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Quantile Function for Rayleigh Distribution Kapasitans-Voltaj (C-V)

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Abstract

Rayleigh distribution is often used in wind speed, energy, physics, communication and lifetime analysis. This distribution and it's relation to other distribution is discussed in this paper. Quantile function of Rayleigh distribution is derived. Population and sample quantiles are calculated under certain conditions.

Rayleigh Dağılımının Yüzdelik Fonksiyonu

Anahtar kelimeler
Rayleigh dağılımı; Yüzdelik fonksiyon; Ters birikimli fonksiyon

Özet

Rayleigh dağılımı genellikle rüzgar hızı, enerji, fizik ve yaşam zamanı çözümlemelerinde kullanılır. Bu çalışmada bu dağılım ve diğer dağılımlar ile ilişkisi incelenmiştir. Rayleigh dağılımının yüzdelik fonksiyonu çıkarsanmıştır. Bazı koşullar altında kitle ve örneklem yüzdelikleri hesaplanmıştır.

1. Rayleigh Distribution

Rayleigh distribution is a member of continious probability distributions and it was introduced by Lord Rayleigh in 1880. It is often used in wind speed, energy, physics, communication and lifetime analysis. For example, it is used to model scattered signals that reach a receiver by multiple paths. Rayleigh (1980) derived it from the amplitude of sound resulting from many independent sources. This distribution is also connected with one or two dimensions and is sometimes referred to as "random walk" frequency distribution. The Rayleigh distribution is also used as a model for wind speed. The model describes the distribution of wind speed over the period of a year. This type of analysis is used for estimating the energy recovery from a wind turbine. The Rayleigh distribution is encountered in applications of probability theory (Johnson, 1994).

One parameter Rayleigh distribution is defined as

$$f(x)=\frac{x}{\sigma^2}exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right) \qquad (1)$$

$$0 \le x < \infty$$

where, σ is a non-negative scale parameter ($\sigma > 0$) (Papoulis,1984).

The mean and variance of a Rayleigh random variable are expressed as:

$$E(X)=\sigma\sqrt{\frac{\pi}{2}}$$

and,

$$Var(X)=\frac{4-\pi}{2}\sigma^2 .$$

It can be also defined that if X and Y are independent random variables with mean zero and standard deviation sigma, then $\sqrt{X^2+Y^2}$ is distributed as Rayleigh distribution with parameter sigma (http://en.wikipedia.org/wiki/Rayleigh_distribution). Rayleigh distribution is a right skewed distribution as shown in Figure 1. Figure 2 displays distribution for different values of σ.

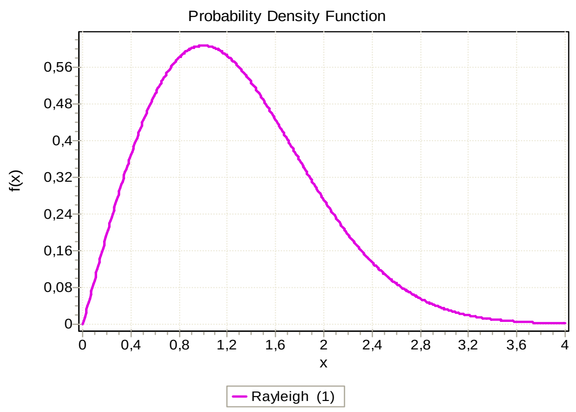

Figure 1. Probability density function of Rayleigh distribution for σ=1

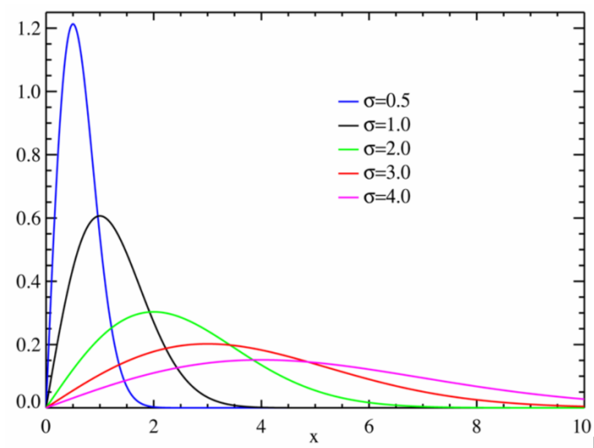

Figure 2. Probability density function of Rayleigh distribution for different values of σ.

Relation to other distributions

The relation between Rayleigh and other distribution can be described as follows:

- Let X and Y independent random variables having normal distribution with mean zero and variance σ^2, then Z is a Rayleigh distribution with parameter σ if $W=\sqrt{X^2+Y^2}$.
- Z is distributed Rayleigh with parameter 1, then Z^2 has a chi-square distribution with two degrees of freedom: $Z^2 \sim \chi_2^2$.
- If X has an exponential distribution with parameter λ, then $Y=\sqrt{2X\sigma^2\lambda}$ is distributed Rayleigh (σ).
- If Z~Rayleigh(σ), then $\sum_{i=1}^{w} Z_i^2$ has a gamma distribution with parameters ω and $2\sigma^2$:

$$Y=\sum_{i=1}^{w} Z_i^2 \sim \Gamma\left(w,2\sigma^2\right).$$

The Rayleigh distribution is a also special case of the Weibull distribution. If A and B are the parameters of the Weibull distribution, then the Rayleigh distribution with parameter b is equivalent to the Weibull distribution with parameters $A=\sqrt{2b}$ and B=2 (http://en.wikipedia.org/wiki/Rayleigh_distribution)

2. Quantiles

The often used method for summarizing data is to calculate the descriptive statistics such as mean, variance, standard deviation and mod. But in particular situations, quantiles provide more suitable information. The sample quantile is based on order statistics and calculated regardless of underlying distribution. Furthermore, a quantile function of a probability distribution is the inverse of its cumulative distribution function (Wackerly et.al.,2008). The p^{th} quantile of a dataset represents a summarizing value having less than or equal to p, where, $0 \le p \le 1$. We mean that, the quantiles are values which divide the distribution such that there is a given proportion of observations below the quantile. For example, the median is the p=0.50th quantile of data. The median is the central point of the distribution. Median value shows half the points are less than or equal to it and half are greater than or equal to it. We can estimate all quantiles from the underlying cumulative frequency distribution.

