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Some new fixed point results for monotone enriched nonexpansive mappings in ordered Banach spaces

Rahul Shukla[a], Rajendra Pant[a]

[a] *Department of Mathematics and Applied Mathematics, University of Johannesburg, Johannesburg, South Africa.*

Abstract

We study monotone enriched nonexpansive mappings and present some new existence and convergence theorems for these mappings in the setting of ordered Banach spaces. More precisely, we employ the Krasnosel'skiĭ iterative method to approximate fixed points of enriched nonexpansive under different conditions. This way a number of results from the literature have been extended, generalized and complemented.

Keywords: Nonexpansive mapping; Enriched nonexpansive mapping; Banach space.
2010 MSC: 47H10; 47H09.

1. Introduction

Let $(\mathcal{B}, \|.\|)$ be a Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if for each pair of elements $\vartheta, \nu \in \mathcal{C}$, we have

$$\|\xi(\vartheta) - \xi(\nu)\| \leq \|\vartheta - \nu\|.$$

A point $\vartheta^\dagger \in \mathcal{C}$ is said to be a fixed point of ξ if $\xi(\vartheta^\dagger) = \vartheta^\dagger$. The class of nonexpansive mapping need not have a fixed point in the case of general Banach spaces. However, in 1965, Browder [7], Göhde [12] Kirk [13], independently proved the first fixed point result for nonexpansive mappings in Banach spaces having ceratin geometrical properties. After these results, a number of nonlinear mappings have been appeared in the literature to enlarge the class of nonexpansive mappings [11, 15, 9, 25, 18] (see also the references therein).

Email addresses: `rshukla.vnit@gmail.com`; `rshukla@uj.ac.za` (Rahul Shukla), `pant.rajendra@gmail.com`; `rpant@uj.ac.za` (Rajendra Pant)

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Very recently, Berinde [3] introduced a new class of nonexpansive mappings known as enriched nonexpansive mappings which generalizes nonexpansive mappings. It is shown in [3, 4] that this class of mappings has strong connection with the nonexpansive mappings.

On the other hand, a number of fixed point theorems have appeared in the literature where the nonexpansive condition on mapping needs to satisfy only for comparable elements in partially ordered spaces. The motivation behind this approach is to determine the nature of the solution whether it is positive or negative and this approach has fruitful applications. Ran and Reurings obtained the solution of matrix equations through a generalization of the Banach contraction principle, see [19]. Nieto and Rodríguez-López [16] applied similar type of fixed point theorem to find the solution of some differential equations, for more applications of the fixed point theory for monotone mappings, see [8]. Thereafter, many authors developed a metric fixed point theory for monotone nonexpansive mappings, see [6, 5, 23, 22, 21, 2, 24].

Motivated by Berinde [3, 4] and others, we extend the class of enriched nonexpansive mappings in the setting of ordered Banach spaces and establish some existence and convergence results for enriched nonexpansive mappings. We employ Krasnosel'skiĭ iterative method to approximate the fixed points in ordered Banach spaces under certain assuptions. Our results complement, extend, and generalize certain results from [3, 4, 5, 24, 2].

2. Preliminaries

Let $\mathcal{B}$ be a Banach space with a partial order $\preceq$ compatible with the linear structure of $\mathcal{B}$, that is,

$$\begin{aligned} \vartheta &\preceq \nu \text{ implies } \vartheta+\zeta \preceq \nu+\zeta, \\ \vartheta &\preceq \nu \text{ implies } \lambda\vartheta \preceq \lambda\nu \end{aligned}$$

for every $\vartheta, \nu, \zeta \in \mathcal{B}$ and $\lambda \geq 0$. It follows that all order intervals $[\vartheta, \rightarrow] = \{\zeta \in \mathcal{B} : \vartheta \preceq \zeta\}$ and $[\leftarrow, \nu] = \{\zeta \in \mathcal{B} : \zeta \preceq \nu\}$ are convex. Moreover, we will assume that each $[\vartheta, \rightarrow]$ and $[\leftarrow, \nu]$ is closed. We will say that $(\mathcal{B}, \|\cdot\|, \preceq)$ is an ordered Banach space.

