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An application of the iterative method to study multi-dimensional fractional order Navier-Stokes equations

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Abstract

In this article, a hybrid method called iteration Shehu transform method has been implemented to solve fractional-order NavierâĂŞStokes equation. Atangana-Balenu operator describes fractional-order derivatives. The analytical solutions of three distinct examples of the time- fractional Navier-Stokes equations are determined by using Iterative shehu transform method. Further, we present the effectiveness and accuracy of the proposed method by comparison of analytical solutions to the exact solutions and the results are represented graphically and numerically.

Keywords: Fractional order Navier-Stokes equations Iterative Shehu transform method Atangana-Balenu derivative.
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1. Introduction

The Navier-Stokes equation is known as Newton second Law for fluid substance, has been derived in 1822 by C. L. Navier and G. Stokes. The N-S equations are useful in describing the physics of many scientific and engineering phenomena of interest. This equation identifies several physical things around the wings of the aircraft, such as liquid flow in pipes, blood flow and air flow [1, 2, 3, 4, 5].The N-S equation create the

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relation between pressure and fluid-acting external forces to the fluid flow response [6, 7, 8, 9].
The exact solution of N-S equations is possible in only a few cases due to its nonlinear nature. In these cases, we have to consider a simple configuration for the flow pattern and certain assumptions need to be made about the state of the fluid. In most practical situations, these equations follow the fractional order and not the integer order. El-Shahed et al. [10] generalized the classical N-S equation with switching integer-order derivative to non integer-order derivative ζ $(0 < \zeta \leq 1)$. The fractional order N-S equation for an in-compressible fluid flow of density ρ and kinematic viscosity $\nu = \frac{\phi}{\rho}$. It is given as:

$$\begin{cases} D_\tau^\zeta F + (F.\nabla) F = \rho_0 \nabla^2 F - \frac{1}{\rho}\nabla P, \\ \nabla.F = 0, \\ F = 0, \qquad on \ \ \Omega \times (0, T). \end{cases} \tag{1}$$

Here, $F = (\vartheta, \nu, \omega)$, ϕ, P, and τ indicates the fluid vector,dynamic viscosity, pressure and time, respectively. (θ, α, η) indicate the spatial components in Ω.
Fractional calculus (FC),including integration and differentiation of arbitrary non-integer order, is the generalization of classical integration and differentiation.The most important advantage of using fractional derivative model is that we can calculate memory, history or non-local effects through this, which is difficult to models through integer order derivatives. The fractional derivative with Mittag-Leffler function as the non-local and non-singular kernel was proposed by Atangana and Baleanu and is called Atangana-Baleanu (A-B) fractional derivative [11]. The advantage of the fractional derivative operator with the non-singular kernel is that it remodels the stretched exponential to power as waiting time, remodels the Gaussian to non-Gaussian as density distribution. For a detailed study see [12, 13, 14, 15].
The main aim for the choice of including the A-B fractional derivative operator is to include into the mathematical formulation of the the dynamical system the effect of non-local fading memory. Equation (1) can be rewritten with the help of A-B fractional order derivative as:

$$\begin{cases} {}_0^{AB}D_\tau^\zeta \vartheta(\theta,\alpha,\eta,\tau) + \vartheta \frac{\partial\vartheta}{\partial\theta} + \nu \frac{\partial\vartheta}{\partial\alpha} + \omega \frac{\partial\vartheta}{\partial\eta} = \rho \left[\frac{\partial^2\vartheta}{\partial\theta^2} + \frac{\partial^2\vartheta}{\partial\alpha^2} + \frac{\partial^2\vartheta}{\partial\eta^2}\right] - \frac{1}{\rho}\frac{\partial P}{\partial\theta}, \\ \\ {}_0^{AB}D_\tau^\zeta \nu(\theta,\alpha,\eta,\tau) + \vartheta \frac{\partial\nu}{\partial\theta} + \nu \frac{\partial\nu}{\partial\alpha} + \omega \frac{\partial\nu}{\partial\eta} = \rho \left[\frac{\partial^2\nu}{\partial\theta^2} + \frac{\partial^2\nu}{\partial\alpha^2} + \frac{\partial^2\nu}{\partial\eta^2}\right] - \frac{1}{\rho}\frac{\partial P}{\partial\alpha}, \\ \\ {}_0^{AB}D_\tau^\zeta \omega(\theta,\alpha,\eta,\tau) + \vartheta \frac{\partial\omega}{\partial\theta} + \nu \frac{\partial\omega}{\partial\alpha} + \omega \frac{\partial\omega}{\partial\eta} = \rho \left[\frac{\partial^2\omega}{\partial\theta^2} + \frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\eta^2}\right] - \frac{1}{\rho}\frac{\partial P}{\partial\eta}. \end{cases} \tag{2}$$

Mathematically, these equations are a problematic arrangement of nonlinear equations within sight of viscous flows [16].
Different methods are adopted to solve multi dimensional fractional order N-S equations. For example, Shah et. al [17] have been investigated N-S equations with the help of NVIM and NDM. The numerical simulation of fractional order multi-dimensional N-S equation presented by Singh et. al [18]. Prakash et. al [19] have been solved fractional coupled NâĂŞS equations by q-HATM. Chu et. al[20], and Mahmood et. al [21] have been Analyzed the fractional order NâĂŞS equations with the help of VIM and ADM via Laplace transform, respectively. Recently, Hajira et. al [22] have been presented analytical solution of the NâĂŞS equations by ADM via Elzaki transform.
Gejji et al. [23] have been introduced and described the new iterative method in 2006. The main aim of this article is to extend the application of iterative Shehu transform method, based on the new iterative method and Shehu transform method [24] and obtain the analytical and numerical solutions of fractional order N-S equations. We take the value of $B(\zeta)$ in all numerical results is $B(\zeta) = 1 - \zeta + \frac{\zeta}{\Gamma\zeta}$.

