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# A MACWILLIAMS TYPE IDENTITY FOR M-SPOTTY GENERALIZED LEE WEIGHT ENUMERATORS OVER $\mathbb{Z}_q$

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Abstract. Burst errors are very common in practice. There have been many designs in order to control and correct such errors. Recently, a new class of byte error control codes called spotty byte error control codes has been specifically designed to fit the large capacity memory systems that use high-density random access memory (RAM) chips with input/output data of 8, 16, and 32 bits. The MacWilliams identity describes how the weight enumerator of a linear code and the weight enumerator of its dual code are related. Also, Lee metric which has attracted many researchers due to its applications. In this paper, we combine these two interesting topics and introduce the m-spotty generalized Lee weights and the m-spotty generalized Lee weight enumerators of a code over $\mathbb{Z}_q$ and prove a MacWilliams type identity. This generalization includes both the case of the identity given in the paper [I. Siap, MacWilliams identity for m-spotty Lee weight enumerators, Appl. Math. Lett. 23 (1) (2010) 13-16] and the identity given in the paper [M. Özen, V. Şiap, The MacWilliams identity for m-spotty weight enumerators of linear codes over finite fields, Comput. Math. Appl. 61 (4) (2011) 1000-1004] over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as special cases.

## 1. Introduction

Large-capacity high-speed memory systems often adopt high-density RAM chips with wide input/output data. Because of their high-density nature, these RAM chips are strongly vulnerable to $\alpha-$particles, neutrons, and so forth. In particular, the large-capacity memory systems need to be protected from high-energy neutrons and cosmic rays. Because of these facts, in order to be able to correct multiple errors a new spotty byte error called m-spotty byte error is introduced in [1] for binary codes. Construction of codes correcting byte errors and properties of such codes are also studied. Some of the related work can be found in [2], [3]. However, the Lee metric was developed as an alternative to the Hamming metric for certain noisy channels that use phase-shift keying modulation [4]. The literature on codes in the Lee metric is very extensive, e.g. [4], [5], [6]. The interest in Lee codes has been increased in the last decade due to many new and diverse applications

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of these codes. Some examples are multidimensional burst-error-correction [7], and error-correction for flash memories [8]. So, we combine these two interesting topics, Lee weight and m-spotty. Recently, Suzuki *et al.* proved the MacWilliams for m-spotty weight enumerator of m-spotty byte error control codes and the MacWilliams identity for the Hamming weight enumerator as a special case [9]. Siap adopted a definition similar to Hamming definition given in [9] for m-spotty Lee weights and proved the MacWilliams identity for m-spotty Lee weight enumerators [10]. Then, Özen and Şiap extended the definition of m-spotty weights originally introduced in [9] for binary codes to codes over finite fields and proved the MacWilliams identity for m-spotty enumerators of linear codes over finite fields [11]. In this paper, we introduce the m-spotty generalized Lee weights and the m-spotty generalized Lee weight enumerators of a code over $\mathbb{Z}_q$ and prove a MacWilliams type identity. Also, we show that this identity is a general of both identities given in [10] and in [11] over $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Let $\mathbb{Z}_q$ be the ring of integers modulo $q$. Let $F_q^n$ be set of all $n-$tuples over $\mathbb{Z}_q$. Then, $F_q^n$ is a module over $\mathbb{Z}_q$. Let $F$ be a submodule of the module $F_q^n$ over $\mathbb{Z}_q$. For $q$ prime, $\mathbb{Z}_q$ becomes a field and correspondingly $F_q^n$ and $F$ become vector space and subspace respectively over the field $\mathbb{Z}_q$. Also, we define modular value $|c|$ of an element $c \in \mathbb{Z}_q$ by

$$|c| = \begin{cases} c & \text{if } 0 \le c \le q/2 \\ q-c & \text{if } q/2 < c \le q-1, \end{cases} \tag{1.1}$$

and then a given vector $c = (c_0, c_1, ..., c_{n-1}), c_i \in \mathbb{Z}_q$, the generalized Lee weight [12] $w_{GL}(c)$ of $c$ is given by

$$w_{GL}(c) = \sum_{i=0}^{n-1} |c_i|. \tag{1.2}$$

The generalized Lee distance $d_{GL}$ between two vectors $u$ and $v$ is defined as the generalized Lee weight of their difference vector, i.e.

$$d_{GL}(u,v) = w_{GL}(u-v). \tag{1.3}$$

**Observation.** Over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, generalized Lee weight (distance) coincides with Hamming weight (distance).

