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Hagen-Poiseuille Flow in Circular Cylinder when Temperature is Exponential and Sinusoidal Function of Length

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Abstract

In this paper, we have investigated the heat transfer in a circular cylindrical pipe for Hagen-Poiseuille flow and used MATLAB as a scientific tool to plot the graphs. The calculations for the axial heat conduction and the temperature gradient have been performed for both upstream and downstream flows. In this experiment, the results are plotted graphically for the different fluids like Air, Water, Milk, Glycerin and Mercury. The physical trends of the plotted curves represent the values of heat transfer that were different in Hydrogen and Air; on the contrary rest of the fluids were behaving similarly when temperature was taken as an exponential function and for sinusoidal function all the fluids were behaving in a similar manner.

Keywords: Heat Transfer, Hagen - Poiseuille Flow, Fluid Flow.
2010 MSC: Subject Classification 80M25.

Symbols and their meanings

- z: Horizontal Distance
- R: Radius of Cylinder
- ρ: Density
- c_v: Specific Heat at constant volume
- κ: Coefficient of thermal conductivity

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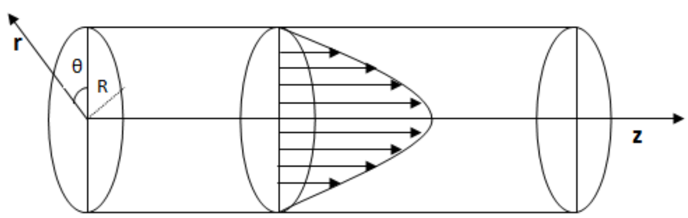
Nu: Nusselt Number
T: Temperature
r: Distance of fluid particle from the axis of the cylinder
$(v_z)_m$: Maximum Velocity of fluid
a: Constant

1. Introduction

G.H.L. Hagen, a German hydraulician and a French physiologist J.L.M. Poiseuille have discovered the fundamental law of laminar flow in pipes. This experiment is diversely used in the field of science and technology where laminar flow through pipes occurs and widely used to know the blood flow through veins and arteries, and for liquids of a large range of viscosities. Laminar flow in pipes is well illustrated for agreement of theory and experiment in classical physics which is studied by students worldwide. The English physicist O. Reynolds [9] gave a detailed examination on its limits at the transition to turbulent flow. With extensive research, the result of flow was linearly stable for all Reynolds numbers i.e. non-axisymmetric modes. The numerical calculations (see for reference) are the basis of the statement on the stability of the flow. Von Kerczek and Tozzi [7] studied a slight change in the Hagen-Poiseuille setting, i.e. small oscillations superpose to the stationary pressure gradient. This resulted in finding that oscillations can have stabilizing and destabilizing effects. Catherine Loudon and Katherine Mcculloh [8] described the use of Hagen-Poiseuille Equation to fluid feeding through short tubes. Erdogan [2] obtained the exact solutions for the motion of viscous fluid due to sine and cosine oscillations of a vertical plate. T. Hayat Ehal [5] interpreted the exact solutions of five problems including time-periodic poiseuille flow due to an oscillating pressure gradient. Hayat et. al. and Fetecau et. al. [3][4] further extended the study of motion of fluids in various geometrical scenarios for sine oscillations, cosine oscillations, longitudinal and torsional oscillations, etc. Harold Salwen et. al. [10] [11] studied the stability of Poisuelle flow in a pipe of circular cross-sector to the horizontal angular distance from a certain direction together with axisymmetric disturbance through a matrix differential equation and showed that pipe flow is stable to infinitesimal disturbances for all taken values, then Salwen and Grosch made corrections in matrix elements and new results confined the stability. J.L. Bansal [1] studied the Hagen Poiseuille flow in a circular pipe for both velocity and temperature distribution assuming the temperature of the wall to be constant and varying uniformly.

The early research was concentrated on the stability of Hagen-Poiseuille flow with reference to the choice of boundary conditions. In spite of extensive research on the stability of the flow with the no-slip boundary condition, in order to understand the stability of fluid flows, Kang C et.al [6] had pioneered these experiments with the support of Hagen-Poiseuille flow. The main focus of current study is to analyze the relation between temperature distribution and the length of the pipe for different fluids such as air, hydrogen, water, milk, mercury, glycerin and study them graphically where temperature is a transcendental variable.



