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Research Article

Some results on Kenmotsu statistical manifolds

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Abstract

In this paper, we first investigate the Kenmotsu statistical structures built on a Kenmotsu space form and determine some special Kenmotsu statistical structures under two curvature conditions. Secondly, we show that if the holomorphic sectional curvature of the hypersurface orthogonal to the structure vector in a Kenmotsu statistical manifold is constant, then the $\phi-$sectional curvature of the ambient Kenmotsu statistical manifold must be constant -1, and the constant holomorphic sectional curvature of the hypersurface is 0. In addition, some non-trivial examples are given to illustrate the results of this paper.

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1. Introduction

In 1969, S. Tanno showed that the maximal dimension of the automorphism group of $(2m+1)-$dimensional connected almost contact Riemannian manifolds is $(m+1)^3$. When the maximum is attained, the manifolds are classified in three classes[18]. One of the three classes in Tanno's classification is given by the warped product of a Kähler manifold with the real line. In 1972, K. Kenmotsu investigated the properties of this warped product and characterized it by tensor equations[9], nowadays called Kenmotsu geometry. This is a branch of differential geometry with many applications in geometrical optics, thermodynamics and geometric quantization[13, 14]. In [9], K. Kenmotsu proved that the sectional curvature of the Kenmotsu manifold of constant $\phi-$sectional curvature is constant -1. We state it as follows:

Theorem 1.1. [9] Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold and ∇^0 be the Levi-Civita connection of g. Denote the curvature tensor field of ∇^0 by R^0. If M is of constant $\phi-$sectional curvature c, that is,

$$\begin{aligned} R^0(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{1.1}$$

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for any $X, Y, Z \in C^\infty(TM)$, then $c = -1$. Furthermore, M is of constant sectional curvature -1 as well.

The notion of statistical structure was initially introduced from the treatment of statistical inference problems in information geometry by S. Amari in 1985[1]. From then on, the geometry of statistical manifolds has developed in close relations with affine differential geometry[11] and Hessian geometry[15]. By definition, a Riemannian structure is a trivial statistical structure with the difference tensor field $K = 0$. Recently, some of the classical Riemann manifolds have been generalized to the corresponding statistical manifolds by endowing them with some suitable statistical structures. For example, in [4], a holomorphic statistical manifold was obtained by endowing a Kähler manifold with a holomorphic statistical structure; in [7], a Sasakian statistical manifold was obtained by endowing a Sasakian manifold with a Sasakian statistical structure; and in [6], a Kenmotsu statistical manifold was obtained by endowing a Kenmotsu manifold with a Kenmotsu statistical structure. Moreover, I. K. Erken focused on almost cosymplectic statistical manifolds in [3], G. E. Vîlcu investigated statistical manifolds endowed with almost product structures in [20].

Since statistical structures can be considered as a generalization of the Riemannian structures, it is natural to consider whether some of the classical results in Riemannian geometry still hold in the geometry of statistical manifolds or not. For example, H. Furuhata[6] generalized Theorem 1.1 to Kenmotsu statistical manifolds and proved the following theorem:

Theorem 1.2. [6] Let $(M, \phi, \xi, \eta, g, \nabla)$ be a Kenmotsu statistical manifold and S be the statistical curvature tensor field of M. If M is of constant $\phi-$sectional curvature c, that is,

$$\begin{aligned} S(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned} \tag{1.2}$$

for any $X, Y, Z \in C^\infty(TM)$, then $c = -1$.

Note that several curvature properties of statistical submanifolds in Kenmotsu statistical manifolds of constant $\phi-$sectional curvature were recently obtained by Y. J. Suh et al. [17] and S. Decu et al. [2].

In addition, H. Furuhata found a special Kenmotsu statistical structure on the odd dimensional hyperbolic space H^{2n+1} of constant sectional curvature -1 such that the $\phi-$sectional curvature with respect to the statistical curvature tensor field S is -1 as well.

Example 1.3. [6] (Hyperbolic space H^{2n+1}) Set $H^{2n+1} = \{(x^1, \dots, x^n, y^1, \dots, y^n, z) \in \mathbb{R}^{2n+1} \mid z > 0\}$.
(i) H^{2n+1} can be endowed with the classical Kenmotsu structure $(\phi, \xi, \eta, \tilde{g}, \tilde{\nabla}^0)$ as follows:

$$\tilde{g} = \frac{1}{z^2}\left\{(dx^1)^2 + \cdots + (dx^n)^2 + (dy^1)^2 + \cdots + (dy^n)^2 + (dz)^2\right\}, \tag{1.3}$$

$$\xi = -z\frac{\partial}{\partial z}, \quad \phi\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad \phi\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \quad \phi\frac{\partial}{\partial z} = 0. \tag{1.4}$$

It can be verified that H^{2n+1} is of constant sectional curvature -1 with respect to the above Riemannian metric. Certainly, H^{2n+1} is of constant $\phi-$sectional curvature -1 as well.
(ii) Set $\tilde{K}(X,Y) = \lambda\eta(X)\eta(Y)\xi$, where $X, Y \in C^\infty(TH^{2n+1})$, $\lambda \in C^\infty(H^{2n+1}, \mathbb{R})$. Then $(\tilde{\nabla} = \tilde{\nabla}^0 + \tilde{K}, \tilde{g})$ is a Kenmotsu statistical structure on H^{2n+1}, and its $\phi-$sectional curvature with respect to the statistical curvature tensor field $\tilde{S}$ is also the constant -1.

Remark 1.4. In fact, the above Kenmotsu statistical structure can be built on any Kenmotsu space form, and the $\phi-$sectional curvature with respect to the statistical curvature tensor field is constant -1 as well.

In view of the above example, the first question we consider in this paper is whether the Kenmotsu statistical structure of constant $\phi-$sectional curvature built on a Kenmotsu space form is unique. We give an affirmative answer to this question in Theorem 3.2. Then we show that under a stronger curvature condition, i.e., when the $\phi-$curvature is constant, the Kenmotsu statistical structure built on a Kenmotsu space form must be trivial (see Theorem 3.6). Moreover, we give an example to show that Theorem 3.2 and Theorem 3.6 do not hold for Kenmotsu statistical structures built on a non-Kenmotsu space form.

In 2009, H. Furuhata showed in [6] that a hypersurface orthogonal to the structure vector in Kenmotsu statistical manifold has a natural holomorphic statistical structure, and the hypersurface must be totally umbilical. Furthermore, if the $\phi-$sectional curvature of the ambient Kenmotsu statistical manifold is constant, then the holomorphic sectional curvature of the hypersurface is constant 0. Inspired by H. Furuhata's result, we consider the converse problem and prove that if the holomorphic sectional curvature of the hypersurface is constant, then the $\phi-$sectional curvature of the ambient Kenmotsu statistical manifold must be constant -1, and the constant holomorphic sectional curvature of the hypersurface is 0 (see Theorem 4.3). In addition, we show some examples to illustrate this result.

2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇^0 be the Levi-Civita connection of g on M. Throughout this paper, we denote the set of all smooth tangent vector fields on M by $C^\infty(TM)$ and the set of all smooth normal vector fields on M by $C^\infty(T^\perp M)$. Besides, $C^\infty(M, \mathbb{R})$ denotes the set of all smooth functions on M.