Let $c = (c_{11}, c_{12}, ..., c_{1b}, ..., c_{n1}, c_{n2}, ..., c_{nb}) \in \mathbb{Z}_q^{nb}$ be a codeword of $N = nb$. The first byte of $c$ is the first $b$ entries denoted by $(c_{11}, c_{12}, ..., c_{1b})$. Hence, the ith byte of $c$ will be denoted by $c_i = (c_{i1}, c_{i2}, ..., c_{ib})$.

**Definition 1.1.** [2] An error is called a $t/b-$error if $t$ or fewer bits within a $b-$bit byte are in error, where $0 \le t \le b$.

**Definition 1.2.** [13] An error is called an m-spotty byte error if at least one $t/b-$error is present in a byte.

Now, we extend the definition of m-spotty weights originally introduced in [10] from codes over $\mathbb{Z}_4$ to codes over $\mathbb{Z}_q$.

**Definition 1.3.** Let $e \in \mathbb{Z}_q^N$ be an error vector and $e_i \in \mathbb{Z}_q^b$ be the ith byte of $e$ where $1 \le i \le n$. The number of $t/b-$ errors in $e$, denoted by $w_{MGL}(e)$, and called m-spotty generalized weight is defined as

$$w_{MGL}(e) = \sum_{i=1}^{n} \left\lceil \frac{w_{GL}(e_i)}{t} \right\rceil, \tag{1.4}$$

where $\lceil x \rceil$ shows the smallest integer larger than or equal to $x$. If $t = 1$, this weight, defined by $w_{MGL}$, is equal to the generalized Lee weight.

**Definition 1.4.** Let $c$ and $v$ be codewords of m-spotty byte error control code $C$. Here, $c_i$ and $v_i$ are the ith bytes of $c$ and $v$, respectively. Then, m-spotty generalized distance between $c$ and $v$, denoted by $d_{MGL}(c, v)$, is defined as follows:

$$d_{MGL}(c, v) = \sum_{i=1}^{n} \left\lceil \frac{d_{GL}(c_i, v_i)}{t} \right\rceil . \tag{1.5}$$

It is also straightforward to show that this distance is a metric in $\mathbb{Z}_q^N$.

## 2. The MacWilliams identity

One of the most important results in coding theory is the MacWilliams identity which relates the weight enumerator of a linear code $C$ to the weight enumerator of $C^{\perp}$. Hence knowing the weight enumerator of one of these codes enables one to determine the weight enumerator, and hence the weight distribution, of the other. This is useful in practice if, for example, one of $C$ and $C^{\perp}$ is substantially smaller than the other, and the weight enumerator of the larger is required. One can determine the weight enumerator of the smaller code, perhaps even by exhaustive methods, and then, from this, obtain the (more difficult to determine directly) weight enumerator of the larger code by using the MacWilliams identity [14].

Let $c = (c_1, c_2, ..., c_N)$ and $v = (v_1, v_2, ..., v_N)$ be two elements of $\mathbb{Z}_q^N$. The inner product of $c$ and $v$, denoted by $\langle c, v\rangle$, is defined as follows:

$$\langle c, v\rangle = \sum_{i=1}^{n} \langle c_i, v_i\rangle = \sum_{i=1}^{n} \left( \sum_{j=1}^{b} c_{ij} v_{ij} \right). \tag{2.1}$$

Here, $\langle c_i, v_i\rangle = \sum_{j=1}^{b} c_{ij} v_{ij}$ denotes the inner product of $c_i$ and $v_i$. Also, $c_{ij}$ and $v_{ij}$ are the $j$th bits of $c_i$ and $v_i$, respectively.

The following lemma plays an important role in obtaining the main theorem.