2. Definitions

Definition 2.1: The Atangana-Balenu (A-B) derivative operator of fractional order ζ $(0 < \zeta \leq 1)$ introduced by [11, 25] and defined as:

$$ {}_{0}^{AB}D_{\tau}^{\zeta}\left[\varphi\left(\tau\right)\right] = \frac{B\left(\zeta\right)}{1-\zeta}\int\limits_{0}^{\tau} E_{\zeta}\left(\frac{-\zeta\left(\tau-\xi\right)}{1-\zeta}\right) D\left[\varphi\left(\xi\right)\right]d\xi\,, \tag{3} $$

where, ${}_{0}^{AB}D_{\tau}^{\zeta}$ is the AB fractional derivative of order ζ, $B\left(\zeta\right)$ is the normalization function such that $B(0) = B(1) = 1$, E_{ζ} represent the Mittag-Leffler function and define as:

$$ E_{\zeta}\left(\tau\right) = \sum_{n=0}^{\infty}\frac{\tau^{n}}{\Gamma(n\zeta+1)}\,, \quad \zeta > 0, \quad \zeta,\tau \in \Re. $$

Definition 2.2: The Shehu transform of continuous function $\varphi\left(\tau\right)$ is expressed by as:

$$ S\left[\varphi\left(\tau\right)\right] = V\left(s,\mu\right) = \int\limits_{0}^{\infty}\exp\left(-\frac{s\tau}{\mu}\right)\varphi\left(\tau\right)\,d\tau, \tag{4} $$

Definition 2.3: If $V\left(s,\mu\right)$ is the Shehu transform of $\varphi\left(\tau\right)$, then the Shehu transform of A-B fractional order derivative is given by [26]:

$$ S\left({}_{0}^{AB}D_{\tau}^{\zeta}\left[\varphi\left(\tau\right)\right]\right) = \frac{B\left(\zeta\right)}{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}\left(V\left(s,\mu\right)-\frac{\mu}{s}\varphi\left(0\right)\right)\,, \tag{5} $$

3. Basic idea of proposed technique:

This section describes the ISTM solution of fractional multi- dimensional Naiver-Stokes equation with fractional atangana-Balenue derivative written as:

$$ \begin{cases} {}^{AB}D_{\tau}^{\zeta}\vartheta\left(\theta,\alpha,\eta,\tau\right) = \rho\left[\frac{\partial^{2}\vartheta}{\partial\theta^{2}}+\frac{\partial^{2}\vartheta}{\partial\alpha^{2}}+\frac{\partial^{2}\vartheta}{\partial\eta^{2}}\right]+q_{1}-\left[\vartheta\frac{\partial\vartheta}{\partial\theta}+\nu\frac{\partial\vartheta}{\partial\alpha}+\omega\frac{\partial\vartheta}{\partial\eta}\right] \\ {}^{AB}D_{\tau}^{\zeta}\nu\left(\theta,\alpha,\eta,\tau\right) = \rho\left[\frac{\partial^{2}\nu}{\partial\theta^{2}}+\frac{\partial^{2}\nu}{\partial\alpha^{2}}+\frac{\partial^{2}\nu}{\partial\eta^{2}}\right]+q_{2}-\left[\vartheta\frac{\partial\nu}{\partial\theta}+\nu\frac{\partial\nu}{\partial\alpha}+\omega\frac{\partial\nu}{\partial\eta}\right] \\ {}^{AB}D_{\tau}^{\zeta}\omega\left(\theta,\alpha,\eta,\tau\right) = \rho\left[\frac{\partial^{2}\omega}{\partial\theta^{2}}+\frac{\partial^{2}\omega}{\partial\alpha^{2}}+\frac{\partial^{2}\omega}{\partial\eta^{2}}\right]+q_{3}-\left[\vartheta\frac{\partial\omega}{\partial\theta}+\nu\frac{\partial\omega}{\partial\alpha}+\omega\frac{\partial\omega}{\partial\eta}\right] \end{cases}, \tag{6} $$

with the initial conditions (ICs):

$$ \begin{cases} \vartheta\left(\theta,\alpha,\eta,0\right) = f\left(\theta,\alpha,\eta\right) \\ \nu\left(\theta,\alpha,\eta,0\right) = g\left(\theta,\alpha,\eta\right) \\ \omega\left(\theta,\alpha,\eta,0\right) = h\left(\theta,\alpha,\eta\right) \end{cases} \tag{7} $$

where, $q_{1} = -\frac{1}{\rho}\frac{\partial P}{\partial\theta}$, $q_{2} = -\frac{1}{\rho}\frac{\partial P}{\partial\alpha}$, $q_{3} = -\frac{1}{\rho}\frac{\partial P}{\partial\eta}$, and $0 < \zeta \leq 1$.

Taking the Shehu transform to both sides of the equation (6)and with the help of equation (7), we have:

$$\begin{aligned}
S[\vartheta] &= \tfrac{\mu}{s} f(\theta,\alpha,\eta) + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q_1] \\
&\quad + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\tfrac{\partial^2\vartheta}{\partial\theta^2}+\tfrac{\partial^2\vartheta}{\partial\alpha^2}+\tfrac{\partial^2\vartheta}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\vartheta}{\partial\theta}+\nu\tfrac{\partial\vartheta}{\partial\alpha}+\omega\tfrac{\partial\vartheta}{\partial\eta}\right]\right], \\
S[\nu] &= \tfrac{\mu}{s} g(\theta,\alpha,\eta) + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q_2] \\
&\quad + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\tfrac{\partial^2\nu}{\partial\theta^2}+\tfrac{\partial^2\nu}{\partial\alpha^2}+\tfrac{\partial^2\nu}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\nu}{\partial\theta}+\nu\tfrac{\partial\nu}{\partial\alpha}+\omega\tfrac{\partial\nu}{\partial\eta}\right]\right], \\
S[\omega] &= \tfrac{\mu}{s} h(\theta,\alpha,\eta) + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q_3] \\
&\quad + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\tfrac{\partial^2\omega}{\partial\theta^2}+\tfrac{\partial^2\omega}{\partial\alpha^2}+\tfrac{\partial^2\omega}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\omega}{\partial\theta}+\nu\tfrac{\partial\omega}{\partial\alpha}+\omega\tfrac{\partial\omega}{\partial\eta}\right]\right].
\end{aligned} \tag{8}$$

Next, taking the inverse Shehu transform to both sides of (8), one can get:

$$\begin{aligned}
\vartheta(\theta,\alpha,\eta,\tau) &= \chi_1(\theta,\alpha,\eta,\tau) + S^{-1}\left[\left(\tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right. \\
&\qquad \left. S\left[\rho\left[\tfrac{\partial^2\vartheta}{\partial\theta^2}+\tfrac{\partial^2\vartheta}{\partial\alpha^2}+\tfrac{\partial^2\vartheta}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\vartheta}{\partial\theta}+\nu\tfrac{\partial\vartheta}{\partial\alpha}+\omega\tfrac{\partial\vartheta}{\partial\eta}\right]\right]\right], \\
\nu(\theta,\alpha,\eta,\tau) &= \chi_2(\theta,\alpha,\eta,\tau) + S^{-1}\left[\left(\tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right. \\
&\qquad \left. S\left[\rho\left[\tfrac{\partial^2\nu}{\partial\theta^2}+\tfrac{\partial^2\nu}{\partial\alpha^2}+\tfrac{\partial^2\nu}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\nu}{\partial\theta}+\nu\tfrac{\partial\nu}{\partial\alpha}+\omega\tfrac{\partial\nu}{\partial\eta}\right]\right]\right], \\
\omega(\theta,\alpha,\eta,\tau) &= \chi_3(\theta,\alpha,\eta,\tau) + S^{-1}\left[\left(\tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right. \\
&\qquad \left. S\left[\rho\left[\tfrac{\partial^2\omega}{\partial\theta^2}+\tfrac{\partial^2\omega}{\partial\alpha^2}+\tfrac{\partial^2\omega}{\partial\eta^2}\right]-\left[\vartheta\tfrac{\partial\omega}{\partial\theta}+\nu\tfrac{\partial\omega}{\partial\alpha}+\omega\tfrac{\partial\omega}{\partial\eta}\right]\right]\right],
\end{aligned} \tag{9}$$

here,

$$\begin{cases}
\chi_1(\theta,\alpha,\eta,\tau) = f(\theta,\alpha,\eta) + q_1 \dfrac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}, \\
\chi_2(\theta,\alpha,\eta,\tau) = g(\theta,\alpha,\eta) + q_2 \dfrac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}, \\
\chi_3(\theta,\alpha,\eta,\tau) = h(\theta,\alpha,\eta) + q_3 \dfrac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}.
\end{cases} \tag{10}$$