Schematic figure of the problem

Navier Stockes equations for viscous in-compressible fluid with constant fluid properties in cylindrical system are

$$\frac{1}{r}\frac{\partial\,(rv_r)}{\partial\,r} + \frac{1}{r}\frac{\partial\,v_\theta}{\partial\,\theta} + \frac{\partial\,v_z}{\partial\,z} = 0. \tag{1}$$

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{2}\right)=\rho\, f_r-\frac{\partial\, p}{\partial\, r}+\mu\Big(\Delta^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial\, v_\theta}{\partial\,\theta}\Big). \tag{2}$$

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}\right)=\rho f_\theta-\frac{1}{r}\frac{\partial\, p}{\partial\,\theta}+\mu\Big(\Delta^2 v_\theta+\frac{2}{r^2}\frac{\partial\, v_r}{\partial\,\theta}-\frac{v_\theta}{r^2}\Big). \tag{3}$$

$$\rho\,\frac{Dv_z}{Dt}=\rho f_z-\frac{\partial\, p}{\partial\, z}+\mu\Delta^2 v_z. \tag{4}$$

$$\rho\, c_v\frac{DT}{Dt}=\frac{\partial\, Q}{\partial\, t}+k\Delta^2 T+\phi_c. \tag{5}$$

In our case there is laminar flow without body forces and motion is due to pressure gradient along the axis of the pipe i.e. Z-axis. Let r denote the radial distance and θ denote the angle, then due to axial symmetry

$$\frac{\partial}{\partial\,\theta}(\,)=0. \tag{6}$$

and v_z is the only non zero component of velocity. Thus the above set of equations reduce to

$$\frac{\partial\,(v_z)}{\partial\, z}=0. \tag{7}$$

$$\frac{\partial p}{\partial r}=0. \tag{8}$$

$$\frac{\partial\, p}{\partial\, z}=\mu\Big(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\Big). \tag{9}$$

$$\rho c_v v_z\frac{\partial T}{\partial z}=k\Big(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2}{\partial z^2}\Big)+\mu\Big(\frac{\partial v_z}{\partial r}\Big)^2. \tag{10}$$

2. Heat transfer in a circular pipe with wall

2.1. When temperature is increasing exponentially:

In any circular pipe if wall temperature is an exponential function e^{az} of characteristics length of the pipe , then the temperature of the wall of pipe will increase exponentially. The heat transferred can be obtained with the help of energy equation. Due to axial symmetry:

$$\rho\, c_v\, v_z\,\frac{\partial\, T}{\partial\, z}=k\Big(\frac{\partial^2\, T}{\partial\, r^2}+\frac{1}{r}\frac{\partial\, T}{\partial\, r}+\frac{\partial^2\, T}{\partial\, z^2}\Big)+\mu\Big(\frac{\partial v_z}{\partial r}\Big)^2. \tag{11}$$

From the Navier- Stocks equations the velocity distribution is $v_z=(v_z)_m\Big[1-\Big(\frac{r}{R}\Big)^2\Big]$, where R = Radius of the cylinder and $(v_z)_m=-\frac{R^2}{4\mu}\frac{dp}{dz}$ and it is the maximum value of velocity at $r=0$. Consider temperature to be exponential function of z, i.e.

$$T=Ae^{az}+g(r). \tag{12}$$

where g is any function of r and is independent from z.

Substituting equation (12) in (11) and neglecting the heat due to dissipation, we get

$$\frac{\rho c_v (v_z)_m}{k}\Big[1-\Big(\frac{r}{R}\Big)^2\Big]Ae^{z}=\frac{d^2 g}{dr^2}+\frac{1}{r}\frac{dg}{dr}+Ae^{az}. \tag{13}$$

On solving this differential equation under the boundary conditions *r=0: g=* finite; *r=R: g=0,* we get

$$g = \frac{A\rho c_v (v_z)_m a e^{az}}{4k}\Big(r^2 - \frac{r^4}{4R^2} - \frac{3R^2}{4}\Big) + \frac{Ae^{az}a^2}{4}(R^2 - r^2). \tag{14}$$

On putting the value in equation (12)

$$T = Ae^{az} + \frac{A\rho c_v (v_z)_m a e^{az}}{4k}\Big(r^2 - \frac{r^4}{4R^2} - \frac{3R^2}{4}\Big) + \frac{Ae^{az}a^2}{4}(R^2 - r^2). \tag{15}$$

For maximum temperature ($r = 0$)

$$T_m = Aae^{az} - \frac{3A\rho c_v (v_z)_m a e^{az} R^2}{16k} + \frac{Aa^2 e^{az} R^2}{4}. \tag{16}$$

Now,

$$T_{mean} = \frac{\int_0^R 2T\pi r dr}{\pi R^2}. \tag{17}$$

$$T_{mean} = Ae^{az} + \frac{Aa^2 e^{az} R^2}{8} - \frac{A\rho c_v (v_z)_m a e^{az} R^2}{12k}. \tag{18}$$

Nusselt number is given by

$$Nu = \frac{2R}{T_{mean} - T_w}\Big(\frac{\partial T}{\partial r}\Big)_{r=R}. \tag{19}$$

Using equation (11,15) where $T = Ae^z$

$$Nu = \frac{2R}{\frac{Ae^{az}a^2R^2}{8} - \frac{A\rho c_v (v_z)_m a e^{az} R^2}{12k}}\Big[\frac{Aa^2 e^{az} R}{2} + \frac{A\rho c_v (v_z)_m a e^{az} R}{4k}\Big]. \tag{20}$$

2.2. When the temperature is sinusoidal function of length:

In any circular pipe if wall temperature is sine function, then the temperature of the wall of pipe will increase and decrease periodically. The heat transferred can be obtained with the help of energy equation. Due to axial symmetry:

$$T = ASinz + g(r). \tag{21}$$

where g is any function of r and is independent from z.