**Lemma 2.1.** [14] *Let $f$ be a function defined on $\mathbb{Z}_q^{nb}$. We define*

$$\widetilde{f}(c) = \sum_{v \in \mathbb{Z}_q^{nb}} \chi(\langle c, v\rangle) f(v), \ c \in \mathbb{Z}_q^{nb}. \tag{2.2}$$

*Here, $\chi$ is a character of $\mathbb{Z}_q$ and defined by $\chi(a) = \xi^a$, $a \in \mathbb{Z}_q$, where $\xi = e^{2\pi i/q}$. Then, the following relation holds between $f(v)$ and $\widetilde{f}(c)$ :*

$$\sum_{v \in C^{\perp}} f(v) = \frac{1}{|C|} \sum_{c \in C} \widetilde{f}(c), \tag{2.3}$$

*where the dual code is $C^{\perp} = \left\{ v \in \mathbb{Z}_q^{nb} : \langle c, v\rangle = 0 \text{ for all } c \in C \right\}$.*

Let $\alpha_{i,k} = \#\{j : c_{ij} = k\}$. That is, $\alpha_{i,k}$, $0 \le k \le q-1$, is the number of entries in $c_i$ that equal to $k$. Here, $c_i$ is the $i$th byte of $c$ and $c_{ij}$ is the $j$th bit of $c_i$. The generalized Lee weight distribution vector $S(c_i) = (\alpha_{i,0}, \alpha_{i,1}, ..., \alpha_{i,q-1})$ is determined uniquely for the codeword $c$. For example, let $c = (0, 0, 1, 5, 4, 4, 0, 3, 5, 5, 3, 2) \in \mathbb{Z}_6^{12}$ be a codeword with byte $b = 4$. Then, the generalized Lee weight distribution vectors are $S(c_1) = (2, 1, 0, 0, 0, 1)$, $S(c_2) = (1, 0, 0, 1, 2, 0)$ and $S(c_3) = (0, 0, 1, 1, 0, 2)$.

**Definition 2.1.** The weight enumerator for m-spotty byte error control code $C$ is defined as

$$W(z) = \sum_{c\in C} z^{w_{MGL}(c)}. \tag{2.4}$$

The following theorem holds for the weight enumerator $W(z)$ of the code and that of the dual code $C^{\perp}$, expressed as $W^{\perp}(z)$.

**Theorem 2.1.** *Let $C$ be a linear code over $\mathbb{Z}_q$. The relation between the m-spotty generalized Lee enumerators of $C$ and its dual is given*

$$W^{\perp}(z) = \frac{1}{|C|}\sum_{c\in C} \widetilde{f}(c) = \frac{1}{|C|}\sum_{c\in C}\prod_{i=1}^{n}\left(V_S^{(t,q)}(z)\right), \tag{2.5}$$

*where*
(2.6)

$$V_S^{(t,q)}(z) = \prod_{k=0}^{q-1}\sum_{\sum\limits_{l=0}^{q-1}\beta_{i,l}^k=\alpha_{i,k}}\left(\frac{\alpha_{i,k}!}{\beta_{i,0}^k!\beta_{i,1}^k!\beta_{i,2}^k!...\beta_{i,q-1}^k!}\right)\left(e^{\frac{2\pi i}{q}}\right)^{\left(k\sum\limits_{l=0}^{q-1} l.\beta_{i,l}^k\right)} z^{\left\lceil\left(\sum\limits_{l=0}^{q-1}|l|.\beta_{i,l}^k\right)\Big/ t\right\rceil}$$

*and* $S(c_i) = (\alpha_{i,0}, \alpha_{i,1}, ..., \alpha_{i,q-1})$.

**Proof.** Using Lemma 2.1, we set $f(v) = \prod_{i=1}^{n} z^{\lceil w_{GL}(v_i)/t\rceil}$ where $v_i$ denotes the $i$th byte of $v$. Then,

$$\widetilde{f}(c) = \sum_{v\in\mathbb{Z}_q^{nb}}\chi(\langle c,v\rangle)\prod_{i=1}^{n} z^{\lceil w_{GL}(v_i)/t\rceil} = \sum_{v\in\mathbb{Z}_q^{nb}}\chi(\langle c_1,v_1\rangle + ... + \langle c_n,v_n\rangle)\prod_{i=1}^{n} z^{\lceil w_{GL}(v_i)/t\rceil}$$

$$= \sum_{v_1\in\mathbb{Z}_q^b}\cdots\sum_{v_n\in\mathbb{Z}_q^b}\left(\prod_{i=1}^{n}\chi(\langle c_i,v_i\rangle) z^{\lceil w_{GL}(v_i)/t\rceil}\right) = \prod_{i=1}^{n}\left(\sum_{v_i\in\mathbb{Z}_q^b}\chi(\langle c_i,v_i\rangle) z^{\lceil w_{GL}(v_i)/t\rceil}\right).$$