Further, we use new iterative method introduced by [23]. We take the solution as an infinite series given as:

$$\begin{aligned}
\vartheta(\theta,\alpha,\eta,\tau) &= \sum_{\ell=0}^{\infty} \vartheta_\ell(\theta,\alpha,\eta,\tau), \\
\nu(\theta,\alpha,\eta,\tau) &= \sum_{\ell=0}^{\infty} \nu_\ell(\theta,\alpha,\eta,\tau), \\
\omega(\theta,\alpha,\eta,\tau) &= \sum_{\ell=0}^{\infty} \omega_\ell(\theta,\alpha,\eta,\tau).
\end{aligned} \tag{11}$$

The nonlinear terms consider as follows:

$$\begin{aligned}
\vartheta\tfrac{\partial\vartheta}{\partial\theta} &= \sum_{\ell=0}^{\infty} G_\ell, \quad \nu\tfrac{\partial\vartheta}{\partial\alpha} = \sum_{\ell=0}^{\infty} H_\ell, \quad \omega\tfrac{\partial\vartheta}{\partial\eta} = \sum_{\ell=0}^{\infty} I_\ell, \\
\vartheta\tfrac{\partial\nu}{\partial\theta} &= \sum_{\ell=0}^{\infty} K_\ell, \quad \nu\tfrac{\partial\nu}{\partial\alpha} = \sum_{\ell=0}^{\infty} L_\ell, \quad \omega\tfrac{\partial\nu}{\partial\eta} = \sum_{\ell=0}^{\infty} M_\ell, \\
\vartheta\tfrac{\partial\omega}{\partial\theta} &= \sum_{\ell=0}^{\infty} P_\ell, \quad \nu\tfrac{\partial\omega}{\partial\alpha} = \sum_{\ell=0}^{\infty} Q_\ell, \quad \omega\tfrac{\partial\omega}{\partial\eta} = \sum_{\ell=0}^{\infty} R_\ell,
\end{aligned} \tag{12}$$

whereas, G_ℓ, H_ℓ, I_ℓ, K_ℓ, L_ℓ, M_ℓ, P_ℓ, Q_ℓ, and R_ℓ are further decomposed as follows:

$$\begin{aligned}
G_\ell &= \sum_{i=0}^{\ell} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell}\vartheta_i\right) - \sum_{i=0}^{\ell-1} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell-1}\vartheta_i\right),\\
H_\ell &= \sum_{i=0}^{\ell} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell}\vartheta_i\right) - \sum_{i=0}^{\ell-1} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell-1}\vartheta_i\right),\\
I_\ell &= \sum_{i=0}^{\ell} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell}\vartheta_i\right) - \sum_{i=0}^{\ell-1} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell-1}\vartheta_i\right),\\
K_\ell &= \sum_{i=0}^{\ell} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell}\nu_i\right) - \sum_{i=0}^{\ell-1} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell-1}\nu_i\right),\\
L_\ell &= \sum_{i=0}^{\ell} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell}\nu_i\right) - \sum_{i=0}^{\ell-1} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell-1}\nu_i\right),\\
M_\ell &= \sum_{i=0}^{\ell} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell}\nu_i\right) - \sum_{i=0}^{\ell-1} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell-1}\nu_i\right),\\
P_\ell &= \sum_{i=0}^{\ell} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell}\omega_i\right) - \sum_{i=0}^{\ell-1} \vartheta_i \frac{\partial}{\partial\theta}\left(\sum_{i=0}^{\ell-1}\omega_i\right),\\
Q_\ell &= \sum_{i=0}^{\ell} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell}\omega_i\right) - \sum_{i=0}^{\ell-1} \nu_i \frac{\partial}{\partial\alpha}\left(\sum_{i=0}^{\ell-1}\omega_i\right),\\
R_\ell &= \sum_{i=0}^{\ell} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell}\omega_i\right) - \sum_{i=0}^{\ell-1} \omega_i \frac{\partial}{\partial\eta}\left(\sum_{i=0}^{\ell-1}\omega_i\right),
\end{aligned} \tag{13}$$

Finally, we get the recursive formulas given by:

$$\begin{aligned}
\vartheta_{\ell+1}(\theta,\alpha,\eta,\tau) &= \chi_1(\theta,\alpha,\eta,\tau) + S^{-1}\Bigg[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^\zeta}{B(\zeta)}\right)\\
&\qquad S\left[\rho\left[\frac{\partial^2\vartheta_\ell}{\partial\theta^2}+\frac{\partial^2\vartheta_\ell}{\partial\alpha^2}+\frac{\partial^2\vartheta_\ell}{\partial\eta^2}\right]-\left[\vartheta_\ell\frac{\partial\vartheta_\ell}{\partial\theta}+\nu_\ell\frac{\partial\vartheta_\ell}{\partial\alpha}+\omega_\ell\frac{\partial\vartheta_\ell}{\partial\eta}\right]\right]\Bigg],\\
\nu_{\ell+1}(\theta,\alpha,\eta,\tau) &= \chi_2(\theta,\alpha,\eta,\tau) + S^{-1}\Bigg[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^\zeta}{B(\zeta)}\right)\\
&\qquad S\left[\rho\left[\frac{\partial^2\nu_\ell}{\partial\theta^2}+\frac{\partial^2\nu_\ell}{\partial\alpha^2}+\frac{\partial^2\nu_\ell}{\partial\eta^2}\right]-\left[\vartheta_\ell\frac{\partial\nu_\ell}{\partial\theta}+\nu_\ell\frac{\partial\nu_\ell}{\partial\alpha}+\omega_\ell\frac{\partial\nu_\ell}{\partial\eta}\right]\right]\Bigg],\\
\omega_{\ell+1}(\theta,\alpha,\eta,\tau) &= \chi_3(\theta,\alpha,\eta,\tau) + S^{-1}\Bigg[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^\zeta}{B(\zeta)}\right)\\
&\qquad S\left[\rho\left[\frac{\partial^2\omega_\ell}{\partial\theta^2}+\frac{\partial^2\omega_\ell}{\partial\alpha^2}+\frac{\partial^2\omega_\ell}{\partial\eta^2}\right]-\left[\vartheta_\ell\frac{\partial\omega_\ell}{\partial\theta}+\nu_\ell\frac{\partial\omega_\ell}{\partial\alpha}+\omega_\ell\frac{\partial\omega_\ell}{\partial\eta}\right]\right]\Bigg].
\end{aligned} \tag{14}$$

4. Numerical problems:

In this section, three numerical problems are given to illustrate the efficiency and accuracy of our proposed method.