On solving equation (11) and (21) with neglecting the heat due to dissipation, we get

$$\frac{\rho c_v (v_z)_m}{k}\Big[1 - \Big(\frac{r}{R}\Big)^2\Big]ACosz = \frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} + ASinz. \tag{22}$$

On solving this differential equation under the boundary condition r=0: g= finite; r=R: g=0, we get

$$g = \frac{A\rho c_v (v_z)_m aCosz}{4k}\Big(r^2 - \frac{r^4}{4R^2} - \frac{3R^2}{4}\Big) + ASinz\Big(\frac{R^2}{4} - \frac{r^2}{4}\Big). \tag{23}$$

On putting the value inn equation (21), we get

$$T = ASinz\Big(1 + \frac{R^2}{4} - \frac{r^2}{4}\Big) + \frac{\rho c_v (v_z)_m A}{4k}\Big(r^2 - \frac{r^4}{4R^2} - \frac{3R^2}{4}\Big)Cosz. \tag{24}$$

Now on calculating T_{mean} from equation(17)

$$T_{mean} = ASinz\Big(1 + \frac{R^2}{8}\Big) - \frac{A\rho c_v (v_z)_m CoszR^2}{12k}. \tag{25}$$

Nusselt number(the coefficient of heat transfer at the surface of walls) is

$$Nu = \frac{2R}{\frac{ASinzR^2}{8} - \frac{A\rho c_v (v_z)_m aCoszR^2}{12k}}\Big[\frac{ASinzR}{2} + \frac{A\rho c_v (v_z)_m aCoszR}{4k}\Big]. \tag{26}$$

Table 1: List of Various Parameters for different fluids whose comparative studies to be verified

Fluids	κ (coeff of thermal conductivity)	Density (kg/m^3)	C_v = (J/Kg° C)
Water	0.319	997	4186
Hydrogen	0.1003	0.082	1016
Air	0.014	1.225	721
Glycerin	0.140	1260	2410
Milk	0.560	1033	3930
Mercury	4.74	13593	139

2.3. Result and Discussion

Temperature Distribution for Sinsudial Function: The assumed values for sinusoidal function:
R=1 cm
r=0, 0.3, 0.6, 0.9, 1.0 (in cm)
$(v_z)_m = 2\ cm/sec$
z = 0, 20, 40, 60, 80, 100 (in cm)
a=1

Table 2: For Water: Values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$
00	-3.9	-3.8	-3.5	-3.13	-2.9
20	-3.7	-3.6	-3.3	-2.94	-2.8
40	-3.05	-2.9	-2.7	-2.4	-2.2
60	-1.9	-1.9	-1.7	-1.56	-1.4
80	-6.9	-6.7	-0.6	-0.54	-0.5
100	6.9	6.7	0.6	0.54	0.5

Table 3: For Hydrogen: Values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	T	T	T	T	T
00	-2.5069	-2.4333	-2.2331	-1.9671	-1.8801
20	-1.5006	-1.4469	-1.3049	-1.1319	-1.0827
40	-0.3134	-0.2860	-0.2194	-0.1602	-0.1547
60	0.9116	0.9094	0.8926	0.8308	0.7920
80	2.0267	1.9952	1.8970	1.7216	1.6431
100	2.8973	2.8402	2.6725	2.4048	2.2961

Table 4: For Air Values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	T	T	T	T	T
00	-3.7095	-3.6008	-3.3045	-2.9108	-2.7822
20	-2.6308	-2.5440	-2.3117	-2.0187	-1.9303
40	-1.2347	-1.1803	-1.0401	-0.8832	-0.8457
60	0.3103	0.3257	0.3569	0.3589	0.3410
80	1.8179	1.7924	1.7109	1.5577	1.4865
100	3.1062	3.0430	2.8586	2.5686	2.2961

Table 5: For Glycerin: Values of z and T for different values of r

	$z = 0$	$z = 0.3$	$z = 0.6$	$z = 0.9$	$z = 1$
z	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$
00	-1.2754	-1.2380	-1.1361	-1.0008	-9.5653
20	-1.1984	-1.1633	-1.0676	-0.9404	-8.9884
40	-9.7698	-9.4833	-8.7029	-0.7666	-7.3273
60	-6.3766	-6.1896	-5.6803	−0.5004	-4.7825
80	-2.2144	-2.1495	-1.9726	-0.1738	-1.6608
100	2.2149	2.1495	1.9726	0.1738	1.6608