A sequence $\{\vartheta_n\}$ is said to be an approximate fixed point sequence (a.f.p.s. for short) for a mapping ξ if $\|\xi(\vartheta_n) - \vartheta_n\| \to 0$ as $n \to \infty$. A sequence $\{\vartheta_n\}$ is monotone increasing if $\vartheta_1 \preceq \vartheta_2 \preceq \vartheta_3 \preceq \cdots$. We shall use the following observation (see [5, Lemma 3.1]). Assume that $\{\vartheta_n\}$ is a monotone sequence that has a cluster point, i.e., there is a subsequence $\{\vartheta_{n_j}\}$ that converges to g (with respect to the strong or weak topology). Since the order intervals are (weakly) closed, we have $g \in [\vartheta_n, \rightarrow)$ for each n, that is, g is an upper bound for $\{\vartheta_n\}$. If g_1 is another upper bound for $\{\vartheta_n\}$, then $\vartheta_n \in (\leftarrow, g_1]$ for each n, and hence $g \preceq g_1$. It follows that $\{\vartheta_n\}$ converges to $g = \sup\{\vartheta_n\}$. If $\{\vartheta_n\}$ is a monotone increasing (resp. monotone decreasing) sequence which converges to p, then $\vartheta_n \preceq p$ (resp. $p \preceq \vartheta_n$). We say that $\vartheta, \nu \in \mathcal{B}$ are comparable whenever $\vartheta \preceq \nu$ or $\nu \preceq \vartheta$.

Definition 2.1. *[10]. A Banach space $\mathcal{B}$ is said to be uniformly convex if for every $\varepsilon \in (0, 2]$ there is some $\delta > 0$ so that, for any $\vartheta, \nu \in \mathcal{B}$ with $\|\vartheta\| = \|\nu\| = 1$, the condition $\|\vartheta - \nu\| \geq \varepsilon$ implies that $\left\|\frac{\vartheta+\nu}{2}\right\| \leq 1 - \delta$.*

Definition 2.2. *Let $(\mathcal{B}, \|\cdot\|, \preceq)$ be an ordered Banach space.*

- *[17]. A space $\mathcal{B}$ satisfies weak-Opial property if, for every weakly convergent sequence $\{\vartheta_n\}$ with weak limit $\vartheta \in \mathcal{B}$ it holds:*
$$\liminf_{n\to\infty} \|\vartheta_n - \vartheta\| < \liminf_{n\to\infty} \|\vartheta_n - \nu\|$$
for all $\nu \in \mathcal{B}$ with $\vartheta \neq \nu$.

- *[1] A space $\mathcal{B}$ satisfies the monotone weak-Opial property if, for every monotone weakly convergent sequence $\{\vartheta_n\}$ with weak limit $\vartheta \in \mathcal{B}$ it holds:*
$$\liminf_{n\to\infty} \|\vartheta_n - \vartheta\| < \liminf_{n\to\infty} \|\vartheta_n - \nu\|$$
for all $\nu \in \mathcal{B}$ and ν is greater or less than all the elements of the sequence $\{\vartheta_n\}$.

All finite dimensional Banach spaces, all Hilbert spaces and ℓ^p $(1 \leq p < \infty)$ satisfy the weak-Opial property. But $L_p([0,1])$, for $p > 1$ do not have the Opial property [10]. In [1], it is proved that $L_p([0,1])$, for $p > 1$, satisfy monotone weak-Opial property.

Definition 2.3. *[10]. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be quasi-nonexpansive if for all $\vartheta \in \mathcal{C}$ and $\vartheta^\dagger \in F(\xi) \neq \emptyset$,*

$$\|\xi(\vartheta) - \vartheta^\dagger\| \leq \|\vartheta - \vartheta^\dagger\|.$$

where $F(\xi)$ is the set of all fixed points of ξ.