Problem 1: Consider time-fractional order of two-dimensional Navier-Stock equation with $q_1 = -q_2 = q$ as:

$$\begin{cases} {}^{AB}D_\tau^\zeta\vartheta(\theta,\alpha,\tau) = \rho\left[\frac{\partial^2\vartheta}{\partial\theta^2}+\frac{\partial^2\vartheta}{\partial\alpha^2}\right]+q-\left[\vartheta\frac{\partial\vartheta}{\partial\theta}+\nu\frac{\partial\vartheta}{\partial\alpha}\right] \\ {}^{AB}D_\tau^\zeta\nu(\theta,\alpha,\tau) = \rho\left[\frac{\partial^2\nu}{\partial\theta^2}+\frac{\partial^2\nu}{\partial\alpha^2}\right]-q-\left[\vartheta\frac{\partial\nu}{\partial\theta}+\nu\frac{\partial\nu}{\partial\alpha}\right] \end{cases}, \tag{15}$$

with the initial conditions (ICs):

$$\begin{cases} \vartheta(\theta,\alpha,0) = -\sin(\theta+\alpha) \\ \nu(\theta,\alpha,0) = \sin(\theta+\alpha) \end{cases}, \tag{16}$$

where, $0 < \zeta \leq 1$.

Taking the Shehu transform to both sides of the equation (15)and with the help of equation (16), we have:

$$\begin{aligned} S[\vartheta] &= \tfrac{\mu}{s}\left[-\sin(\theta+\alpha)\right] + \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q] \\ &+ \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\tfrac{\partial^2\vartheta}{\partial\theta^2}+\tfrac{\partial^2\vartheta}{\partial\alpha^2}\right]-\left[\vartheta\tfrac{\partial\vartheta}{\partial\theta}+\nu\tfrac{\partial\vartheta}{\partial\alpha}\right]\right], \\ S[\nu] &= \tfrac{\mu}{s}\left[\sin(\theta+\alpha)\right) - \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q] \\ &+ \tfrac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\tfrac{\partial^2\nu}{\partial\theta^2}+\tfrac{\partial^2\nu}{\partial\alpha^2}\right]-\left[\vartheta\tfrac{\partial\nu}{\partial\theta}+\nu\tfrac{\partial\nu}{\partial\alpha}\right]\right]. \end{aligned} \tag{17}$$

Next, taking the inverse Shehu transform to both sides of (17), one can get:

$$\begin{aligned} \vartheta(\theta,\alpha,\tau) &= -\sin(\theta+\alpha) + q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &+ S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left[\rho\left[\tfrac{\partial^2\vartheta}{\partial\theta^2}+\tfrac{\partial^2\vartheta}{\partial\alpha^2}\right]-\left[\vartheta\tfrac{\partial\vartheta}{\partial\theta}+\nu\tfrac{\partial\vartheta}{\partial\alpha}\right]\right]\right], \\ \nu(\theta,\alpha,\tau) &= \sin(\theta+\alpha) - q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &+ S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left[\rho\left[\tfrac{\partial^2\nu}{\partial\theta^2}+\tfrac{\partial^2\nu}{\partial\alpha^2}\right]-\left[\vartheta\tfrac{\partial\nu}{\partial\theta}\right]\right]\right]. \end{aligned} \tag{18}$$

According to our proposed scheme, we have the following results:

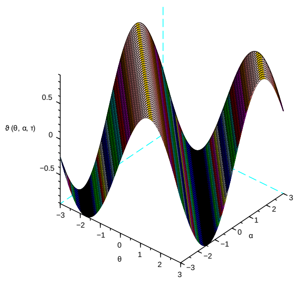
$$\begin{aligned} \vartheta_0(\theta,\alpha,\tau) &= -\sin(\theta+\alpha) + q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \nu_0(\theta,\alpha,\tau) &= \sin(\theta+\alpha) - q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \vartheta_1(\theta,\alpha,\tau) &= 2\rho\sin(\theta+\alpha)\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \nu_1(\theta,\alpha,\tau) &= -2\rho\sin(\theta+\alpha)\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \vartheta_2(\theta,\alpha,\tau) &= \frac{-4\rho^2\sin(\theta+\alpha)}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)\right), \\ \nu_2(\theta,\alpha,\tau) &= \frac{4\rho^2\sin(\theta+\alpha)}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)\right), \end{aligned}$$

Similarly, we obtain next terms in the same manner. Hence, the approximate solution of (15) is given as:

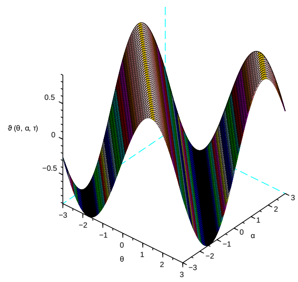
$$\begin{aligned} \vartheta(\theta,\alpha,\tau) &= \vartheta_0(\theta,\alpha,\tau)+\vartheta_1(\theta,\alpha,\tau)+\vartheta_2(\theta,\alpha,\tau)+......, \\ \nu(\theta,\alpha,\tau) &= \nu_0(\theta,\alpha,\tau)+\nu_1(\theta,\alpha,\tau)+\nu_2(\theta,\alpha,\tau)+...... \quad . \\ \vartheta(\theta,\alpha,\tau) &= -\sin(\theta+\alpha) + q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) + 2\rho\sin(\theta+\alpha)\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &- \frac{4\rho^2\sin(\theta+\alpha)}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1-2\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}\right)\right)+..., \\ \nu(\theta,\alpha,\tau) &= \sin(\theta+\alpha) - q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) - 2\rho\sin(\theta+\alpha)\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &+ \frac{4\rho^2\sin(\theta+\alpha)}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1-2\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}\right)\right)+..... \end{aligned} \tag{19}$$

In particular, the analytical solution of equation (15) converge fastly to the exact solution of classical N-S equation for the velocity with q=0 and $\zeta = 1$.

$$\begin{aligned} \vartheta\left(\theta, \alpha, \tau\right) &= -e^{-2\rho\tau}\sin\left(\theta+\alpha\right), \\ \nu\left(\theta, \alpha, \tau\right) &= e^{-2\rho\tau}\sin\left(\theta+\alpha\right) \ . \end{aligned} \tag{20}$$

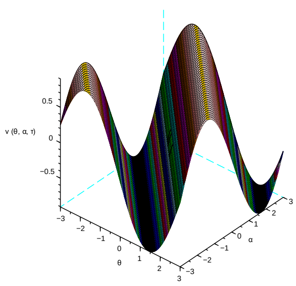


(a) Exact solution

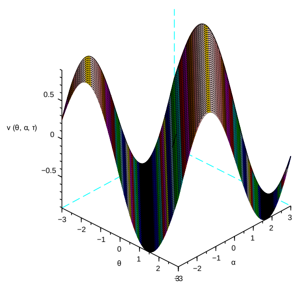


(b) ISTM solution

Figure 1: For problem 1, the velocity profile $\vartheta\left(\theta, \alpha, \tau\right)$ of N-S equation and the exact solution of (15) at $\zeta = 1$, q=0, $\rho = 0.5$, and $\tau = 0.1$.