Table 6: For Milk: Values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$
00	-6.8203	-6.6203	-6.0755	-5.3517	-5.1152
20	-6.4090	-6.2210	-5.7091	-5.0289	-4.8067
40	-5.2246	-5.0714	-4.6541	-4.0996	-3.9185
60	-3.4101	-3.3101	-3.0377	-2.6758	-2.5576
80	-1.1843	-1.1496	-1.0550	-0.9293	-8.8823
100	1.1844	1.1496	1.0550	0.9293	8.8827

Table 7: For Mercury: Values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$
00	-2.6868	-2.6080	-2.3934	-2.1082	-2.0151
20	-6.4090	-6.2210	-5.7091	-5.0289	-4.8067
40	-5.2246	-5.0714	-4.6541	-4.0996	-3.9185
60	-3.4101	-3.3101	-3.0377	-2.6758	-2.5576
80	-1.1843	-1.1496	-1.0550	-0.9293	-8.8823
100	1.1844	1.1496	1.0550	0.9293	8.8827

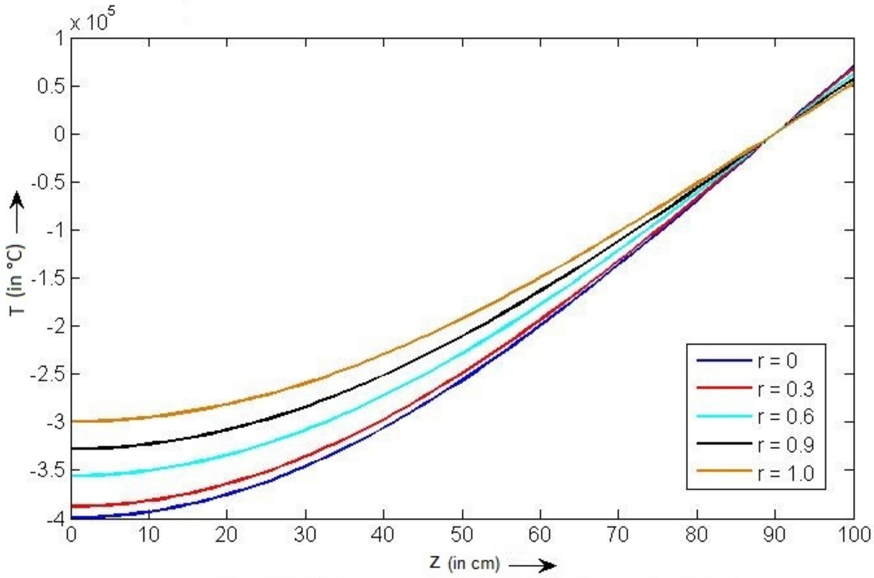


Fig 1(a) Temperature Distribution For Water

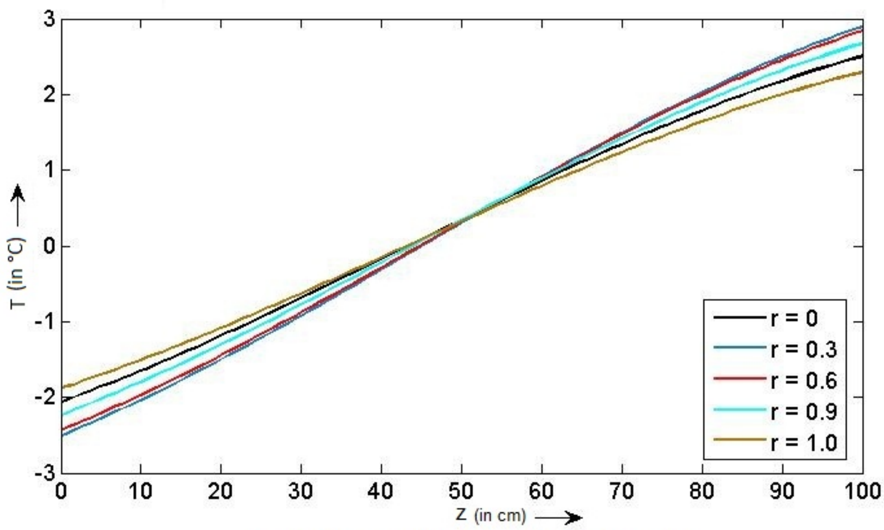


Fig 1(b) Temperature distribution for Hydrogen

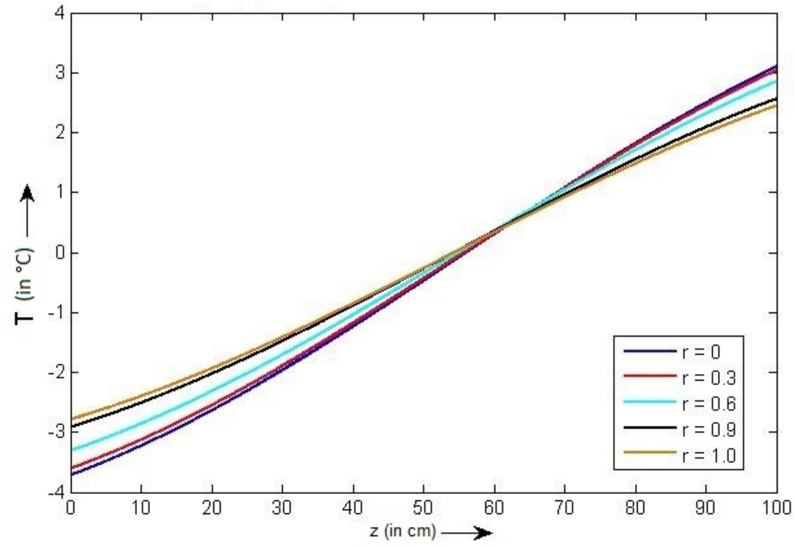


Fig 1(c) Temperature Distribution for Air

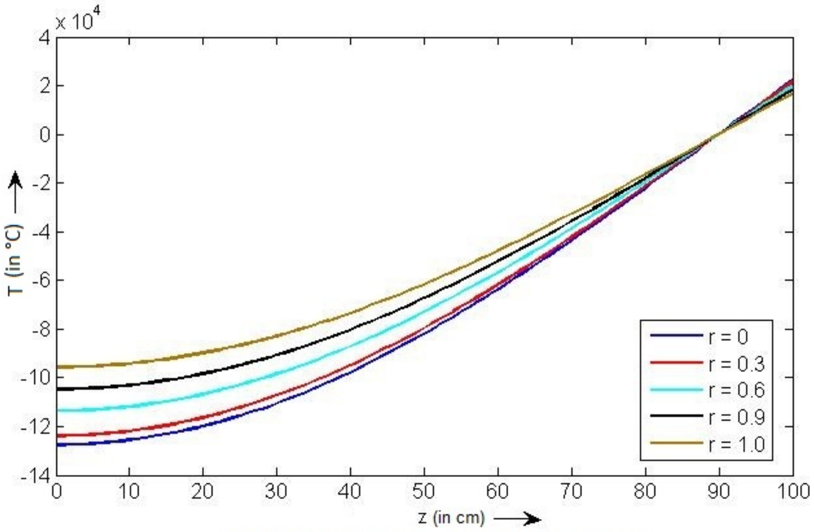


Fig 1(d) Temperature distribution for Glycerin

Table 8: For Water values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$
00	-0.100	-0.088	-0.0562	-0.138	2.000
1	-2.71	-0.240	-0.1529	-0.376	5.4366
2	-7.38	-651	-0.4155	-1.023	14.7781
3	-2.005	-1.770	-1.1295	-2.782	40.1711
4	-5.451	-4.812	-3.0702	-7.561	109.196
5	-14.819	-13.080	-8.3457	-20.554	296.8263