It is well known that a nonexpansive mapping with a fixed point is quasi-nonexpansive. However the converse need not to be true.

Definition 2.4. [20]. *The mapping $\xi : \mathcal{C} \to \mathcal{C}$ with $F(\xi) \neq \emptyset$ satisfies Condition (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that $\|\vartheta - \xi(\vartheta)\| \geq f(d(\vartheta, F(\xi)))$ for all $\vartheta \in \mathcal{C}$, where $d(\vartheta, F(\xi)) = \inf\{\|\vartheta - y\| : \nu \in F(\xi)\}$.*

Let $\mathcal{C}$ be a convex subset of a Banach space $\mathcal{B}$ and $\xi : \mathcal{C} \to \mathcal{C}$ a monotone nonexpansive mapping. The following iteration process is known as the Krasnosel'skiĭ-Mann iteration process (see [14]):

$$\begin{cases} \vartheta_1 \in \mathcal{C} \\ \vartheta_{n+1} = \alpha_n \vartheta_n + (1 - \alpha_n)\xi(\vartheta_n) \end{cases} \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[a,b]$ with $a, b \in (0,1)$.

Lemma 2.5. [5]. *Let $\mathcal{C}$ be a nonempty bounded closed convex subset of an ordered Banach space $(\mathcal{B}, \|\cdot\|, \preceq)$ and $\xi : \mathcal{C} \to \mathcal{C}$ a monotone nonexpansive mapping. Suppose that $\{\vartheta_n\}$ is a sequence defined by* (1) *and $\vartheta_1 \preceq \xi(\vartheta_1)$. Then $\lim_{n\to\infty} \|\vartheta_n - \xi(\vartheta_n)\| = 0$.*

Lemma 2.6. [5]. *Let $\mathcal{C}$ be a nonempty closed convex subset of an ordered convex space $(\mathcal{B}, \|\cdot\|, \preceq)$ and $\xi : \mathcal{C} \to \mathcal{C}$ a monotone mapping. Suppose that $\{\vartheta_n\}$ is a sequence defined by* (1) *and $\vartheta_1 \preceq \xi(\vartheta_1)$. Then*

$$\vartheta_n \preceq \vartheta_{n+1} \preceq \xi(\vartheta_n)$$

for all $n \in \mathbb{N}$.

Lemma 2.7. *Let $\mathcal{C}$ a nonempty convex subset a Banach space $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping, define $S : \mathcal{C} \to \mathcal{C}$ as follows:*

$$S(\vartheta) = (1 - \lambda)\vartheta + \lambda\xi(\vartheta)$$

for all $\vartheta \in \mathcal{C}$ and $\lambda \in (0,1)$. Then $F(S) = F(\xi)$.

Definition 2.8. *Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{B}$.*

- *A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be compact if $\xi(\mathcal{C})$ has a compact closure.*
- *A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be weakly compact if $\xi(\mathcal{C})$ has a weakly compact closure.*

Lemma 2.9. [26] *For given $r > 0$. A Banach space $\mathcal{B}$ is uniformly convex if and only if there exists a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, such that*

$$\|\lambda\vartheta + (1 - \lambda)\nu\|^2 \leq \lambda\|\vartheta\|^2 + (1 - \lambda)\|\nu\|^2 - \lambda(1 - \lambda)\varphi(\|\vartheta - \nu\|) \tag{2}$$

for all $\lambda \in [0,1]$ and $\vartheta, \nu \in \mathcal{M}$ with $\|\vartheta\| \leq r$, $\|\nu\| \leq r$.

3. Main Results

Recently, Berinde [3] introduced the following class of nonlinear mappings.