(a) Exact solution



(b) ISTM solution

Figure 2: For problem 1, the velocity profile $\nu\left(\theta, \alpha, \tau\right)$ of N-S equation and the exact solution of (15) at $\zeta = 1$, q=0, $\rho = 0.5$, and $\tau = 0.1$.

Problem 2: Consider time-fractional order of two-dimensional Navier-Stock equation with $q_1 = -q_2 = q$ as:

$$\begin{cases} {}^{AB}D_{\tau}^{\zeta}\vartheta\left(\theta, \alpha, \tau\right) = \rho\left[\frac{\partial^2\vartheta}{\partial\theta^2} + \frac{\partial^2\vartheta}{\partial\alpha^2}\right] + q - \left[\vartheta\frac{\partial\vartheta}{\partial\theta} + \nu\frac{\partial\vartheta}{\partial\alpha}\right] \\ {}^{AB}D_{\tau}^{\zeta}\nu\left(\theta, \alpha, \tau\right) = \rho\left[\frac{\partial^2\nu}{\partial\theta^2} + \frac{\partial^2\nu}{\partial\alpha^2}\right] - q - \left[\vartheta\frac{\partial\nu}{\partial\theta} + \nu\frac{\partial\nu}{\partial\alpha}\right] \end{cases}, \tag{21}$$

with the initial conditions (ICs):

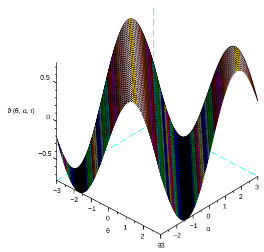
$$\begin{aligned} &\vartheta\left(\theta, \alpha, 0\right) = -e^{(\theta+\alpha)}, \\ &\nu\left(\theta, \alpha, 0\right) = e^{(\theta+\alpha)}. \end{aligned} \tag{22}$$

Table I: Absolute error for the velocity profile $\vartheta(\theta,\alpha,\tau)$ of Navier-Stokes equation at $\rho = 0.5$, $\zeta = 1$, and $\tau = 0.01$ for problem 1.

θ	α	ϑ_{ISTM}	ϑ_{exact}	$\lvert\vartheta_{exa.} - \vartheta_{NISTM}\rvert$
1	1	−0.9002497662580690	−0.9002497662573126	7.5639E-13
2	2	0.7492721846611990	0.7492721846605699	6.2961E-13
3	3	0.2766352675390201	0.2766352675387875	2.3248E-13
4	4	−0.9795139675886706	−0.9795139675878476	8.2301E-13
5	5	0.5386080103925108	0.5386080103720583	4.5253E-13
6	6	0.5312339282610830	0.5312339282606366	4.4631E-13
7	7	−0.9807506478172335	−0.9807506478164094	8.2412E-13
8	8	0.2850386308003215	0.2850386308008200	2.3948E-13
9	9	0.7435147988146682	0.7435147988140435	6.2472E-13
10	10	−0.9038612937057399	−0.9038612937049804	7.5950E-13

Table II: Absolute error for the velocity profile $\nu(\theta,\alpha,\tau)$ of Navier-Stokes equation at $\rho = 0.5$, $\zeta = 1$, and $\tau = 0.01$ for problem 1.

θ	α	ϑ_{ISTM}	ϑ_{exact}	$\lvert\vartheta_{exa.} - \vartheta_{NISTM}\rvert$
1	1	0.9002497662580690	0.9002497662573126	7.5639E-13
2	2	−0.7492721846611990	−0.7492721846605699	6.2961E-13
3	3	−0.2766352675390201	−0.2766352675387875	2.3248E-13
4	4	0.9795139675886706	0.9795139675878476	8.2301E-13
5	5	−0.5386080103925108	−0.5386080103720583	4.5253E-13
6	6	−0.5312339282610830	−0.5312339282606366	4.4631E-13
7	7	0.9807506478172335	0.9807506478164094	8.2412E-13
8	8	−0.2850386308003215	−0.2850386308008200	2.3948E-13
9	9	−0.7435147988146682	−0.7435147988140435	6.2472E-13
10	10	0.9038612937057399	0.9038612937049804	7.5950E-13


(a) $\vartheta(\theta,\alpha,\tau)$

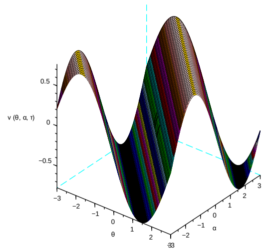

(b) $\nu(\theta,\alpha,\tau)$

Figure 3: For problem 1, the velocity profiles $\vartheta(\theta,\alpha,\tau)$ and $\nu(\theta,\alpha,\tau)$ of N-S equation (15) at $\zeta = 0.5$, q=0, $\rho = 0.5$, and $\tau = 0.1$.

where, $0 < \zeta \leq 1$.

Taking the Shehu transform to both sides of the equation (21)and with the help of equation (22), we have:

$$\begin{aligned} S[\vartheta] &= \tfrac{\mu}{s}\left[-e^{(\theta+\alpha)}\right] + \frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q] \\ &+ \frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\frac{\partial^2\vartheta}{\partial\theta^2}+\frac{\partial^2\vartheta}{\partial\alpha^2}\right] - \left[\vartheta\frac{\partial\vartheta}{\partial\theta}+\nu\frac{\partial\vartheta}{\partial\alpha}\right]\right], \\ S[\nu] &= \tfrac{\mu}{s}\left[e^{(\theta+\alpha)}\right) - \frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S[q] \\ &+ \frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)} S\left[\rho\left[\frac{\partial^2\nu}{\partial\theta^2}+\frac{\partial^2\nu}{\partial\alpha^2}\right] - \left[\vartheta\frac{\partial\nu}{\partial\theta}+\nu\frac{\partial\nu}{\partial\alpha}\right]\right]. \end{aligned} \tag{23}$$