2.1. Statistical manifold

Definition 2.1. [5] Let ∇ be an affine connection on a Riemannian manifold (M, g). The affine connection ∇^* is called the dual connection of ∇ with respect to g if

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y) \tag{2.1}$$

for any $X, Y, Z \in C^\infty(TM)$.

Obviously, $(\nabla^*)^* = \nabla$. Moreover, if ∇ and ∇^* are both torsion free, then[11]

$$\nabla + \nabla^* = 2\nabla^0, \tag{2.2}$$

where ∇^0 is the Levi-Civita connection of g on M.

Definition 2.2. [11] Let (M, g) be a Riemannian manifold and ∇ be an affine connection on M. The pair (∇, g) is called a statistical structure or a Codazzi structure, if ∇ is torsion free and the Codazzi equation

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$$

holds for any $X, Y, Z \in C^\infty(TM)$. In this case, (M, ∇, g) is said to be a statistical manifold or a Codazzi manifold.

By definition, a Riemannian structure (∇^0, g) is a special statistical structure, which is called a Riemannian statistical structure or a trivial statistical structure[4]. In fact, the Levi-Civita connection ∇^0 is self-dual with respect to the Riemannian metric g. Besides, if (∇, g) is a statistical structure on M, so is (∇^*, g).

Proposition 2.3. [7] Let (M, ∇, g) be a statistical manifold and ∇^0 be the Levi-Civita connection of g on M. For any $X, Y, Z \in C^\infty(TM)$, the tensor field K of type $(1, 2)$ defined by $K := \nabla - \nabla^0$ satisfies:

$$K_X Y = K_Y X, \quad g(K_X Y, Z) = g(K_X Z, Y). \tag{2.3}$$

Conversely, if a $(1, 2)-$tensor field K on M satisfies (2.3), then $(M, \nabla^0 + K, g)$ is a statistical manifold.

Remark 2.4. $K := \nabla - \nabla^0$ is called the difference tensor field of the statistical manifold. For simplicity, we also write $K_X Y$ by $K(X, Y)$. The Riemannian structure, as a trivial statistical structure, has the difference tensor field $K = 0$.

Definition 2.5. [7] Let (M, ∇, g) be a statistical manifold and ∇^* be the dual connection of ∇ with respect to g. Denote the curvature tensor field of ∇(resp. ∇^*) by R(resp. R^*), i.e., for any $X, Y, Z \in C^\infty(TM)$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z.$$

Define

$$S(X, Y)Z = \frac{1}{2}\{R(X, Y)Z + R^*(X, Y)Z\}, \tag{2.4}$$

and call S the statistical curvature tensor field of (M, ∇, g).

Obviously, the statistical curvature tensor of the Levi-Civita connection ∇^0 is the classical Riemann curvature tensor field R^0. Furthermore, one can verify that the statistical curvature tensor field S satisfies[5]:

$$g(S(X, Y)Z, W) + g(S(Y, X)Z, W) = 0, \tag{2.5}$$

$$g(S(X, Y)Z, W) + g(S(X, Y)W, Z) = 0, \tag{2.6}$$

$$g(S(X, Y)Z, W) - g(S(Z, W)X, Y) = 0, \tag{2.7}$$

$$S(X, Y)Z + S(Y, Z)X + S(Z, X)Y = 0. \tag{2.8}$$

Remark 2.6. [5] R does not have enough symmetries like the statistical curvature tensor field S. In fact, for any $X, Y, Z, W \in C^\infty(TM)$,

$$g(R(X, Y)Z, W) = -g(R^*(X, Y)W, Z). \tag{2.9}$$

Proposition 2.7. [4] Let $(M, \nabla = \nabla^0 + K, g)$ be a statistical manifold, denote the curvature tensor field of ∇(resp. ∇^0) by R(resp. R^0). Then the following formula holds:

$$R(X, Y)Z = R^0(X, Y)Z + (\nabla^0_X K)(Y, Z) - (\nabla^0_Y K)(Z, X) + [K_X, K_Y]Z, \tag{2.10}$$

where $X, Y, Z \in C^\infty(TM)$, $[K_X, K_Y] = K_X K_Y - K_Y K_X$.

Proposition 2.8. [7] Let $(M, \nabla = \nabla^0 + K, g)$ be a statistical manifold, and S be the statistical curvature tensor field. Denote the curvature tensor field of ∇^0 by R^0. Then the following formula holds:

$$S(X, Y)Z = R^0(X, Y)Z + [K_X, K_Y]Z, \tag{2.11}$$

where $X, Y, Z \in C^\infty(TM)$.

In 1990, T. Kurose[10] defined the statistical manifold of constant curvature in terms of the curvature tensor field R. In 2016, Furuhata[5] introduced the concept of the statistical manifold of constant sectional curvature in terms of the statistical curvature tensor field S.

Definition 2.9. [10] A statistical manifold (M, ∇, g) is said to be of constant curvature $c \in \mathbb{R}$ if

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} \tag{2.12}$$

for any $X, Y, Z \in C^\infty(TM)$, where R is the curvature tensor field of ∇.

Definition 2.10. [5] A statistical manifold (M, ∇, g) is said to be of constant sectional curvature $c \in \mathbb{R}$ if

$$S(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} \tag{2.13}$$

for any $X, Y, Z \in C^\infty(TM)$, where S is the statistical curvature tensor field on M.

According to (2.9), if (M, ∇, g) is a statistical manifold of constant curvature c, then (M, ∇^*, g) is of constant curvature c as well. Further, by (2.4), (M, ∇, g) is of constant sectional curvature c.

2.2. Kenmotsu statistical manifold

We first introduce the knowledge of Kenmotsu manifold.

Definition 2.11. [9] Let M be an odd dimensional Riemainian manifold, g be the Riemannian metric on M and ϕ, ξ, η respectively represent a $(1, 1)-$tensor field, a vector field and a $1-$form on M. If the following equations hold for any $X, Y \in C^\infty(TM)$:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi), \tag{2.14}$$

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{2.15}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.16}$$

then the quadruple (ϕ, ξ, η, g) is called an almost contact metric structure, and (M, ϕ, ξ, η, g) is called an almost contact metric manifold.

Definition 2.12. [6] Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, and ∇^0 be the Levi-Civita connection of g on M. For any $X, Y \in C^\infty(TM)$, if

$$(\nabla^0_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.17}$$

then $(\phi, \xi, \eta, g, \nabla^0)$ is called a Kenmotsu structure, $(M, \phi, \xi, \eta, g, \nabla^0)$ is called a Kenmotsu manifold, and ξ is called the structure vector field.

Remark 2.13. [6] If $(M, \phi, \xi, \eta, g, \nabla^0)$ is a Kenmotsu manifold, then for any $X \in C^\infty(TM)$, we have

$$\nabla^0_X \xi = X - \eta(X)\xi. \tag{2.18}$$

Definition 2.14. [9] Let $(M, \phi, \xi, \eta, g, \nabla^0)$ be a Kenmotsu manifold, and R^0 be the curvature tensor field of ∇^0. Then $(M, \phi, \xi, \eta, g, \nabla^0)$ is said to be of constant $\phi-$sectional curvature $c \in \mathbb{R}$ if

$$\begin{aligned} R^0(X, Y)Z = &\frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \end{aligned} \tag{2.19}$$

for any $X, Y, Z \in C^\infty(TM)$.