Definition 3.1. *Let $(\mathcal{B}, \|.\|)$ be a Banach space. A mapping $\xi : \mathcal{B} \to \mathcal{B}$ is said to be b-enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that for all $\vartheta, \nu \in \mathcal{B}$*

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \leq (b+1)\|\vartheta - \nu\|. \tag{3}$$

It is shown that every nonexpansive mapping ξ is a 0-enriched mapping. The classes of b-enriched nonexpansive mappings and that of quasi-nonexpansive mappings are independent in nature. The following two examples illustrate this fact.

Example 3.2. *[3]. Let $\mathcal{C} = \left[\frac{1}{2}, 2\right] \subset \mathbb{R}$ and $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping defined as $\xi(\vartheta) = \frac{1}{\vartheta}$. Then $F(\xi) = \{1\}$ and ξ is a $\frac{3}{2}$-enriched nonexpansive mapping. On the other hand at $\vartheta = \frac{1}{2}$*

$$|\xi(\vartheta) - 1| = |2 - 1| = 1 > \frac{1}{2} = |\vartheta - 1|.$$

Thus ξ is not a quasi-nonexpansive mapping.

Example 3.3. *Let $\mathcal{C} = [0, 4] \subset \mathbb{R}$ and $\xi : \mathcal{C} \to \mathcal{C}$ be a mapping defined as*

$$\xi(\vartheta) = \begin{cases} 0, & \text{if } \vartheta \neq 4 \\ 3, & \text{if } \vartheta = 4. \end{cases}$$

Then $F(\xi) = \{0\}$ and ξ is a quasi-nonexpansive mapping. On the other hand at $\vartheta = 3$ and $\nu = 4$, ξ is not a b-enriched nonexpansive mapping for any $b \in [0, \infty)$.

The following useful definition is due to [6]:

Definition 3.4. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be monotone if*

$$\vartheta \preceq \nu \text{ implies } \xi(\vartheta) \preceq \xi(\nu),$$

where $\vartheta, \nu \in \mathcal{C}$.

Now, we extend Definition 3.1 in the setting of partially ordered Banach spaces as follows:

Definition 3.5. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{B}$. A mapping $\xi : \mathcal{C} \to \mathcal{C}$ is said to be monotone b-enriched nonexpansive mapping if ξ is monotone and there exists $b \in [0, \infty)$ such that*

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \leq (b+1)\|\vartheta - \nu\| \tag{4}$$

for all $\vartheta, \nu \in \mathcal{C}$ with ϑ and ν are comparable.

It can be seen that every monotone nonexpansive mapping ξ is a monotone 0-enriched mapping.

Theorem 3.6. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered uniformly convex Banach space and $\mathcal{C}$ a nonempty bounded closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a monotone b-enriched nonexpansive mapping. Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that ϑ_1 and $\xi(\vartheta_1)$ are comparable. Then $F(\xi) \neq \emptyset$.*

Moreover, for given $\lambda \in \left(0, \frac{1}{b+1}\right)$ the sequence $\{\vartheta_n\}$ defined by (Krasnosel'skiĭ iterative method)

$$\vartheta_{n+1} = (1 - \lambda)\vartheta_n + \lambda\xi(\vartheta_n) \tag{5}$$

converges weakly to a point in $F(\xi)$.

Proof. By the definition of monotone b-enriched nonexpansive mapping, we have

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \leq (b+1)\|\vartheta - \nu\| \tag{6}$$

for all $\vartheta \preceq \nu$. Take $\mu = \frac{1}{b+1} \in (0,1)$ and put $b = \frac{1-\mu}{\mu}$ in (6) then the above inequality is equivalent to

$$\|(1-\mu)(\vartheta - \nu) + \mu(\xi(\vartheta) - \xi(\nu))\| \leq \|\vartheta - \nu\|. \tag{7}$$

Define the mapping S as follows:

$$S(\vartheta) = (1-\mu)\vartheta + \mu\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}.$$