Next, taking the inverse Shehu transform to both sides of (23), one can get:

$$\begin{aligned} \vartheta(\theta,\alpha,\tau) &= -e^{(\theta+\alpha)} + q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &+ S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left[\rho\left[\frac{\partial^2\vartheta}{\partial\theta^2}+\frac{\partial^2\vartheta}{\partial\alpha^2}\right] - \left[\vartheta\frac{\partial\vartheta}{\partial\theta}+\nu\frac{\partial\vartheta}{\partial\alpha}\right]\right]\right], \\ \nu(\theta,\alpha,\tau) &= e^{(\theta+\alpha)} - q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right) \\ &+ S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left[\rho\left[\frac{\partial^2\nu}{\partial\theta^2}+\frac{\partial^2\nu}{\partial\alpha^2}\right] - \left[\vartheta\frac{\partial\nu}{\partial\theta}\right]\right]\right]. \end{aligned} \tag{24}$$

According to our proposed scheme, we have the following results:

$$\begin{aligned} \vartheta_0(\theta,\alpha,\tau) &= -e^{(\theta+\alpha)} + q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \nu_0(\theta,\alpha,\tau) &= e^{(\theta+\alpha)} - q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \vartheta_1(\theta,\alpha,\tau) &= -2\rho e^{(\theta+\alpha)}\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \nu_1(\theta,\alpha,\tau) &= 2\rho e^{(\theta+\alpha)}\left(\frac{1-\zeta+\zeta\left(\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right), \\ \vartheta_2(\theta,\alpha,\tau) &= \frac{-4\rho^2 e^{(\theta+\alpha)}}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right)\right), \end{aligned}$$

$$\nu_2(\theta,\alpha,\tau)=\frac{4\rho^2e^{(\theta+\alpha)}}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

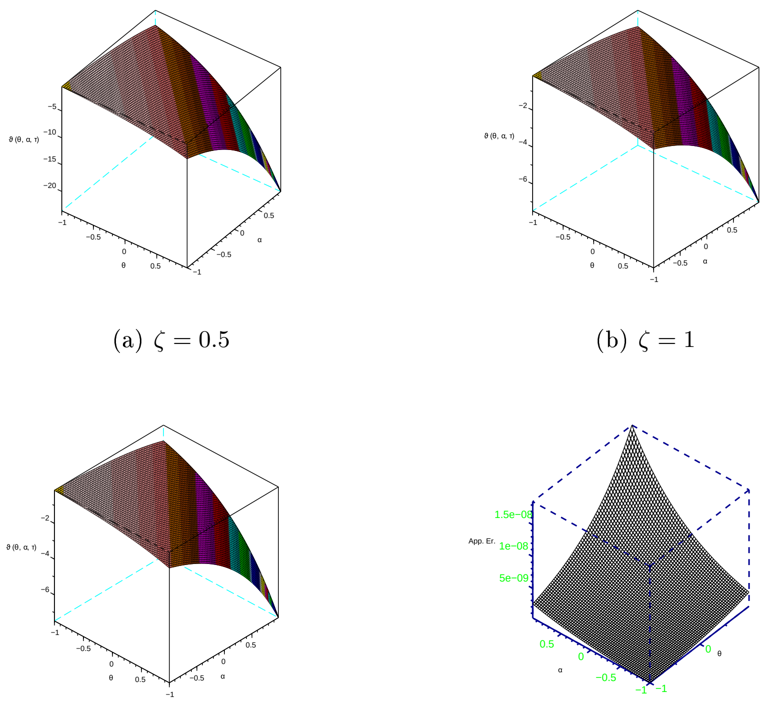
Similarly, we obtain next terms in the same manner. Hence, the approximate solution of (21) is given as:

$$\vartheta(\theta,\alpha,\tau)=\vartheta_0(\theta,\alpha,\tau)+\vartheta_1(\theta,\alpha,\tau)+\vartheta_2(\theta,\alpha,\tau)+......,$$
$$\nu(\theta,\alpha,\tau)=\nu_0(\theta,\alpha,\tau)+\nu_1(\theta,\alpha,\tau)+\nu_2(\theta,\alpha,\tau)+...... .$$

$$\begin{aligned}\vartheta(\theta,\alpha,\tau)=&-e^{(\theta+\alpha)}+q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right)-2\rho e^{(\theta+\alpha)}\left(\frac{1-\zeta+\zeta\left(\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right)\\&-\frac{4\rho^2e^{(\theta+\alpha)}}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1-2\frac{\tau^\zeta}{\Gamma(\zeta+1)}+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}\right)\right)+...,\\\nu(\theta,\alpha,\tau)=&\,e^{(\theta+\alpha)}-q\left(\frac{1-\zeta+\zeta\left(\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right)+2\rho e^{(\theta+\alpha)}\left(\frac{1-\zeta+\zeta\left(\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)}{B(\zeta)}\right)\\&+\frac{4\rho^2e^{(\theta+\alpha)}}{(B(\zeta))^2}\left(1+2\zeta\left(1+\frac{\tau^\zeta}{\Gamma(\zeta+1)}\right)+\zeta^2\left(1-2\frac{\tau^\zeta}{\Gamma(\zeta+1)}+\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)}\right)\right)+.....\end{aligned}\tag{25}$$

In particular, the analytical solution of equation (21) converge fastly to the exact solution of classical N-S equation for the velocity with q=0 and $\zeta=1$.

$$\begin{aligned}\vartheta(\theta,\alpha,\tau)&=-e^{2\rho\tau+\theta+\alpha},\\\nu(\theta,\alpha,\tau)&=e^{2\rho\tau+\theta+\alpha}.\end{aligned}\tag{26}$$

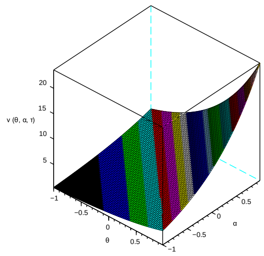


(a) $\zeta=0.5$ (b) $\zeta=1$

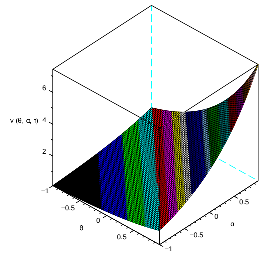
(c) Exact solution (d) Absolute error

Figure 4: For problem 2, the behaviour of velocity profile $\vartheta(\theta,\alpha,\tau)$ of N-S equation for $\zeta=0.5$, $\zeta=1$, exact solution, and absolute error $|\vartheta_{exa.}(\theta,\alpha,\tau)-\vartheta_{app.}(\theta,\alpha,\tau)|$,respectively with $\rho=0.5$, q=0, and $\tau=0.05$.