A Kenmotsu manifold of constant $\phi-$sectional curvature is usually called a Kenmotsu space form.

Proposition 2.15. [8] Let $(M, \phi, \xi, \eta, g, \nabla^0)$ be a Kenmotsu manifold, and R^0 be the curvature tensor field of ∇^0. Then the following formulas hold:

$$R^0(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.20}$$

$$R^0(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.21}$$

$$R^0(\xi, X)\xi = X - \eta(X)\xi, \tag{2.22}$$

where $X, Y \in C^\infty(TM)$.

H. Furuhata[6] introduced the notion of Kenmotsu statistical manifold by endowing a Kenmotsu manifold with a suitable statistical structure.

Definition 2.16. [6] Let $(M, \phi, \xi, \eta, g, \nabla^0)$ be a Kenmotsu manifold, $(\nabla = \nabla^0 + K, g)$ be a statistical structure on M. Then $(M, \phi, \xi, \eta, g, \nabla)$ is called a Kenmotsu statistical manifold if

$$K(X, \phi Y) + \phi K(X, Y) = 0 \tag{2.23}$$

for any $X, Y \in C^\infty(TM)$.

Definition 2.17. [6] Let $(M, \phi, \xi, \eta, g, \nabla)$ be a Kenmotsu statistical manifold, S be the statistical curvature tensor field. Then $(M, \phi, \xi, \eta, g, \nabla)$ is said to be of constant $\phi-$sectional curvature $c \in \mathbb{R}$ if

$$\begin{aligned} S(X, Y)Z = &\frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \end{aligned} \tag{2.24}$$

for any $X, Y, Z \in C^\infty(TM)$.

2.3. Holomorphic statistical manifold

We first review the notion of Kähler manifold.

Definition 2.18. [22] Let (M, g) be an even dimensional Riemannian manifold, ∇^0 be the Levi-Civita connection of g, and J be a $(1, 1)-$tensor field on M. If

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad \nabla^0_X JY = J\nabla^0_X Y \tag{2.25}$$

for any $X, Y \in C^\infty(TM)$, then (M, J, g, ∇^0) is called a Kähler manifold.

H. Furuhata[4] introduced the notion of holomorphic statistical manifold by endowing a Kähler manifold with a suitable statistical structure.

Definition 2.19. [4] Let (M, J, g, ∇^0) be a Kähler manifold and (∇, g) be a statistical structure on M. Then (M, J, g, ∇) is called a holomorphic statistical manifold if the difference tensor field K satisfies

$$K(X, JY) + JK(X, Y) = 0 \tag{2.26}$$

for any $X, Y \in C^\infty(TM)$.

Definition 2.20. [5] A holomorphic statistical manifold (M, J, g, ∇) is said to be of constant holomorphic sectional curvature $c \in \mathbb{R}$ if its statistical curvature tensor field satisfies

$$S(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\} \tag{2.27}$$

for any $X, Y, Z \in C^\infty(TM)$.

2.4. Statistical submanifold

Now we review some basics of statistical submanifolds.

Definition 2.21. [12] Let $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical manifold and $f : M \to \tilde{M}$ be an immersion. Denote the tangent mapping and the pullback mapping of f by f_* and f^*, respectively. Define g and ∇ on M by

$$g = f^*\tilde{g}, \quad g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z).$$

Then the pair (∇, g) is a statistical structure on M, which is called the induced statistical structure by f from $(\tilde{\nabla}, \tilde{g})$.

Definition 2.22. [12] Let (M, ∇, g) and $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be two statistical manifolds. An immersion $f : M \to \tilde{M}$ is called a statistical immersion if (∇, g) coincides with the induced statistical structure by f from $(\tilde{\nabla}, \tilde{g})$. Also, (M, ∇, g) is called a statistical submanifold of $(\tilde{M}, \tilde{\nabla}, \tilde{g})$.

Similar to the theory of Riemannian submanifolds, the statistical submanifolds also have the Gauss and Weingarten formulas[16]. Let (M, ∇, g) be a statistical submanifold of $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, then we have:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{2.28}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \tilde{\nabla}^*_X N = -A^*_N X + \nabla^{*\perp}_X N, \tag{2.29}$$

where $X, Y \in C^\infty(TM), N \in C^\infty(T^\perp M)$. In the above formulas, h and h^* are the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively; A and A^* are the shape operators with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively; $\nabla^\perp$ and $\nabla^{*\perp}$ are the normal connections with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. Besides we have the following[21]:

$$h(X, Y) = h(Y, X), \quad h^*(X, Y) = h^*(Y, X), \tag{2.30}$$

$$g(A_N X, Y) = \tilde{g}(h^*(X, Y), N), \quad g(A^*_N X, Y) = \tilde{g}(h(X, Y), N). \tag{2.31}$$

In addition, the statistical submanifolds also have the Gauss, Codazzi and Ricci equations.

Proposition 2.23. [5] Let (M, ∇, g) be a statistical submanifold of $(\tilde{M}, \tilde{\nabla}, \tilde{g})$. Denote the curvature tensor field of $\nabla^\perp$(resp. $\nabla^{*\perp}$) by $R^\perp$(resp. $R^{*\perp}$). Set $S^\perp = \frac{1}{2}\{R^\perp + R^{*\perp}\}$, then the following equations hold for any $X, Y, Z \in C^\infty(TM)$, and $N \in C^\infty(T^\perp M)$:

$$\begin{aligned} 2[\tilde{S}(X, Y)Z]^\top = 2S(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ + A^*_{h^*(X,Z)}Y - A^*_{h^*(Y,Z)}X, \end{aligned} \tag{2.32}$$

$$\begin{aligned} 2[\tilde{S}(X, Y)Z]^\perp =& (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) \\ &+ (\tilde{\nabla}^*_X h^*)(Y, Z) - (\tilde{\nabla}^*_Y h^*)(X, Z), \end{aligned} \tag{2.33}$$

$$\begin{aligned} 2[\tilde{S}(X, Y)N]^\top =& (\tilde{\nabla}_Y A)(N, X) - (\tilde{\nabla}_X A)(N, Y) \\ &+ (\tilde{\nabla}^*_Y A^*)(N, X) - (\tilde{\nabla}^*_X A^*)(N, Y), \end{aligned} \tag{2.34}$$

$$\begin{aligned} 2[\tilde{S}(X, Y)N]^\perp = 2S^\perp(X, Y)N + h(Y, A_N X) - h(X, A_N Y) \\ + h^*(Y, A^*_N X) - h^*(X, A^*_N Y), \end{aligned} \tag{2.35}$$

where $[\cdot]^\top$ and $[\cdot]^\perp$ are the tangent component and the normal component of the vector field " $\cdot$ ", respectively.

Remark 2.24. Equation (2.32) is the Gauss equation, (2.33), (2.34) are the Codazzi equations, (2.35) is the Ricci equation.