Since ξ is monotone, for all $\vartheta \preceq \nu$

$$S(\vartheta) = (1-\mu)\vartheta + \mu\xi(\vartheta) \preceq (1-\mu)\vartheta + \mu\xi(\nu) \preceq (1-\mu)\nu + \mu\xi(\nu) = S(\nu)$$

and S is monotone. Then from (7), we get

$$\|S(\vartheta) - S(\nu)\| \leq \|\vartheta - \nu\|$$

for all $\vartheta \preceq \nu$. Thus S is a monotone nonexpansive mapping. Since $\vartheta_1 \preceq \xi(\vartheta_1)$

$$\vartheta_1 = (1-\mu)\vartheta_1 + \mu\vartheta_1 \preceq (1-\mu)\vartheta_1 + \mu\xi(\vartheta_1) = S(\vartheta_1).$$

Thus all the assumptions of [6, Theorem 4.1] are satisfied and S has a fixed point in $\mathcal{C}$. From Lemma 2.7, $F(S) = F(\xi) \neq \emptyset$.

Next, for given $\vartheta_1 \in \mathcal{C}$ and any $\lambda \in (0,1)$, consider the sequence

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda S(\vartheta_n). \tag{8}$$

From Lemma 2.6 (with $\lambda = \alpha_n$ for all $n \in \mathbb{N}$)

$$\vartheta_n \preceq \vartheta_{n+1} \preceq S(\vartheta_n)$$

for all $n \in \mathbb{N}$. Again from Lemma 2.5

$$\lim_{n\to\infty} \|\vartheta_n - S(\vartheta_n)\| = 0.$$

Therefore $\{\vartheta_n\}$ is an a.f.p.s. for a monotone nonexpansive mapping S and all the assumptions of [24, Theorem 1] are fulfilled. Hence $\{\vartheta_n\}$ converges weakly to a fixed point of S. But $F(S) = F(\xi)$ and

$$(1-\lambda)\vartheta + \lambda S(\vartheta) = (1-\lambda\mu)\vartheta + \lambda\mu\xi(\vartheta)$$

for all $\vartheta \in \mathcal{C}$. Since $\lambda \in (0,1)$ and $\mu = \frac{1}{b+1}$. This implies that $\lambda\mu \in \left(0, \frac{1}{b+1}\right)$. Therefore for any $\lambda \in \left(0, \frac{1}{b+1}\right)$, the sequence $\{\vartheta_n\}$ defined by (5) converges weakly to a point in $F(\xi)$. □

Theorem 3.7. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered uniformly convex Banach space and $\mathcal{C}$ a nonempty bounded closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a monotone b-enriched nonexpansive mapping. Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that ϑ_1 and $\xi(\vartheta_1)$ are comparable. Then $F(\xi) \neq \emptyset$.*

Moreover, the sequence $\{\vartheta_n\}$ defined by

$$\vartheta_{n+1} = \left(1 - \frac{1}{b+1}\right)\vartheta_n + \frac{1}{b+1}\xi(\vartheta_n)$$

converges weakly to a point in $F(\xi)$.

Proof. Following the same proof technique as in Theorem 3.6, we can define a mapping $S : \mathcal{C} \to \mathcal{C}$ as follows:

$$S(\vartheta) = \left(1 - \frac{1}{b+1}\right)\vartheta + \frac{1}{b+1}\xi(\vartheta) \text{ for all } \vartheta \in \mathcal{C}$$

and S is a monotone nonexpansive mapping with $\vartheta_1 \preceq S(\vartheta_1)$. Then all the assumptions of [24, Theorem 5] are satisfied, hence $\{S^n(\vartheta_1)\}$ converges weakly to a fixed point of S. But $F(S) = F(\xi)$ and

$$S^n(\vartheta_1) = \left(1 - \frac{1}{b+1}\right)\vartheta_n + \frac{1}{b+1}\xi(\vartheta_n)$$

for all $n \in \mathbb{N}$. This completes the proof. □

Remark 3.8. *In Theorem 3.7, we extend the value of λ to $\frac{1}{b+1}$. In [4, Theorem 3.2], the value of λ lies in $\left(0, \frac{1}{b+1}\right)$.*