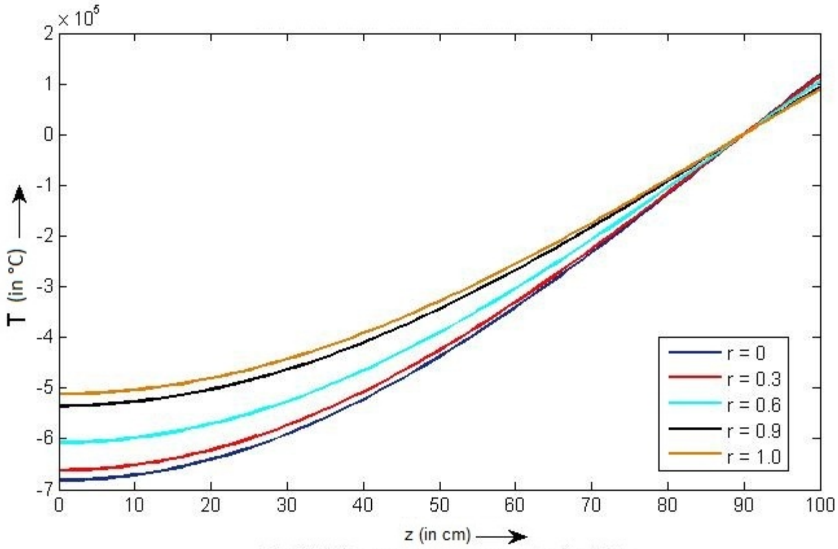


Fig 1(e) Temperature distribution for Milk

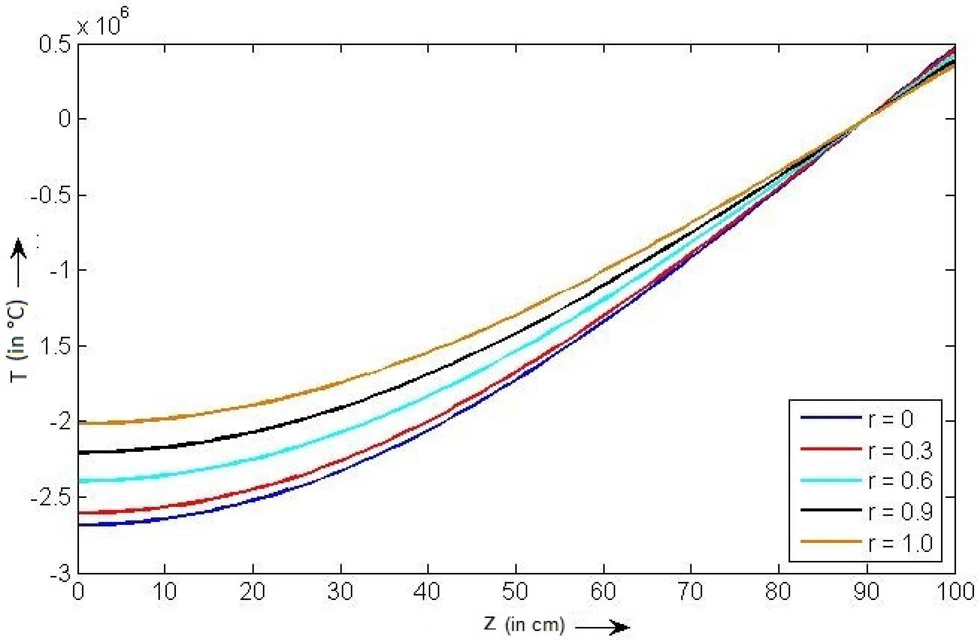


Fig 1(f) Temperature distribution for Mercury

Table 9: For Hydrogen values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	T	T	T	T	T
00	1.8733	1.9018	1.9670	2.0081	2.0000
1	5.0921	5.1696	5.3470	5.4585	5.4366
2	13.8418	14.0525	14.5345	14.8378	14.7781
3	37.6259	38.1987	39.5089	40.3333	40.1711
4	102.2779	103.8347	107.3964	109.6372	109.1963
5	278.0202	282.2520	291.9338	298.0247	296.8263

Table 10: For Air values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	T	T	T	T	T
00	1.5726	1.6364	1.7977	1.9664	2.000
1	4.2748	4.4482	4.8866	5.3452	5.4366
2	11.6201	12.0914	13.2833	14.5296	14.7781
3	31.5868	32.8679	36.1077	39.4956	40.1711
4	85.8618	89.3442	98.1509	107.3602	109.196
5	233.3966	242.8627	266.8018	291.8354	296.8263

Table 11: For Glycerin values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$	$T \times (10)^5$
00	-0.319	-0.281	-0.180	-0.0442	2.000
1	-0.867	-0.765	-0.488	-0.1202	5.4366
2	-2.356	-2.079	-1.327	-0.3266	14.7781
3	-6.404	-5.654	-3.606	-0.8878	40.1711
4	-17.407	-15.365	-9.803	-2.4134	109.196
5	-47.317	-41.766	-26.647	-6.5602	296.8263

Table 12: For Milk values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$
00	-0.171	-0.151	0.096	-0.0236	2.000
1	-0.463	-0.409	0.261	-0.0643	5.4366
2	-1.260	-1.112	0.710	-0.1747	14.7781
3	-3.425	-3.023	1.929	-0.4750	40.1711
4	-9.309	-8.217	5.243	-1.2911	109.196
5	-25.305	-22.337	14.252	-3.5096	296.8263

Table 13: For Mercury values of z and T for different values of r

	$r = 0$	$r = 0.3$	$r = 0.6$	$r = 0.9$	$r = 1$
z	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$	$T \times (10)^6$
00	-0.672	-0.593	-0.378	-0.0093	2.000
1	-1.826	-1.612	-1.028	-0.0253	5.4366
2	-4.963	-4.381	-2.795	-0.0688	14.7781
3	-13.491	-11.909	-7.598	-0.1871	40.1711
4	-36.673	-32.371	-20.654	-0.5086	109.196
5	-99.687	-87.994	-56.144	-1.3826	296.8263