Remark 2.25. If M is a statistical hypersurface of $\tilde{M}$ and N is the unit normal vector field on M, then the Gauss equation can be written as

$$\begin{aligned}2[\tilde{S}(X,Y)Z]^{\top} = 2S(X,Y)Z + g(A^*X,Z)AY - g(A^*Y,Z)AX \\ + g(AX,Z)A^*Y - g(AY,Z)A^*X,\end{aligned} \tag{2.36}$$

where we respectively denote A_N and A^*_N by A and A^* for simplicity.

3. Statistical structures on Kenmotsu space forms

In this section, we investigate the Kenmotsu statistical structures built on a Kenmotsu space form and determine the Kenmotsu statistical structures under two curvature conditions. First we consider the Kenmotsu statistical structures of constant $\phi-$sectional curvature on a Kenmotsu space form. Before stating and proving the main theorems, we prove the following lemma first.

Lemma 3.1. *Let $(M,\phi,\xi,\eta,g,\nabla)$ be a Kenmotsu statistical manifold, and K be the difference tensor field on M. Then*

$$K(X,\xi) = \lambda\eta(X)\xi \tag{3.1}$$

holds for any $X \in C^\infty(TM)$, where $\lambda = g(K(\xi,\xi),\xi) \in C^\infty(M,\mathbb{R})$.

Proof. Put $Y = \xi$ in (2.23), then $\phi K(X,\xi) = 0$. Letting ϕ act on both sides of it, by (2.14), we have

$$K(X,\xi) = g(K(X,\xi),\xi)\xi. \tag{3.2}$$

In particular,

$$K(\xi,\xi) = \lambda\xi,$$

where $\lambda = g(K(\xi,\xi),\xi) \in C^\infty(M,\mathbb{R})$. Thus by (3.2) and (2.3), we have

$$K(X,\xi) = g(K(\xi,\xi),X)\xi = \lambda\eta(X)\xi.$$

□

Theorem 3.2. *Let $(M,\phi,\xi,\eta,g,\nabla^0)$ be a Kenmotsu space form, and $(\nabla = \nabla^0 + K, g)$ be a Kenmotsu statistical structure on M. If the $\phi-$sectional curvature of $(M,\phi,\xi,\eta,g,\nabla = \nabla^0 + K)$ is constant, then*

$$K(X,Y) = \lambda\eta(X)\eta(Y)\xi, \tag{3.3}$$

where $X,Y \in C^\infty(TM)$, $\lambda \in C^\infty(M,\mathbb{R})$.

Proof. Let $(\nabla = \nabla^0 + K, g)$ be a Kenmotsu statistical structure of constant $\phi-$sectional curvature on a Kenmotsu space form $(M,\phi,\xi,\eta,g,\nabla^0)$. We denote the statistical curvature tensor field of M by S, and the curvature tensor field of ∇^0 by R^0. By Theorem 1.1 and Theorem 1.2, we have

$$S(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.4}$$

$$R^0(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.5}$$

where $X,Y,Z \in C^\infty(TM)$. Thus by using (2.11), we have

$$[K_X,K_Y]Z = 0. \tag{3.6}$$

Taking the inner product of both sides of (3.6) with any tangent vector field W, one gets

$$g(K(X,K(Y,Z)),W) - g(K(Y,K(X,Z)),W) = 0. \tag{3.7}$$

From (2.3), (3.7) is equivalent to

$$g(K(X,W),K(Y,Z)) - g(K(X,Z),K(Y,W)) = 0. \tag{3.8}$$

By Lemma 3.1, for $U \in C^\infty(TM), U \perp \xi$, we have

$$K(U,\xi) = 0. \tag{3.9}$$

Taking $X = Z = U,\ Y = W = \phi U,\ U \perp \xi$ in (3.8) and applying (2.3) and (3.9), we obtain

$$K(U,U) = 0. \tag{3.10}$$

Further, taking $X = Z = U,\ Y = W = V,\ U \perp \xi,\ V \perp \xi$ in (3.8) and applying (3.10), we get

$$K(U,V) = 0. \tag{3.11}$$

Finally for any $X, Y \in C^\infty(TM)$, X, Y can be decomposed orthogonally as: $X = U + \eta(X)\xi,\ Y = V + \eta(Y)\xi$, thus from (3.9), (3.11) and Lemma 3.1, we get (3.3) immediately. □

Now we consider the curvature tensor field R of a Kenmotsu statistical manifold $(M, \phi, \xi, \eta, g, \nabla)$ and introduce the following concept.

Definition 3.3. Let $(M, \phi, \xi, \eta, g, \nabla)$ be a Kenmotsu statistical manifold, R be the curvature tensor field of ∇. $(M, \phi, \xi, \eta, g, \nabla)$ is said to be of constant $\phi-$curvature $c \in \mathbb{R}$ if

$$\begin{aligned} R(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{3.12}$$

where $X, Y, Z \in C^\infty(TM)$.

Remark 3.4. For other types of statistical manifolds, there are similar concepts. For example, T. Kurose[10] define the statistical manifold of constant curvature in terms of R (see Definition 2.9); H.Furuhata[4] define the holomorphic statistical manifold of constant holomorphic curvature in a similar way.

Lemma 3.5. *Let $(M, \phi, \xi, \eta, g, \nabla)$ be a Kenmotsu statistical manifold of constant $\phi-$curvature c, then $(M, \phi, \xi, \eta, g, \nabla)$ is a Kenmotsu statistic manifold of constant $\phi-$sectional curvature c as well.*

Proof. Denote the curvature tensor fields of ∇, ∇^* by R, R^* and the statistical curvature tensor field of M by S, respectively. Since M is of constant $\phi-$curvature c, we have

$$\begin{aligned} R(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{3.13}$$

where $X, Y, Z \in C^\infty(TM)$. Taking the inner product of both sides of (3.13) with any tangent vector field W and applying (2.9), one gets

$$\begin{aligned} &g(R^*(X,Y)W,Z) \\ =& \frac{c-3}{4}\{g(X,Z)(Y,W) - g(Y,Z)g(X,W)\} \\ &+ \frac{c+1}{4}\{\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W) \\ &+ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &+ g(\phi X,Z)g(\phi Y,W) - g(\phi Y,Z)g(\phi X,W) + 2g(\phi X,Y)g(\phi Z,W)\}, \end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned} R^*(X,Y)W =& \frac{c-3}{4}\{g(Y,W)X - g(X,W)Y\} + \frac{c+1}{4}\{\eta(X)\eta(W)Y \\ &- \eta(Y)\eta(W)X + g(X,W)\eta(Y)\xi - g(Y,W)\eta(X)\xi \\ &+ g(\phi Y,W)\phi X - g(\phi X,W)\phi Y - 2g(\phi X,Y)\phi W\}. \end{aligned} \tag{3.15}$$

Thus $(M, \phi, \xi, \eta, g, \nabla^*)$ is also of constant $\phi-$curvature c. Further by (2.4), M is of constant $\phi-$sectional curvature c as well. □

Now we prove that a Kenmotsu statistical structure of constant $\phi-$curvature built on a Kenmotsu space form must be trivial.