Theorem 3.9. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered Banach space having the weak-Opial property and $\mathcal{C}$ a nonempty weakly compact convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a monotone b-enriched nonexpansive mapping. Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that ϑ_1 and $\xi(\vartheta_1)$ are comparable. Then $F(\xi) \neq \emptyset$.*

Moreover, for given $\lambda \in \left(0, \frac{1}{b+1}\right)$ the sequence $\{\vartheta_n\}$ defined by (Krasnosel'skiĭ iterative method)

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n)$$

converges weakly to a point in $F(\xi)$.

Proof. Following largely the proof of Theorem 3.6, we can define a monotone nonexpansive mapping S with $\vartheta_1 \preceq S(\vartheta_1)$. Thus all the assumptions of [5, Theorem 3.3] are satisfied and it is guaranteed that S has at least one fixed point. From Lemma 2.7, $F(S) = F(\xi) \neq \emptyset$. For given $\vartheta_1 \in \mathcal{C}$ and for any $\lambda \in (0, 1)$, consider a sequence

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda S(\vartheta_n). \tag{9}$$

From [5, Theorem 3.3], $\{\vartheta_n\}$ converges weakly to a fixed point of S. But $F(S) = F(\xi)$, the rest of proof directly follows from Theorem 3.6. □

Theorem 3.10. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered Banach space having the monotone weak-Opial property and $\mathcal{C}$ a nonempty bounded closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a weakly compact monotone b-enriched nonexpansive mapping. Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that ϑ_1 and $\xi(\vartheta_1)$ are comparable. Then $F(\xi) \neq \emptyset$.*

Moreover, for given $\lambda \in \left(0, \frac{1}{b+1}\right)$ the sequence $\{\vartheta_n\}$ defined by (Krasnosel'skiĭ iterative method)

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n)$$

converges weakly to a point in $F(\xi)$.

Proof. From the proof of Theorem 3.6, we can define a monotone nonexpansive mapping S with $\vartheta_1 \preceq S(\vartheta_1)$. For given $\vartheta_1 \in \mathcal{C}$ and for any $\lambda \in (0, 1)$, consider a sequence

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda S(\vartheta_n). \tag{10}$$

From Lemma 2.6 (with $\lambda = \alpha_n$ for all $n \in \mathbb{N}$)

$$\vartheta_n \preceq \vartheta_{n+1} \preceq S(\vartheta_n)$$

for all $n \in \mathbb{N}$. Again from Lemma 2.5

$$\lim_{n\to\infty} \|\vartheta_n - S(\vartheta_n)\| = 0$$

and

$$\lim_{n\to\infty} \|\vartheta_n - \xi(\vartheta_n)\| = 0. \tag{11}$$

Since the range of $\mathcal{C}$ under ξ is contained in a weakly compact set, there exists a subsequence $\{\xi(\vartheta_{n_j})\}$ of $\{\xi(\vartheta_n)\}$ converges weakly to $\vartheta^\dagger \in \mathcal{C}$. By (11), the subsequence $\{\vartheta_{n_j}\}$ converges weakly to $\vartheta^\dagger$. Since $\{\vartheta_n\}$ is monotone increasing, the sequences $\{\vartheta_n\}$ and $\{S(\vartheta_n)\}$ converge weakly to $\vartheta^\dagger$. Thus for each $n \in \mathbb{N}$, $\vartheta_n \preceq S(\vartheta_n) \preceq \vartheta^\dagger$. By the monotonicity of S, for each $n \in \mathbb{N}$, $S(\vartheta_n) \preceq S(\vartheta^\dagger)$. Suppose that $S(\vartheta^\dagger) \neq \vartheta^\dagger$, by the monotone weak-Opial property, we get

$$\liminf_{n\to\infty} \|\vartheta_n - \vartheta^\dagger\| < \liminf_{n\to\infty} \|\vartheta_n - S(\vartheta^\dagger)\|. \tag{12}$$