Here, $c_i$ is fixed and the sum $\sum_{v_i\in\mathbb{Z}_q^b}\chi(\langle c_i,v_i\rangle) z^{\lceil w_{GL}(v_i)/t\rceil}$ runs over all $v_i\in\mathbb{Z}_q^b$. Therefore, we categorize the components of $v_i$ with respect to the fixed vector $u_i$ in the following way. Let $\beta_{i,0}^k, \beta_{i,1}^k, ..., \beta_{i,q-1}^k$ be the numbers of components of $v_i$ with value zero, one,...,$q-1$ respectively that share the same index with the components of $c_i$ whose number equals to $\alpha_{i,k}$.

For example, let $c = (0,0,1,5,4,4,0,3,5,5,3,2) \in \mathbb{Z}_6^{12}$ with $b = 4$. Suppose that $v = (0,0,1,3,4,0,0,1,2,2,3,2) \in \mathbb{Z}_6^{12}$, where $v_1 = (0,0,1,3)$, $v_2 = (4,0,0,1)$, and $v_3 = (2,2,3,2)$. Then, the corresponding values of $v_1$ are $\beta_{1,0}^1 = 2$, $\beta_{1,1}^1 = 1$, $\beta_{1,3}^5 = 1$, and all other cases are equal to zero. The corresponding values of $v_2$ are $\beta_{2,4}^4 = 1$, $\beta_{2,0}^4 = 1$, $\beta_{2,0}^0 = 1$, $\beta_{2,1}^3 = 1$, and the remaining values are equal to zero. Finally, the corresponding values of $v_3$ are $\beta_{3,2}^5 = 2$, $\beta_{3,3}^3 = 1$, $\beta_{3,2}^2 = 1$, and the remaining values are equal to zero.

The inner product defined in (2.1) can be interpreted as

$$\sum\nolimits_{j=1}^{b} c_{ij}v_{ij} = \sum\nolimits_{k=0}^{q-1}\left(k\sum\nolimits_{l=0}^{q-1} l.\beta_{i,l}^k\right).$$

However, the generalized Lee weight of $v_i$ given in (1.2) is equal to $\sum_{k=0}^{q-1}\left(\sum_{l=0}^{q-1}|l|.\beta_{i,l}^k\right)$.

Next, we can split the sum according to the index set of each fixed $b-$ byte components $c_i$ and by interpreting the inner sum accordingly, we have

$$\widetilde{f}(c) = \prod_{i=1}^{n}\prod_{k=0}^{q-1}\sum_{\sum_{l=0}^{q-1}\beta_{i,l}^{k}=\alpha_{i,k}}\left(\frac{\alpha_{i,k}!}{\beta_{i,0}^{k}!\beta_{i,1}^{k}!\ldots\beta_{i,q-1}^{k}!}\right)\left(e^{\frac{2\pi i}{q}}\right)^{\left(k\sum_{l=0}^{q-1}l.\beta_{i,l}^{k}\right)} z^{\left\lceil\left(\sum_{l=0}^{q-1}|l|.\beta_{i,l}^{k}\right)/t\right\rceil}.$$

**Corollary 2.1.** *The MacWilliams identity given in* [10] *is a special case of Theorem 2.1.*

**Proof.** If we take $q = 4$ in Theorem 2.1, the proof is complete.

**Corollary 2.2.** *Theorem 2.1 coincides with the MacWillams identity given in* [11] *over* $\mathbb{Z}_2$ *and* $\mathbb{Z}_3$.

**Proof.** Since the generalized Lee weight (distance) coincides with Hamming weight (distance) over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, the proof is complete.