Theorem 3.6. *Let $(M, \phi, \xi, \eta, g, \nabla^0)$ be a Kenmotsu space form, and $(\nabla = \nabla^0 + K, g)$ be a Kenmotsu statistical structure on M. If the $\phi-$curvature of $(M, \phi, \xi, \eta, g, \nabla = \nabla^0 + K)$ is constant, then $K = 0$.*

Proof. Let $(M, \phi, \xi, \eta, g, \nabla^0)$ be a Kenmotsu space form, and $(\nabla = \nabla^0 + K, g)$ be a Kenmotsu statistical structure of constant $\phi-$curvature $c \in \mathbb{R}$ on M. Denote the curvature tensor fields of $\nabla, \nabla^*, \nabla^0$ by R, R^*, R^0, respectively, and denote the statistical curvature tensor field of M by S. According to Theorem 1.1,

$$R^0(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y\}. \tag{3.16}$$

Since M is of constant $\phi-$curvature c, by Lemma 3.5, M is of constant $\phi-$sectional curvature c as well. It follows from Theorem 1.2 that $c = -1$. By Theorem 3.2, we know that

$$K(X, Y) = \lambda\eta(X)\eta(Y)\xi. \tag{3.17}$$

On the other hand, substituting $c = -1$ into (3.12) we obtain

$$R(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y\}. \tag{3.18}$$

From (2.10), (3.16) and (3.18), one gets

$$(\nabla^0_X K)(Y, Z) - (\nabla^0_Y K)(Z, X) + [K_X, K_Y]Z = 0. \tag{3.19}$$

From (3.17), it is easy to see that $[K_X, K_Y]Z = 0$. Thus (3.19) is reduced to

$$(\nabla^0_X K)(Y, Z) - (\nabla^0_Y K)(Z, X) = 0. \tag{3.20}$$

Taking the inner product of the two sides of (3.20) with any tangent vector W, we obtain

$$\begin{aligned} &Xg(K(Y, Z), W) - g(K(Y, Z), \nabla^0_X W) - g(K(\nabla^0_X Y, Z), W) - g(K(Y, \nabla^0_X Z), W) \\ &- Yg(K(Z, X), W) + g(K(Z, X), \nabla^0_Y W) + g(K(\nabla^0_Y Z, X), W) + g(K(Z, \nabla^0_Y X), W) \\ &= 0. \end{aligned} \tag{3.21}$$

Taking $X = Z = U,\ Y = W = \xi,\ U \perp \xi$ in (3.21) and applying (2.3) and (3.17), we have

$$g(K(\xi, \xi), \nabla^0_U U) = 0. \tag{3.22}$$

From (3.17), the above equation is just reduced to $\lambda g(\xi, \nabla^0_U U) = 0$, i.e.,

$$\lambda(Ug(\xi, U) - g(\nabla^0_U \xi, U)) = 0. \tag{3.23}$$

Substituting (2.18) into (3.23) and noting that $U \perp \xi$, we get

$$\lambda g(U, U) = 0, \tag{3.24}$$

which implies that $\lambda = 0$. This together with (3.17) gives that $K = 0$. □

Remark 3.7. The following example shows that Theorem 3.2 and Theorem 3.6 do not hold for Kenmotsu statistical structures built on a non-Kenmotsu space form.

Example 3.8. Let $\tilde{M} = \{(x, y, z) \in \mathbb{R}^3 | x > 0, y > 0\}$, then we can define a Kenmotsu structure $(\phi, \xi, \eta, \tilde{g})$ on $\tilde{M}$ whose $\phi-$sectional curvature is not constant as follows:

$$\tilde{g} = xe^{2z}\left\{(dx)^2 + (dy)^2\right\} + (dz)^2,$$

$$\xi = \frac{\partial}{\partial z}, \quad \eta(X) = \tilde{g}(X, \xi),$$

$$\phi\frac{\partial}{\partial x}=\frac{\partial}{\partial y},\quad \phi\frac{\partial}{\partial y}=-\frac{\partial}{\partial x},\quad \phi\frac{\partial}{\partial z}=0.$$

Furthermore, define the affine connection $\tilde{\nabla}$ on $\tilde{M}$ as follows:

$$\begin{aligned}
&\tilde{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=\frac{2-\sqrt{2}}{4x}\frac{\partial}{\partial x}+\frac{1}{2\sqrt{2}x}\frac{\partial}{\partial y}-xe^{2z}\xi,\quad \tilde{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\frac{1}{2\sqrt{2}x}\frac{\partial}{\partial x}+\frac{2+\sqrt{2}}{4x}\frac{\partial}{\partial y},\\
&\tilde{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}=\frac{1}{2\sqrt{2}x}\frac{\partial}{\partial x}+\frac{2+\sqrt{2}}{4x}\frac{\partial}{\partial y},\quad \tilde{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{\sqrt{2}-2}{4x}\frac{\partial}{\partial x}-\frac{1}{2\sqrt{2}x}\frac{\partial}{\partial y}-xe^{2z}\xi,\\
&\tilde{\nabla}_{\frac{\partial}{\partial x}}\xi=\tilde{\nabla}_{\xi}\frac{\partial}{\partial x}=\frac{\partial}{\partial x},\quad \tilde{\nabla}_{\frac{\partial}{\partial y}}\xi=\tilde{\nabla}_{\xi}\frac{\partial}{\partial y}=\frac{\partial}{\partial y},\quad \tilde{\nabla}_{\xi}\xi=0.
\end{aligned}\tag{3.25}$$

Then $(\phi,\xi,\eta,\tilde{g},\tilde{\nabla})$ is the Kenmotsu statistical structure of constant $\phi-$curvature -1 and constant $\phi-$sectional curvature -1, but the difference tensor field is different from the one defined by (3.3).

Proof. Write $e_1=\frac{\partial}{\partial x},\ e_2=\frac{\partial}{\partial y}$ for simplicity. According to the definition of $\tilde{g}$,

$$\begin{aligned}
\tilde{g}(e_1,e_1)=\tilde{g}(e_2,e_2)=xe^{2z},\ \tilde{g}(\xi,\xi)=1,\\
\tilde{g}(e_1,e_2)=\tilde{g}(e_1,\xi)=\tilde{g}(e_2,\xi)=0.
\end{aligned}\tag{3.26}$$

Obviously, $(\phi,\xi,\eta,\tilde{g})$ satisfies (2.14)-(2.16), thus $(\tilde{M},\phi,\xi,\eta,\tilde{g})$ is an almost contact metric manifold.