By the triangle inequality and using the fact that S is monotone nonexpansive mapping,

$$\|\vartheta_n - S(\vartheta^\dagger)\| \leq \|\vartheta_n - S(\vartheta_n)\| + \|S(\vartheta_n) - S(\vartheta^\dagger)\| \leq \|\vartheta_n - S(\vartheta_n)\| + \|\vartheta_n - \vartheta^\dagger\|$$

and

$$\liminf_{n\to\infty} \|\vartheta_n - S(\vartheta^\dagger)\| \leq \liminf_{n\to\infty} \|\vartheta_n - \vartheta^\dagger\|$$

a contradiction from (12). Thus $S(\vartheta^\dagger) = \vartheta^\dagger$, and the rest of proof directly follows from Theorem 3.6. □

Theorem 3.11. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered uniformly convex Banach space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a monotone b-enriched nonexpansive mapping and ξ satisfies Condition (I). Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that $\vartheta_1 \preceq \xi(\vartheta_1)$, $F(\xi) \neq \emptyset$ and $\vartheta_1 \preceq \zeta$ for all $\zeta \in F(\xi)$. For given $\lambda \in \left(0, \frac{1}{b+1}\right)$ the sequence $\{\vartheta_n\}$ defined by (Krasnosel'skiĭ iterative method)*

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n)$$

converges strongly to a point in $F(\xi)$.

Proof. Following largely the proof of Theorem 3.6, we can define a monotone nonexpansive mapping S with $\vartheta_1 \preceq S(\vartheta_1)$. Let $\lambda \in (0,1)$ and define

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda S(\vartheta_n). \tag{13}$$

Since $\vartheta_1 \preceq \zeta$ for all $\zeta \in F(\xi) = F(S)$ and S is monotone mapping, $S(\vartheta_1) \preceq S(\zeta) = \zeta$ and

$$\vartheta_2 = (1-\lambda)\vartheta_1 + \lambda S(\vartheta_1) \preceq (1-\lambda)\vartheta_1 + \lambda\zeta \preceq (1-\lambda)\zeta + \lambda\zeta = \zeta$$

similarly, it can be seen that $\vartheta_n \preceq \zeta$ for all $\zeta \in F(S)$ and $n \in \mathbb{N}$. □

Now, we show that $\lim_{n\to\infty} d(\vartheta_n, F(S)) = 0$. For any $\zeta \in F(S)$,

$$\|S(\vartheta_n) - \zeta\| \leq \|\vartheta_n - \zeta\| \text{ for all } n \geq 1. \tag{14}$$

Thus

$$\|\vartheta_{n+1} - \zeta\| \leq (1-\lambda)\|\vartheta_n - \zeta\| + \lambda\|S(\vartheta_n) - \zeta\| \leq \|\vartheta_n - \zeta\|.$$

Hence the sequences $\{\|\vartheta_n - \zeta\|\}$ and $\{d(\vartheta_n, F(S))\}$ are monotone nonincreasing and $\lim_{n\to\infty} \|\vartheta_n - \zeta\|$, $\lim_{n\to\infty} d(\vartheta_n, F(S))$ exist. Again

$$\begin{aligned}
\|\vartheta_{n+1} - \zeta\|^2 &= \|(1-\lambda)(\vartheta_n - \zeta) + \lambda(S(\vartheta_n) - \zeta)\|^2 \\
&\leq (1-\lambda)\|\vartheta_n - \zeta\|^2 + \lambda\|S(\vartheta_n) - \zeta)\|^2 - \lambda(1-\lambda)\varphi(\|\vartheta_n - S(\vartheta_n)\|) \\
&\leq \|\vartheta_n - \zeta\|^2 - \lambda(1-\lambda)\varphi(\|\vartheta_n - S(\vartheta_n)\|).
\end{aligned}$$

