Henri Poincaré, 2001,2, p.109-178. gr-qc/0001003.

[11] Duggal, K. L. *On scalar curvature in lightlike geometry*, J. Geom. Phy., 57 (2007) 473-481.

[12] Duggal, K. L. and Bejancu, A. *Lightlike submanifolds of semi-Riemannian Manifolds and Applications.* Kluwer Acad. Publishers, Dordrecht, Volume 364, 1996.

[13] Katsuno, K. *Null Hypersurfaces in Lorentzian Manifolds.* –Math. Proc. Cab. Phil. Soc. **88** (1980), p. 175-182.

[14] Kupeli, D. N. *Singular Semi-Riemannian Geometry.* Kluwer Acad. Publishers, Dordrecht, volume 366, 1996.

[15] Larsen, J. C. *Singular Semi-Riemannian Geometry.* –J. Geom. Phys. **9** (1992), No. 1, p. 3-23.

[16] Larsen, J. C. *Submanifold Geometry.*– J. Geom. Anal. **4** (1994),No. 2, p. 189-205.

[17] Penrose, R. *The Twistor Geometry of Light Rays.*– Geometry and Physics, Clas- sical Quantum Gravity (1997), 14 (1A), p. A299-A323.

[18] Pinl, M. *Zur Theorie del Halbisotrope Flachen im R4.* –J. Geom. Phys **1** (1992), 9, p. 3-23.

[19] Taub, A. H. *Singular Hypersurfaces in General Rrelativity.* –Illinois J. Math **1** (1957), p.378-388.

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