Denote the Levi-Civita connection on $\tilde{M}$ by $\tilde{\nabla}^0$. By using Koszul's formula[19]:

$$2\tilde{g}(\tilde{\nabla}^0_XY,Z)=X\tilde{g}(Y,Z)+Y\tilde{g}(Z,X)-Z\tilde{g}(X,Y)-\tilde{g}(X,[Y,Z])+\tilde{g}(Y,[Z,X])+\tilde{g}(Z,[X,Y]),$$

we get:

$$\begin{aligned}
\tilde{\nabla}^0_{e_1}e_1=\frac{1}{2x}e_1-xe^{2z}\xi,\quad \tilde{\nabla}^0_{e_1}e_2=\frac{1}{2x}e_2,\\
\tilde{\nabla}^0_{e_2}e_1=\frac{1}{2x}e_2,\quad \tilde{\nabla}^0_{e_2}e_2=-\frac{1}{2x}e_1-xe^{2z}\xi,\\
\tilde{\nabla}^0_{e_1}\xi=\nabla^0_{\xi}e_1=e_1,\quad \tilde{\nabla}^0_{e_2}\xi=\tilde{\nabla}^0_{\xi}e_2=e_2,\quad \tilde{\nabla}^0_{\xi}\xi=0.
\end{aligned}\tag{3.27}$$

By using the above formulas and the definition of ϕ, one can verify that $(\phi,\xi,\eta,\tilde{g},\tilde{\nabla}^0)$ satisfies (2.17). Hence $(\tilde{M},\phi,\xi,\eta,\tilde{g},\tilde{\nabla}^0)$ is a Kenmotsu manifold. Denote the curvature tensor field of $\tilde{\nabla}^0$ by $\tilde{R}^0$. By (3.27), we calculate

$$\tilde{R}^0(e_1,e_2)e_1=\tilde{\nabla}^0_{e_1}\tilde{\nabla}^0_{e_2}e_1-\tilde{\nabla}^0_{e_2}\tilde{\nabla}^0_{e_1}e_1-\tilde{\nabla}^0_{[e_1,e_2]}e_1=(-\frac{1}{2x^2}+xe^{2z})e_2.$$

Comparing the above formula with (2.19), it can be seen that the $\phi-$sectional curvature of $(\tilde{M},\phi,\xi,\eta,\tilde{g},\tilde{\nabla}^0)$ is not constant.

Let $\tilde{K}=\tilde{\nabla}-\tilde{\nabla}^0$. By (3.25) and (3.27), we have

$$\begin{aligned}
\tilde{K}(e_1,e_1)=-\frac{1}{2\sqrt{2}x}e_1+\frac{1}{2\sqrt{2}x}e_2,\\
\tilde{K}(e_2,e_2)=\frac{1}{2\sqrt{2}x}e_1-\frac{1}{2\sqrt{2}x}e_2,\\
\tilde{K}(e_1,e_2)=\tilde{K}(e_2,e_1)=\frac{1}{2\sqrt{2}x}e_1+\frac{1}{2\sqrt{2}x}e_2,\\
\tilde{K}(e_1,\xi)=\tilde{K}(\xi,e_1)=\tilde{K}(e_2,\xi)=\tilde{K}(\xi,e_2)=\tilde{K}(\xi,\xi)=0.
\end{aligned}\tag{3.28}$$

It is easy to see that $\tilde{K}=\tilde{\nabla}-\tilde{\nabla}^0$ satisfies (2.3) and (2.23), so $(\tilde{\nabla}=\tilde{\nabla}^0+\tilde{K},\tilde{g})$ is a Kenmotsu statistical structure on $\tilde{M}$. Obviously, $\tilde{K}$ differs from the difference tensor field defined by (3.3).

Denote the curvature tensor field of $\tilde{\nabla}$ by $\tilde{R}$. By using (3.25), one gets

$$\begin{aligned}
&\tilde{R}(e_1, e_2)e_1 = xe^{2z}e_2, \quad \tilde{R}(e_1, e_2)e_2 = -xe^{2z}e_1, \quad \tilde{R}(e_1, e_2)\xi = 0,\\
&\tilde{R}(e_1, \xi)e_1 = xe^{2z}\xi, \quad \tilde{R}(e_1, \xi)e_2 = 0, \quad \tilde{R}(e_1, \xi)\xi = -e_1,\\
&\tilde{R}(e_2, \xi)e_1 = 0, \quad \tilde{R}(e_2, \xi)e_2 = xe^{2z}\xi, \quad \tilde{R}(e_2, \xi)\xi = -e_2,\\
&\tilde{R}(e_i, e_i)e_j = 0.
\end{aligned}$$

Combined these formulas with (3.26), $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)Z = -\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\}, \tag{3.29}$$

which implies $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla} = \tilde{\nabla}^0 + \tilde{K})$ is of constant $\phi-$curvature -1. By Lemma 3.5, $\tilde{M}$ is of constant $\phi-$sectional curvature -1 as well. □

4. Hypersurfaces in Kenmotsu statistical manifolds

In this section, we study hypersurfaces in Kenmotsu statistical manifolds. H. Furuhata showed in [6] that the hypersurface orthogonal to the structure vector field in a Kenmotsu statistical manifold has a natural holomorphic statistical structure, and the hypersurface must be totally umbilical.

Proposition 4.1. [6] Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ be a Kenmotsu statistical manifold, M be a hypersurface in $\tilde{M}$ orthogonal to ξ. Denote the induced metric of $\tilde{g}$ on M by g. Set

$$JU = \phi U, \tag{4.1}$$

$$\tilde{\nabla}_U V = \nabla_U V + \alpha(U, V)\xi, \tag{4.2}$$

$$\tilde{\nabla}_U \xi = -AU + \tau(U)\xi, \tag{4.3}$$

where $U, V \in C^\infty(TM)$, $\nabla_U V$ and $-AU$ are the tangent components of $\tilde{\nabla}_U V$ and $\tilde{\nabla}_U \xi$, respectively. Then (J, g, ∇) is a holomorphic statistical structure on M, and the following formulas hold:

$$\alpha(U, V) = -g(U, V), \quad AU = -U, \quad \tau(U)\xi = \nabla^\perp_U \xi.$$

Furthermore, if the $\phi-$sectional curvature of the ambient Kenmotsu statistical manifold is constant, then according to the Gauss equation and Theorem 1.2, we immediately have the following interesting result.

Theorem 4.2. [6] Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ be a Kenmotsu statistical manifold, M be a hypersurface in $\tilde{M}$ orthogonal to ξ. Let (J, g, ∇) be the holomorphic statistical structure on M as in Proposition 4.1. If $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ is of constant $\phi-$sectional curvature, then (M, J, g, ∇) is of constant holomorphic sectional curvature 0.

In this section we consider the converse problem and prove the following Theorem.

Theorem 4.3. *Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ be a $(2n+1)-$dimensional Kenmotsu statistical manifold, $n \geqslant 2$, M be a hypersurface in $\tilde{M}$ orthogonal to ξ. Let (J, g, ∇) be the holomorphic statistical structure on M as in Proposition 4.1. If (M, J, g, ∇) is of constant holomorphic sectional curvature, then $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ is of constant $\phi-$sectional curvature -1, and the constant holomorphic sectional curvature of (M, J, g, ∇) is 0.*

Proof. Let ∇^* and $\tilde{\nabla}^*$ be the dual connections of ∇ and $\tilde{\nabla}$, S and $\tilde{S}$ be the statistical curvature tensor fields of M and $\tilde{M}$, respectively. Set

$$\tilde{\nabla}^*_U \xi = -A^*U + \tau^*(U)\xi \tag{4.4}$$

for any $U \in C^\infty(TM)$, where $-A^*U$ and $\tau^*(U)\xi = \nabla^{*\perp}_U \xi$ are the tangent component and the normal component of $\tilde{\nabla}^*_U \xi$, respectively. We first show that

$$A^*U = -U, \quad \tau(U) + \tau^*(U) = 0. \tag{4.5}$$

In fact, for $U, V \in C^\infty(TM)$, we calculate

$$\tilde{g}(A^*U, V) = -\tilde{g}(\tilde{\nabla}^*_U\xi, V) = -U\tilde{g}(\xi, V) + \tilde{g}(\xi, \tilde{\nabla}_U V) = \tilde{g}(\xi, \alpha(U, V)\xi) = -g(U, V), \quad (4.6)$$

$$U\tilde{g}(\xi, \xi) = \tilde{g}(\tilde{\nabla}_U\xi, \xi) + \tilde{g}(\xi, \tilde{\nabla}^*_U\xi) = \tau(U) + \tau^*(U), \quad (4.7)$$

where we have used (4.2) in the third equality of (4.6), and we have used (4.3) in the second equality of (4.7). Then (4.6) and (4.7) yield (4.5) immediately.