**Example 2.1.** Let

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the generator matrix of a linear code $C$ over $\mathbb{Z}_6$. $C$ has 12 codewords. The dual code of $C$ is a linear code over $\mathbb{Z}_6$ and it has 3888 codewords.

Now, we first demonstrate how to apply the formulae. It is clear that the codeword $c = (1, 3, 0, 0, 0, 0)$ belongs to $C$. Let $b = 3$ and $t = 2$. Then, $c = (c_1, c_2)$. The generalized Lee weight distribution vectors corresponding to $c_1$ and $c_2$ are $S(c_1) = (1, 1, 0, 1, 0, 0)$ and $S(c_2) = (3, 0, 0, 0, 0, 0)$, respectively. The generalized Lee weight distribution vectors of the codewords in the code $C$, and polynomials $V_S^{(2,6)}$ for $t = 2$ and $q = 6$ are shown in Table 1 for the necessary computations to apply to the main theorem.

By Eq. (1.4), we obtain the $m-$spotty generalized Lee weight enumerator of $C$ as

$$W(z) = 1 + 4z + 4z^2 + 3z^3. \tag{2.7}$$

By Theorem 2.1 and Table 1, we get

(2.8)

$$\begin{aligned} W^{\perp}(z) &= \frac{1}{|C|}\sum_{c\in C}\prod_{i=1}^{2}\left(V_S^{(2,6)}(z)\right) \\ &= 1 + 30z + 236z^2 + 799z^3 + 1317z^4 + 1058z^5 + 388z^6 + 57z^7 + 2z^8. \end{aligned}$$

In Table 1, $a = 1+24z+83z^2+83z^3+24z^4+z^5, b = 1+4z+z^2-z^3-4z^4-z^5, c = 1+16z+19z^2-19z^3-16z^4-z^5, d = 1-z^2-z^3+z^5, e = 1+6z-7z^2-7z^3+6z^4+z^5$, and $f = 1 - 2z + z^2 - z^3 + 2z^4 - z^5$.

## 3. Conclusion

In this paper, we prove a MacWilliams type identity for $m-$spotty generalized Lee weight enumerators over $\mathbb{Z}_q$. Further, we show that this identity is a general case of the identities given in [10] and in [11] over $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Finally, we conclude the paper by giving an illustration of the main theorem.

| Codewords | $S(c_1), S(c_2)$ | $V_S^{(2,6)}(z) = V_{S(c_1)}^{(2,6)}(z)\, V_{S(c_2)}^{(2,6)}(z)$ |
|---|---|---|
| $(0,0,0,0,0,0)$ | $(3,0,0,0,0,0), (3,0,0,0,0,0)$ | $a^2$ |
| $(0,3,0,0,0,0)$ | $(2,0,0,1,0,0), (3,0,0,0,0,0)$ | $ba$ |
| $(1,0,0,0,0,0)$ | $(2,1,0,0,0,0), (3,0,0,0,0,0)$ | $ca$ |
| $(1,3,0,0,0,0)$ | $(1,1,0,1,0,0), (3,0,0,0,0,0)$ | $da$ |
| $(2,0,0,0,0,0)$ | $(2,0,1,0,0,0), (3,0,0,0,0,0)$ | $ea$ |
| $(2,3,0,0,0,0)$ | $(1,0,1,1,0,0), (3,0,0,0,0,0)$ | $fa$ |
| $(3,0,0,0,0,0)$ | $(2,0,0,1,0,0), (3,0,0,0,0,0)$ | $ba$ |
| $(3,3,0,0,0,0)$ | $(1,0,0,2,0,0), (3,0,0,0,0,0)$ | $da$ |
| $(4,0,0,0,0,0)$ | $(2,0,0,0,1,0), (3,0,0,0,0,0)$ | $ea$ |
| $(4,3,0,0,0,0)$ | $(1,0,0,1,1,0), (3,0,0,0,0,0)$ | $fa$ |
| $(5,0,0,0,0,0)$ | $(2,0,0,0,0,1), (3,0,0,0,0,0)$ | $ca$ |
| $(5,3,0,0,0,0)$ | $(1,0,0,1,0,1), (3,0,0,0,0,0)$ | $da$ |

Table 1. The codewords and their corresponding expressions.

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