Thus

$$\lambda(1-\lambda)\varphi(\|\vartheta_n - S(\vartheta_n)\|) \leq \|\vartheta_{n+1} - \zeta\|^2 - \|\vartheta_n - \zeta\|^2 \to 0 \text{ as } n \to \infty$$

and

$$\|\vartheta_n - S(\vartheta_n)\| \to 0 \text{ as } n \to \infty. \tag{15}$$

Since $S(\vartheta) = (1-\mu)\vartheta + \mu\xi(\vartheta)$ for all $\vartheta \in \mathcal{C}$,

$$\vartheta - S(\vartheta) = \mu(\vartheta - \xi(\vartheta)) \text{ for all } \vartheta \in \mathcal{C}. \tag{16}$$

Since ξ satisfies Condition (I), and (16), we obtain

$$\frac{\|\vartheta_n - S(\vartheta_n)\|}{\mu} = \|\vartheta_n - \xi(\vartheta_n)\| \geq f(d(\vartheta_n, F(\xi))) = f(d(\vartheta_n, F(S))).$$

From (15), $\lim_{n\to\infty} f(d(\vartheta_n, F(S))) = 0$ and

$$\lim_{n\to\infty} d(\vartheta_n, F(S)) = 0. \tag{17}$$

Now, it can be seen that the sequence $\{\vartheta_n\}$ is Cauchy. For the sake of completeness we include the argument. For given $\varepsilon > 0$, in view of (17), there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d(\vartheta_n, F(S)) < \frac{\varepsilon}{4}.$$

In particular,

$$\inf\{\|\vartheta_{n_0} - \zeta\| : \zeta \in F(S)\} < \frac{\varepsilon}{4},$$

and there exists $\zeta \in F(S)$ such that

$$\|\vartheta_{n_0} - \zeta\| < \frac{\varepsilon}{2}.$$

Therefore, for all $m, n \geq n_0$,

$$\|\vartheta_{n+m} - \vartheta_n\| \leq \|\vartheta_{n+m} - \zeta\| + \|\zeta - \vartheta_n\| \leq \|\vartheta_n - \zeta\| < 2\frac{\varepsilon}{2} = \varepsilon,$$

and the sequence $\{\vartheta_n\}$ is Cauchy. Since $\mathcal{C}$ is a closed subset of $\mathcal{B}$, so $\{\vartheta_n\}$ converges to a point $\vartheta^\dagger \in \mathcal{C}$ and

$$\vartheta_n \preceq \vartheta^\dagger \text{ for all } n \in \mathbb{N}.$$

$$\begin{aligned}
\|\vartheta^\dagger - S(\vartheta^\dagger)\| &\leq \|\vartheta^\dagger - \vartheta_n\| + \|\vartheta_n - S(\vartheta_n)\| + \|S(\vartheta_n) - S(\vartheta^\dagger)\| \\
&\leq 2\|\vartheta^\dagger - \vartheta_n\| + \|\vartheta_n - S(\vartheta_n)\|
\end{aligned}$$

from (15), $\vartheta^\dagger = S(\vartheta^\dagger)$. Hence, the sequence $\{\vartheta_n\}$ converges strongly to a point in $F(\xi)$.

Theorem 3.12. *Let $(\mathcal{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathcal{C}$ a nonempty bounded closed convex subset of $\mathcal{B}$. Let $\xi : \mathcal{C} \to \mathcal{C}$ be a compact monotone b-enriched nonexpansive mapping. Suppose that there exists a point ϑ_1 in $\mathcal{C}$ such that $\vartheta_1 \preceq \xi(\vartheta_1)$. For given $\lambda \in \left(0, \frac{1}{b+1}\right)$ the sequence $\{\vartheta_n\}$ defined by (Krasnosel'skiĭ iterative method)*

$$\vartheta_{n+1} = (1-\lambda)\vartheta_n + \lambda\xi(\vartheta_n)$$

converges strongly to a point in $F(\xi)$.

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