Assume that the holomorphic sectional curvature of M is constant c. We will prove that the ϕ−sectional curvature of $\tilde{M}$ is constant $c-1$. By Definition 2.17, that is, for any $X, Y, Z \in C^\infty(T\tilde{M})$,

$$\begin{aligned}\tilde{S}(X, Y)Z =& \frac{c-4}{4}\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\} + \frac{c}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + \tilde{g}(X, Z)\eta(Y)\xi - \tilde{g}(Y, Z)\eta(X)\xi \\ & + \tilde{g}(\phi Y, Z)\phi X - \tilde{g}(\phi X, Z)\phi Y - 2\tilde{g}(\phi X, Y)\phi Z\}.\end{aligned} \quad (4.8)$$

In the following we prove (4.8) in four cases.

Case i. Assume that $X = U, Y = V \in C^\infty(TM), Z = \xi$. According to the Ricci equation and noting that $AU = -U, A^*U = -U$, we have

$$\begin{aligned}2[\tilde{S}(U, V)\xi]^\perp &= 2S^\perp(U, V)\xi + h(V, AU) - h(U, AV) + h^*(V, A^*U) - h^*(U, A^*V) \\ &= 2S^\perp(U, V)\xi.\end{aligned} \quad (4.9)$$

Note that

$$2S^\perp(U, V)\xi = R^\perp(U, V)\xi + R^{*\perp}(U, V)\xi. \quad (4.10)$$

We calculate

$$\begin{aligned}R^\perp(U, V)\xi &= \nabla^\perp_U\nabla^\perp_V\xi - \nabla^\perp_V\nabla^\perp_U\xi - \nabla^\perp_{[U,V]}\xi = \nabla^\perp_U\tau(V)\xi - \nabla^\perp_V\tau(U)\xi - \tau([U, V])\xi \\ &= U(\tau(V))\xi - V(\tau(U))\xi - \tau([U, V])\xi,\end{aligned}$$

where we have used $\nabla^\perp_U\xi = \tau(U)\xi$ in the second and the third equalities. In the same way, by using $\nabla^{*\perp}_U\xi = \tau^*(U)\xi$ and $\tau(U) + \tau^*(U) = 0$, we get

$$R^{*\perp}(U, V)\xi = -U(\tau(V))\xi + V(\tau(U))\xi + \tau([U, V])\xi.$$

Substituting the above two equations into (4.10) yields $S^\perp(U, V)\xi = 0$. Thus from (4.9), we obtain

$$[\tilde{S}(U, V)\xi]^\perp = 0. \quad (4.11)$$

Furthermore, according to the Codazzi equation and noting that $AU = -U, A^*U = -U$,

$$\begin{aligned}2[\tilde{S}(U, V)\xi]^\top =& (\nabla_V A)U - (\nabla_U A)V + (\nabla^*_V A^*)U - (\nabla^*_U A^*)V \\ & + \tau(V)(A^* - A)U - \tau(U)(A^* - A)V \\ =& \nabla_V AU - A(\nabla_V U) - \nabla_U AV + A(\nabla_U V) \\ & + \nabla^*_V A^*U - A^*(\nabla^*_V U) - \nabla^*_U A^*V + A^*(\nabla^*_U V) \\ =& 0.\end{aligned} \quad (4.12)$$

Combining (4.11) and (4.12), we have

$$\tilde{S}(U, V)\xi = 0. \quad (4.13)$$

On the other hand, taking $X = U, Y = V, Z = \xi$ in the right-hand side of (4.8), the result is also 0, which implies that (4.8) holds in Case i.

Case ii. Assume that $X = U, Y = V, Z = W \in C^\infty(TM)$. By using (2.6) and (4.13), for any $W \in C^\infty(TM)$,

$$\tilde{g}(\tilde{S}(U, V)W, \xi) = -\tilde{g}(\tilde{S}(U, V)\xi, W) = 0. \quad (4.14)$$

By using the Gauss equation (2.36), and noting (2.27), $AU = -U$ and $A^*U = -U$, we get

$$\begin{aligned}2[\tilde{S}(U,V)W]^\top &= 2S(U,V)W - 2\{g(V,W)U - g(U,W)V\}\\ &= \frac{c}{2}\{g(V,W)U - g(U,W)V + g(JV,W)JU - g(JU,W)JV\\ &\quad - 2g(JU,V)JW\} - 2\{g(V,W)U - g(U,W)V\}.\end{aligned}$$

This together with (4.14) yields that

$$\begin{aligned}\tilde{S}(U,V)W =& \frac{c}{4}\{g(V,W)U - g(U,W)V + g(JV,W)JU - g(JU,W)JV\\ &- 2g(JU,V)JW\} - \{g(V,W)U - g(U,W)V\}.\end{aligned}$$

Since $JU = \phi U$, $\eta(U) = 0$ and $\eta(V) = 0$, the above equation is equivalent to

$$\begin{aligned}\tilde{S}(U,V)W =& \frac{c-4}{4}\{g(V,W)U - g(U,W)V\} + \frac{c}{4}\{\eta(U)\eta(W)V\\ &- \eta(V)\eta(W)U + g(U,W)\eta(V)\xi - g(V,W)\eta(U)\xi\\ &+ g(\phi V,W)\phi U - g(\phi U,W)\phi V - 2g(\phi U,V)\phi W\}.\end{aligned} \tag{4.15}$$

Case iii. Assume that $X = \xi, Y = U, Z = V \in C^\infty(TM)$. By using (2.11), we obtain

$$\begin{aligned}\tilde{S}(\xi,U)V =& \tilde{R}^0(\xi,U)V + [\tilde{K}_\xi, \tilde{K}_U]V\\ =& \tilde{R}^0(\xi,U)V + \tilde{K}(\xi, \tilde{K}(U,V)) - \tilde{K}(U, \tilde{K}(\xi,V)).\end{aligned} \tag{4.16}$$

By using Lemma 3.1, (2.3), and noting that $\eta(U) = 0, \eta(V) = 0$, we get

$$\tilde{K}(\xi, \tilde{K}(U,V)) = \lambda\eta(\tilde{K}(U,V))\xi = \lambda\tilde{g}(\tilde{K}(U,V),\xi)\xi = \lambda\tilde{g}(\tilde{K}(U,\xi),V)\xi = 0,$$

$$\tilde{K}(U, \tilde{K}(\xi,V)) = 0.$$

Thus (4.16) gives that

$$\tilde{S}(\xi,U)V = \tilde{R}^0(\xi,U)V.$$

This together with (2.21) yields

$$\tilde{S}(\xi,U)V = -\tilde{g}(U,V)\xi. \tag{4.17}$$

On the other hand, taking $X = \xi, Y = U, Z = V$ in the right-hand side of (4.8), the result is also $-\tilde{g}(U,V)\xi$, which implies that (4.8) holds in Case iii.