Let X_1, X_2,..,X_n be a sequence of i.i.d. random variables; $F(x)$ be a cumulative distribution function and, $p\in(0,1)$; x_p denote the p^{th} quantile which has the property that $F(x_p)=P\left(X \le x_p\right)$. The quantile function of underlying distribution is defined as:

$$Q_p=F^{-1}(p)$$
$$=\inf\left\{x\in R; p\le F(x)\right\}.$$

Quantiles are useful for example, in forecasts, risk assessments, quality control, lifetime analysis and so on.

3. Quantile function of Rayleigh distribution

Cumulative distribution function of Rayleigh distribution is:

$$F(x) = 1 - exp\left(\frac{x^2}{2\sigma^2}\right) \quad (2)$$

Therefore, quantile function of Rayleigh distribution can be derived as an inverse function of cumulative distribution function as follows:

$$F^{-1}\left(p;\sigma^2\right) = \sigma\sqrt{-2\,ln(p-1)} \quad (3)$$

Population quantiles can be calculated using Equation (3).

4. Numerical Examples

Population quantiles are obtained using Equation (3) under Rayleigh distribution by taking quantiles correspond to p=0.05 ; 0.10 ; 0.25 ; 0.50 ; 0.75 ; 0.90 ; 0.95 and σ^2=0.5, 1, 2, 3, 5, 10. Results are illustrated in Table 1. Moreover, random samples under Rayleigh distribution are generated for samples sizes n=10, 30, 50, 100, 500, 1000 and σ^2=1. Several descriptive statistics are calculated for each sample. This procedure is replicated for 1000 times. Mean values of these statistics calculated over 1000 replications are given in Table 2.

Table 1. Population quantiles

x_p	σ^2=0.5	σ^2=1	σ^2=2	σ^2=3	σ^2=5	σ^2=10
5%	0.1601	0.3203	0.6406	0.9608	1.6015	3.2029
10%	0.2295	0.4590	0.9181	1.3771	2.2952	4.5904
25%	0.3792	0.7585	1.5170	2.2755	3.7926	7.5853
50%	0.5887	1.1774	2.3548	3.5322	5.8870	11.7741
75%	0.8326	1.6651	3.3302	4.9953	8.3255	16.6511
90%	1.0729	2.1459	4.2919	6.4378	10.7298	21.4597
95%	1.2238	2.4477	4.8954	7.3432	12.2387	24.4774

Table 2. Mean values of statistics over 1000 replications for σ^2=1

Statistic	**Value**						
n	**10**	**30**	**50**	**100**	**500**	**1000**	**5000**
Range	1.9743	3.4009	3.0328	2.7612	3.7154	3.7617	3.7887
Mean	1.7849	1.1898	1.4217	1.2511	1.2248	1.2431	1.2594
Variance	0.4146	0.5879	0.5650	0.3908	0.4161	0.4005	0.4265
St.Dev.	0.6439	0.7667	0.7516	0.6251	0.6451	0.6328	0.6531
CV	0.3607	0.6444	0.5287	0.4997	0.5266	0.5091	0.5186
St.Error	0.2036	0.1399	0.1063	0.0625	0.0288	0.0200	0.0092
Skewness	-0.9320	1.0272	0.9680	0.4127	0.5215	0.6202	0.6062
Kurtosis	0.1527	1.3620	0.4370	-0.3151	0.1168	0.1691	0.1429
Min	0.6526	0.1017	0.3702	0.1587	0.0677	0.0644	0.0204
%5	0.6527	0.1626	0.4494	0.2675	0.2544	0.3646	0.3179
%10	0.6598	0.4085	0.5523	0.4256	0.4113	0.4704	0.4621
%25(Q_1)	1.3931	0.5271	0.8826	0.8510	0.7698	0.7501	0.7712
%50 (Q_2)	1.8914	1.0792	1.2646	1.1455	1.1719	1.1626	1.1873
%75 (Q_3)	2.2344	1.6938	1.6813	1.7395	1.6734	1.6493	1.6602
%90	2.5966	2.1764	2.5441	2.1376	2.0799	2.0953	2.1417
%95	2.6270	2.9560	3.2122	2.4440	2.3510	2.3829	2.4546
Max	2.6270	3.5027	3.4031	2.9200	3.7832	3.8262	3.8092

In Table 2, while sample size inceases, the sample quantiles give approximate estimation to population quantiles. For example, from Table 2, for n=5000 and σ^2=1, mean value of 5th quantile is 0.3179. From Table 1, 5th quantile is 0.3203 for σ^2=1. Note that, these values are quite close. For large values of parameter σ^2, quantiles also give large values.

In Table 2, skewness values for large sample sizes illustrate that Rayleigh distribution attributes a right skewed shape.

As a conclusion, it is practice way to estimate the quantiles of a complicated distribution by using order statistics. This article has demonstrated how the use of quantile function of Rayleigh distribution.

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