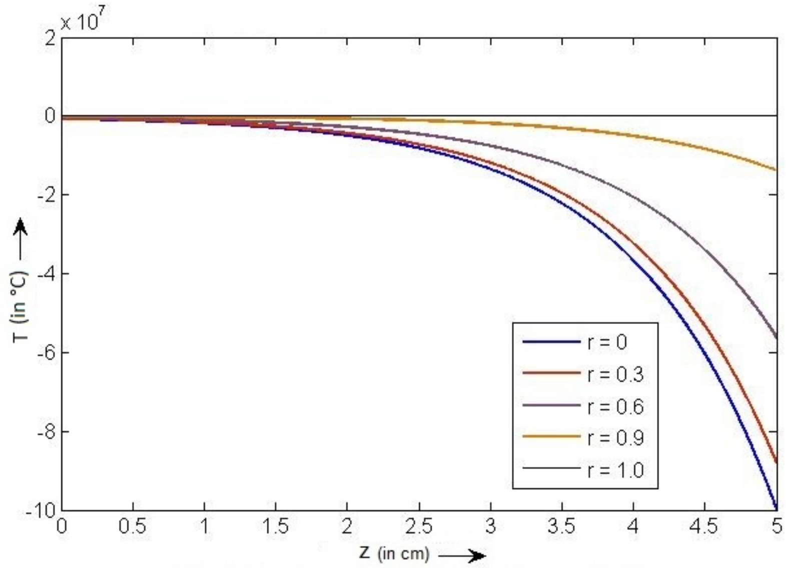


Fig 2(a) Temperature Distribution For Water

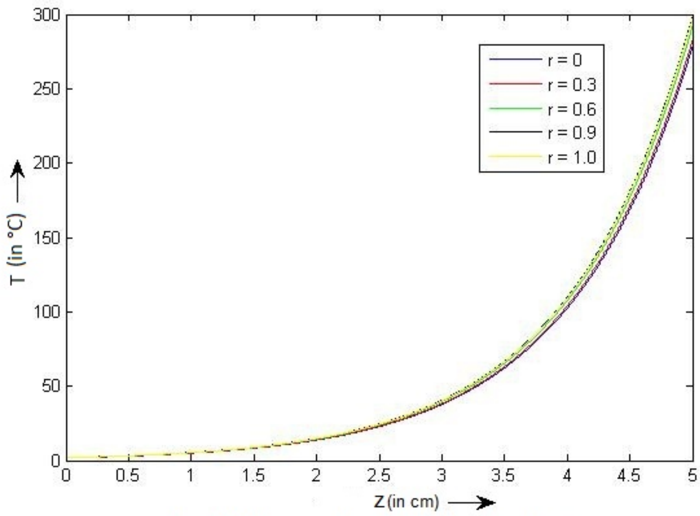


Fig 2(b) Temperature distribution for Hydrogen

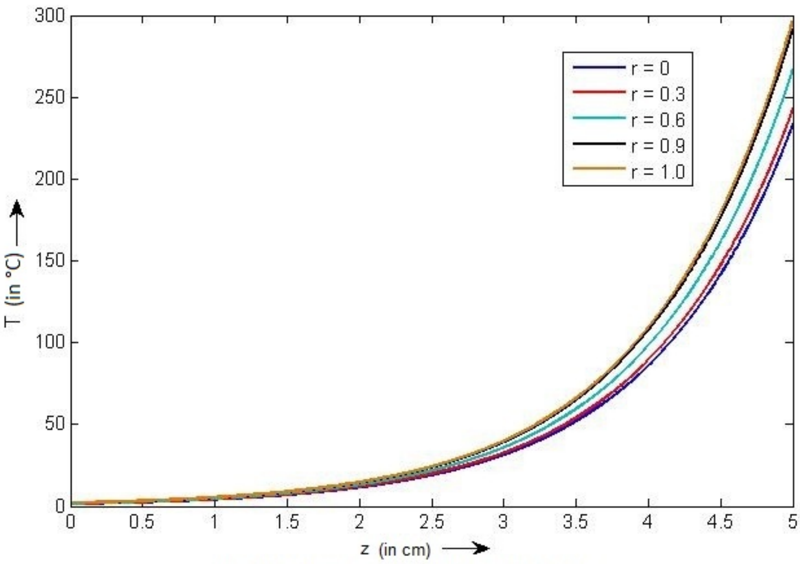


Fig 2(c) Temperature distribution for Air

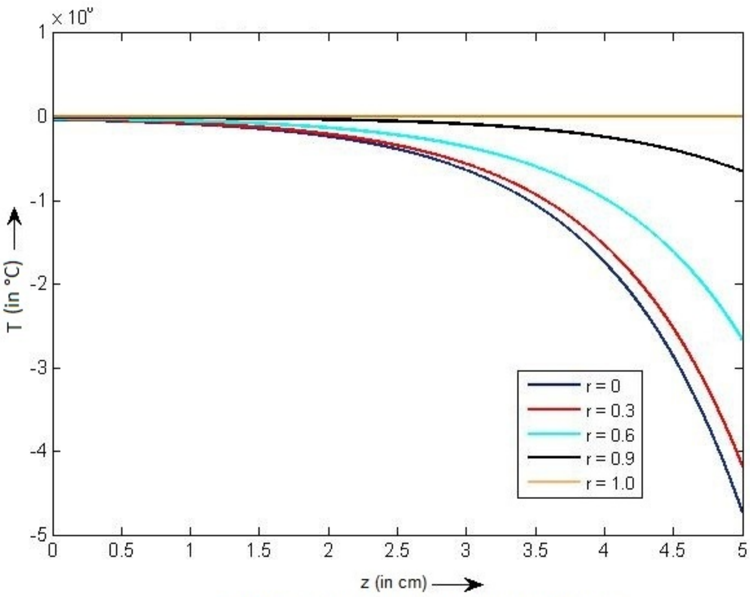


Fig2(d) Temperature distribution for Glycerin

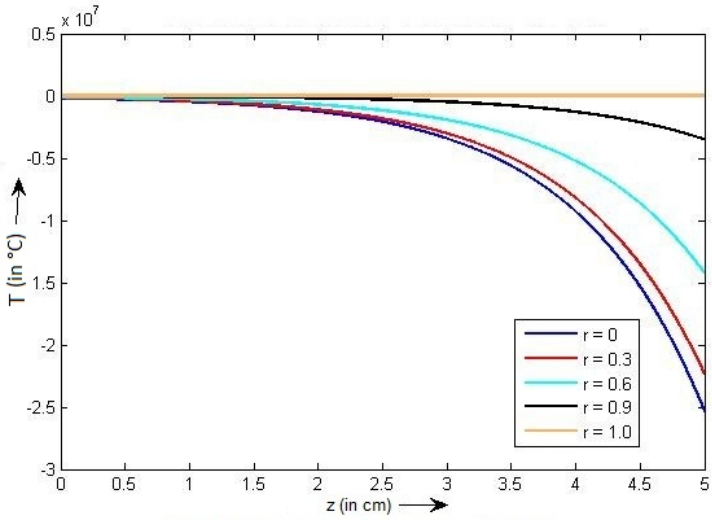


Fig 2(e) Temperature distribution for Milk

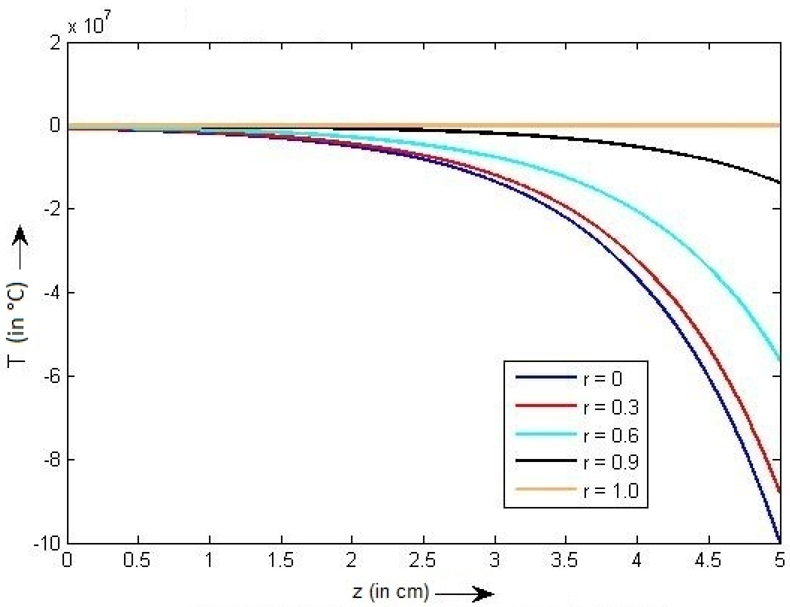


Fig 2(f) Temperature Distribution for mercury

2.4. Conclusions

When T is exponential function fig 2(a,d,e,f) are plotted with increasing temperature function on Y axis and the length variation of cylindrical pipe on X axis. It can be easily seen from the graphs that the both parameters are inversely proportional i.e. with the increment in length of pipe there is decrement in temperature and vice versa. In case of Hydrogen and Air fig 2(b,c) there is exception as seen in pipe there is increase in temperature. A similar study can be done for gases where a similar reaction is found. For liquids (water, milk, glycerin, mercury) the elevating function of the density would show the diminishing behavior of the temperature gradient function. The same study was done for gasses also which represents that the temperature gradient was dropped rapidly as compare to liquids with inflating function of density. We got the same behavior for liquids and gasses both if we enhance the value of thermal conductivity then the temperature gradient is decreased. When T is sinusoidal function of z, It has been observed from the fig 1(a)- 1(f) that for all fluids both the parameters are directly proportional i.e. with the increment in length of pipe there is also an increment in temperature gradient. When density is increased, same behavior observed but the rate of increment of temperature is small.

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