Case iv. Assume that $X = \xi, Y = U \in C^\infty(TM), Z = \xi$. By using (2.11), we have

$$\begin{aligned}\tilde{S}(\xi,U)\xi =& \tilde{R}^0(\xi,U)\xi + [\tilde{K}_\xi, \tilde{K}_U]\xi\\ =& \tilde{R}^0(\xi,U)\xi + \tilde{K}(\xi, \tilde{K}(U,\xi)) - \tilde{K}(U, \tilde{K}(\xi,\xi)).\end{aligned} \tag{4.18}$$

From Lemma 3.1 and $\eta(U) = 0$, we obtain

$$\tilde{K}(\xi, \tilde{K}(U,\xi)) = 0, \quad \tilde{K}(U, \tilde{K}(\xi,\xi)) = \lambda\tilde{K}(U,\xi) = 0.$$

Thus (4.18) gives that

$$\tilde{S}(\xi,U)\xi = \tilde{R}^0(\xi,U)\xi.$$

This together with (2.22) yields

$$\tilde{S}(\xi,U)\xi = U. \tag{4.19}$$

On the other hand, taking $X = \xi, Y = U, Z = \xi$ in the right-hand side of (4.8), the result is also U, which implies that (4.8) holds in Case iv.

Thus (4.8) holds for any $X, Y, Z \in C^\infty(T\tilde{M})$. Hence $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ is of constant $\phi-$sectional curvature $c - 1$. Applying Theorem 1.2, we know that $c = 0$. □

Example 4.4. Let $(\phi, \xi, \eta, \tilde{g}, \tilde{\nabla} = \tilde{\nabla}^0 + \tilde{K})$ be the Kenmotsu statistical structure on H^{2n+1} as in Example 1.3, whose $\phi-$sectional curvature with respect to the statistical curvature tensor field $\tilde{S}$ is constant -1. Set $M = \{(x^1, \ldots, x^n, y^1, \ldots, y^n, 1) \in \mathbb{R}^{2n+1}\}$. Then M is a hypersurface in H^{2n+1} orthogonal to ξ, and the holomorphic sectional curvature of the induced holomorphic statistical structure is constant 0.

Proof. Denote the induced metric of $\tilde{g}$ on M by g. Then from (1.3),

$$g = (dx^1)^2 + \cdots + (dx^n)^2 + (dy^1)^2 + \cdots + (dy^n)^2. \tag{4.20}$$

Denote the Levi-Civita connection on M by ∇^0, and set

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}. \tag{4.21}$$

It is obvious that (M, J, g, ∇^0) is just the complex Euclidean space.

Let K be the difference tensor field of (M, J, g, ∇), where ∇ is the induced affine connection of $\tilde{\nabla}$ of H^{2n+1}. Then for every $X, Y \in C^\infty(TM)$,

$$K(X,Y) = \nabla_X Y - \nabla^0_X Y = (\tilde{\nabla}_X Y)^\top - (\tilde{\nabla}^0_X Y)^\top = (\tilde{K}(X,Y))^\top, \tag{4.22}$$

where $(\cdot)^\top$ denotes the tangent component of " $\cdot$ ". Noting that $\tilde{K}(X,Y) = \lambda\eta(X)\eta(Y)\xi$ and M is orthogonal to ξ, thus $K = 0$, which implies the statistical structure (∇, g) on M is trivial. Hence the statistical curvature tensor field S of (M, J, g, ∇) is just the Riemannian curvature tensor field R^0 of (M, g). From (4.20), it is obvious that $R^0 = 0$. Therefore the holomorphic sectional curvature of the induced holomorphic statistical structure is 0.

□

Example 4.5. Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ be the Kenmotsu statistical manifold as in Example 3.8, whose $\phi-$sectional curvature is constant -1. Set $M = \{(x, y, 0) \in \mathbb{R}^3 | x > 0, y > 0\}$, then M is a hypersurface in $\tilde{M}$ orthogonal to ξ, and the holomorphic sectional curvature of the induced holomorphic statistical structure is constant 0.

Proof. Write $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$ for simplicity. Denote the induced metric of $\tilde{g}$ on M by g, it is easy to see that

$$g = x\left\{(dx)^2 + (dy)^2\right\}. \tag{4.23}$$

Denote the Levi-Civita connection on M by ∇^0, then

$$\begin{aligned} \nabla^0_{e_1} e_1 &= \frac{1}{2x} e_1, \quad \nabla^0_{e_1} e_2 = \frac{1}{2x} e_2, \\ \nabla^0_{e_2} e_1 &= \frac{1}{2x} e_2, \quad \nabla^0_{e_2} e_2 = -\frac{1}{2x} e_1. \end{aligned} \tag{4.24}$$

Set

$$Je_1 = e_2, \quad Je_2 = -e_1. \tag{4.25}$$

By using (4.23)-(4.25), it can be verified that (J, g, ∇^0) satisfies (2.25), thus (M, J, g, ∇^0) is a Kähler manifold.

Denote the induced affine connection of $\tilde{\nabla}$ on M by ∇, then

$$\begin{aligned} \nabla_{e_1} e_1 &= \frac{2-\sqrt{2}}{4x} e_1 + \frac{1}{2\sqrt{2}x} e_2, \quad \nabla_{e_1} e_2 = \frac{1}{2\sqrt{2}x} e_1 + \frac{2+\sqrt{2}}{4x} e_2, \\ \nabla_{e_2} e_1 &= \frac{1}{2\sqrt{2}x} e_1 + \frac{2+\sqrt{2}}{4x} e_2, \quad \nabla_{e_2} e_2 = \frac{-2+\sqrt{2}}{4x} e_1 - \frac{1}{2\sqrt{2}x} e_2. \end{aligned} \tag{4.26}$$

Let $K = \nabla - \nabla^0$, according to (4.24) and (4.26), we get

$$\begin{aligned} K(e_1, e_1) &= -\frac{1}{2\sqrt{2}x}e_1 + \frac{1}{2\sqrt{2}x}e_2, \\ K(e_2, e_2) &= \frac{1}{2\sqrt{2}x}e_1 - \frac{1}{2\sqrt{2}x}e_2, \\ K(e_1, e_2) &= K(e_2, e_1) = \frac{1}{2\sqrt{2}x}e_1 + \frac{1}{2\sqrt{2}x}e_2. \end{aligned} \tag{4.27}$$

It is easy to see that $K = \nabla - \nabla^0$ satisfies (2.3) and (2.26), thus $(J, g, \nabla = \nabla^0 + K)$ is a holomorphic statistical structure on M.

Denote the curvature tensor field of ∇ by R. By using (4.26), we calculate

$$R(e_1, e_2)e_1 = 0, \quad R(e_1, e_2)e_2 = 0,$$

thus $R = 0$. By using (2.4) and (2.9), we get $S = 0$. Hence the holomorphic sectional curvature of M is constant 0. □

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