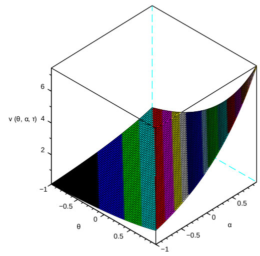
Problem 3: Consider $q_1=q_2=q_3=0$, then the time fractional $(2+1)$ dimensional NavierâĂŞStokes equation with atangana-Balenue fractional derivative is given as:



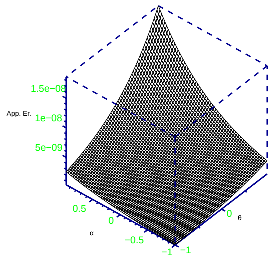
(a) $\zeta = 0.5$



(b) $\zeta = 1$



(c) Exact solution



(d) Absolute error

Figure 5: For problem 2, the behaviour of velocity profile $\nu(\theta,\alpha,\tau)$ of N-S equation for $\zeta = 0.5$, $\zeta = 1$, exact solution, and absolute error $|\vartheta_{exa.}(\theta,\alpha,\tau) - \vartheta_{app.}(\theta,\alpha,\tau)|$,respectively with $\rho = 0.5$, q=0, and $\tau = 0.05$.

$$\begin{aligned}
{}^{AB}D_{\tau}^{\zeta}\vartheta(\theta,\alpha,\eta,\tau) &= \rho\left[\frac{\partial^2\vartheta}{\partial\theta^2} + \frac{\partial^2\vartheta}{\partial\alpha^2} + \frac{\partial^2\vartheta}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\vartheta}{\partial\theta} + \nu\frac{\partial\vartheta}{\partial\alpha} + \omega\frac{\partial\vartheta}{\partial\eta}\right],\\
{}^{AB}D_{\tau}^{\zeta}\nu(\theta,\alpha,\eta,\tau) &= \rho\left[\frac{\partial^2\nu}{\partial\theta^2} + \frac{\partial^2\nu}{\partial\alpha^2} + \frac{\partial^2\nu}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\nu}{\partial\theta} + \nu\frac{\partial\nu}{\partial\alpha} + \omega\frac{\partial\nu}{\partial\eta}\right],\\
{}^{AB}D_{\tau}^{\zeta}\omega(\theta,\alpha,\eta,\tau) &= \rho\left[\frac{\partial^2\omega}{\partial\theta^2} + \frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\omega}{\partial\theta} + \nu\frac{\partial\omega}{\partial\alpha} + \omega\frac{\partial\omega}{\partial\eta}\right].
\end{aligned} \tag{27}$$

with the initial conditions (ICs):

$$\begin{cases}
\vartheta(\theta,\alpha,\eta,0) = -0.5\theta + \alpha + \eta,\\
\nu(\theta,\alpha,\eta,0) = \theta - 0.5\alpha + \eta,\\
\omega(\theta,\alpha,\eta,0) = \theta + \alpha - 0.5\eta .
\end{cases} \tag{28}$$

where, $0 < \zeta \leq 1$.

Taking the Shehu transform to both sides of the equation (27)and with the help of equation (28), we have:

$$\begin{aligned}
S[\vartheta] &= \frac{\mu}{s}[-0.5\theta + \alpha + \eta]\\
&\quad + \left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left(\rho\left[\frac{\partial^2\vartheta}{\partial\theta^2} + \frac{\partial^2\vartheta}{\partial\alpha^2} + \frac{\partial^2\vartheta}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\vartheta}{\partial\theta} + \nu\frac{\partial\vartheta}{\partial\alpha} + \omega\frac{\partial\vartheta}{\partial\eta}\right]\right)\right],\\
S[\nu] &= \frac{\mu}{s}[\theta - 0.5\alpha + \eta]\\
&\quad + \left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left(\rho\left[\frac{\partial^2\nu}{\partial\theta^2} + \frac{\partial^2\nu}{\partial\alpha^2} + \frac{\partial^2\nu}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\nu}{\partial\theta} + \nu\frac{\partial\nu}{\partial\alpha} + \omega\frac{\partial\nu}{\partial\eta}\right]\right)\right],\\
S[\omega] &= \frac{\mu}{s}[\theta + \alpha - 0.5\eta]\\
&\quad + \left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right) S\left(\rho\left[\frac{\partial^2\omega}{\partial\theta^2} + \frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\eta^2}\right] - \left[\vartheta\frac{\partial\omega}{\partial\theta} + \nu\frac{\partial\omega}{\partial\alpha} + \omega\frac{\partial\omega}{\partial\eta}\right]\right)\right],
\end{aligned} \tag{29}$$

Next, taking the inverse Shehu transform to both sides of (29), one can get:

$$\begin{aligned}
\vartheta(\theta,\alpha,\eta,\tau) &= [-0.5\theta+\alpha+\eta] + S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right.\\
&\left. S\left[\rho\left[\frac{\partial^2\vartheta}{\partial\theta^2}+\frac{\partial^2\vartheta}{\partial\alpha^2}+\frac{\partial^2\vartheta}{\partial\eta^2}\right]-\left[\vartheta\frac{\partial\vartheta}{\partial\theta}+\nu\frac{\partial\vartheta}{\partial\alpha}+\omega\frac{\partial\vartheta}{\partial\eta}\right]\right]\right],\\
\nu(\theta,\alpha,\eta,\tau) &= [\theta-0.5\alpha+\eta] + S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right.\\
&\left. S\left[\rho\left[\frac{\partial^2\nu}{\partial\theta^2}+\frac{\partial^2\nu}{\partial\alpha^2}+\frac{\partial^2\nu}{\partial\eta^2}\right]-\left[\vartheta\frac{\partial\nu}{\partial\theta}+\nu\frac{\partial\nu}{\partial\alpha}+\omega\frac{\partial\nu}{\partial\eta}\right]\right]\right],\\
\omega(\theta,\alpha,\eta,\tau) &= [\theta+\alpha-0.5\eta] + S^{-1}\left[\left(\frac{1-\zeta+\zeta\left(\frac{\mu}{s}\right)^{\zeta}}{B(\zeta)}\right)\right.\\
&\left. S\left[\rho\left[\frac{\partial^2\omega}{\partial\theta^2}+\frac{\partial^2\omega}{\partial\alpha^2}+\frac{\partial^2\omega}{\partial\eta^2}\right]-\left[\vartheta\frac{\partial\omega}{\partial\theta}+\nu\frac{\partial\omega}{\partial\alpha}+\omega\frac{\partial\omega}{\partial\eta}\right]\right]\right],
\end{aligned} \tag{30}$$

According to our proposed scheme, we have the following results:

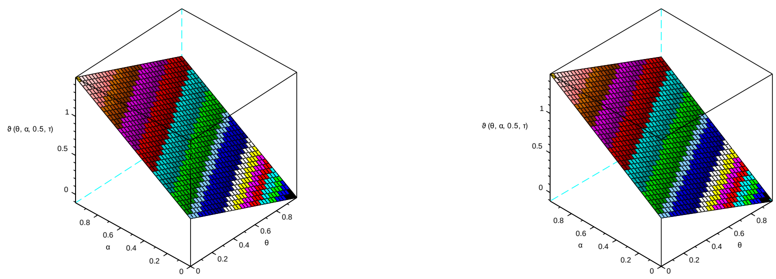
$$\begin{aligned}
\vartheta_0(\theta,\alpha,\eta,\tau) &= -0.5\theta+\alpha+\eta,\\
\nu_0(\theta,\alpha,\eta,\tau) &= \theta-0.5\alpha+\eta,\\
\omega_0(\theta,\alpha,\eta,\tau) &= \theta+\alpha-0.5\eta,\\
\vartheta_1(\theta,\alpha,\eta,\tau) &= \frac{-2.25\theta}{B(\zeta)}\left(1-\zeta+\zeta\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right),\\
\nu_1(\theta,\alpha,\eta,\tau) &= \frac{-2.25\alpha}{B(\zeta)}\left(1-\zeta+\zeta\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right),\\
\omega_1(\theta,\alpha,\eta,\tau) &= \frac{-2.25\eta}{B(\zeta)}\left(1-\zeta+\zeta\frac{\tau^{\zeta}}{\Gamma(\zeta+1)}\right),
\end{aligned}$$

Similarly, we obtain next terms in the same manner. Hence, the approximate solution of (27) is given as:

$$\begin{aligned}
\vartheta(\theta,\alpha,\tau) &= \vartheta_0(\theta,\alpha,\tau)+\vartheta_1(\theta,\alpha,\tau)+\vartheta_2(\theta,\alpha,\tau)+......,\\
\nu(\theta,\alpha,\tau) &= \nu_0(\theta,\alpha,\tau)+\nu_1(\theta,\alpha,\tau)+\nu_2(\theta,\alpha,\tau)+......,\\
\omega(\theta,\alpha,\tau) &= \omega_0(\theta,\alpha,\tau)+\omega_1(\theta,\alpha,\tau)+\omega_2(\theta,\alpha,\tau)+......
\end{aligned} \tag{31}$$

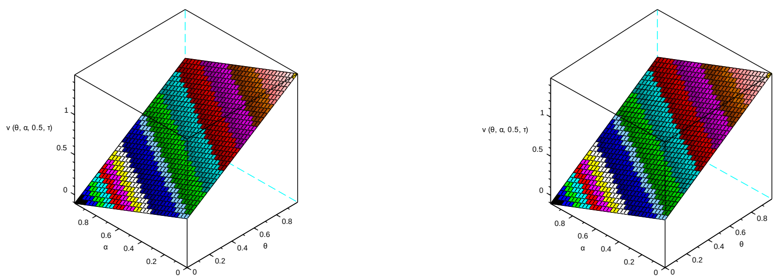
In particular, the analytical solution of equation (27) converge fastly to the exact solution of classical N-S equation for $\zeta = 1$.

$$\begin{aligned}
\vartheta(\theta,\alpha,\eta,\tau) &= \frac{-0.5\theta+\alpha+\eta-2.25\,\theta\tau}{1-2.25\,\tau^2},\\
\nu(\theta,\alpha,\eta,\tau) &= \frac{\theta-0.5\alpha+\eta-2.25\,\alpha\tau}{1-2.25\,\tau^2},\\
\omega(\theta,\alpha,\eta,\tau) &= \frac{\theta+\alpha-0.5\eta-2.25\,\eta\tau}{1-2.25\,\tau^2}.
\end{aligned} \tag{32}$$



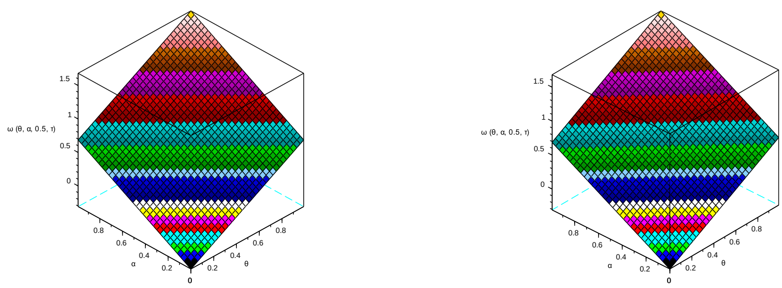
(a) ISTM solution (b) Exact solution

Figure 6: The surface ISTM solutions behaviour for $\vartheta(\theta, \alpha, 0.5, \tau)$ of problem 3 Vs exact solution, with $\rho = 0.5$, $\zeta = 1$ and $\tau = 0.05$.



(a) ISTM solution (b) Exact solution

Figure 7: The surface ISTM solutions behaviour for $\nu(\theta, \alpha, 0.5, \tau)$ of problem 3 Vs exact solution, with $\rho = 0.5$, $\zeta = 1$ and $\tau = 0.05$. .



(a) ISTM solution (b) Exact solution

Figure 8: The surface ISTM solutions behaviour for $\omega(\theta, \alpha, 0.5, \tau)$ of problem 3 Vs exact solution, with $\rho = 0.5$, $\zeta = 1$ and $\tau = 0.05$. .

5. Results and Discussions

In the present study, we analysis from Table I-II that the numerical results gain by the proposed technique for velocity profiles $\vartheta(\theta, \alpha, \tau)$, and $\nu(\theta, \alpha, \tau)$ presents the closed contact to exact solutions of N-S equations of problem 1 at $\rho = 0.5$, ζ =1, and $\tau = 0.01$. Fig. 1-2 depicts the comparative analysis of approximate solutions at q=0, $\zeta = 1$, $\rho = 0.5$, and $\tau = 0.1$. with the exact solution for problem 1. Fig.3 represent the nature of both velocity profiles $\vartheta(\theta, \alpha, \tau)$, and $\nu(\theta, \alpha, \tau)$ of problem 1 , respectively with q=0, $\zeta = 0.5$, $\rho = 0.5$, and $\tau = 0.1$. The behaviour for both velocity profiles $\vartheta(\theta, \alpha, \tau)$, and $\nu(\theta, \alpha, \tau)$ of problem 2 with $\zeta = 0.5$, $\zeta = 1$, exact solution, and absolute error $|\vartheta_{exa.}(\theta, \alpha, \tau) - \vartheta_{app.}(\theta, \alpha, \tau)|$,respectively are depicts in Figures 4 and 5 at $\rho = 0.5$, and $\tau = 0.05$. Moreover, Figures 6-8 represents the behaviour of approximate solutions of the N-S equation at $\zeta = 1$ in comparison with the exact solution for problem 3.

6. Conclusions

In this article, we applied iterative shehu transform method (ISTM), effectively to study time-fractional multidimensional fractional order N-S equations involving Atangana-Balenu derivative in caputo sense, and We are solving three applications of N-S equations in order to present the effectiveness and accuracy of proposed scheme.The numerical and graphical solutions obtained by iterative shehu transform method are presented that the proposed technique is computationally accurate, easy, and the approximate solutions converge rapidly fast to the exact solutions.

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