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On the ℓ_p Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

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Abstract

In this study, we have established upper and lower bounds for the ℓ_p norms of the matrices T and H where T and H are Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices respectively.

Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrislerinin ℓ_p Normları Üzerine

Anahtar kelimeler
Quaternion Cauchy-Toeplitz Matrices, Quaternion Cauchy-Hankel Matrices, Euclidean Norm, ℓ_p norm.
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Özet

Bu çalışmada, T , H sırasıyla Quaternion Cauchy-Toeplitz ve Quaternion Cauchy-Hankel Matrisleri olmak üzere, T ve H matrislerinin ℓ_p normlar için alt ve üst sınırlar elde ettik.

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1.Introduction and Preliminaries

In quantum physic, the family of quaternions plays an important role. But in mathematics they generally play a role in algebraic systems, skew fields or noncommutative division algebras, matrices in commutative rings take attention but, matrices with quaternion entries has not been investigated very much yet. But in recent times quaternions are in order of day.

The main obstacles in the study of quaternion matrices, as expected come from the noncommutative multiplication of quaternions. One will find that working on a quaternion matrix problem is often equivalent to dealing with a pair of complex matrices [Zhang(1997), Lee(1949)]. Recently, the studies concern with matrices norms, has been given by several authors, see for instance [Moenck(1977),Mathias(1990),Visick(2000),Zielke (1988),Horn and Johnson(1991),Bozkurt(1998), Bozkurt(1996),Bozkurt(1996),Türkmen and Bozkurt (2002)] and references cited therein. In this paper, we have obtained some a lower and an upper bounds for the ℓ_p of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices. Now, we need the following definitions and preliminaries.

Definition 1. *Let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of the complex and real numbers respectively. Let $\mathbb{Q}$ be a four-dimensional vector space over $\mathbb{R}$ with an ordered basis, denoted by $\boldsymbol{e}, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. A real quaternion, simply called quaternion, is a vector*

$$x = x_0\boldsymbol{e} + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \in \mathbb{Q}$$

with real coefficients x_0, x_1, x_2 *and* x_3.

Besides the addition and the scalar multiplication of the vector space $\mathbb{Q}$ over $\mathbb{R}$, the product of any two quaternions $\boldsymbol{e}, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are defined by the requirement that $\boldsymbol{e}$ act as a identity and by the table

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Let $M_{m\times n}(\mathbb{Q})$, simply $M_n(\mathbb{Q})$ when $m = n$, denote the collection of all $m\times n$ matrices with quaternion entries.

Definition 2. *Let* $A = A_1 + A_2\boldsymbol{j} \in M_n(\mathbb{Q})$, *where* A_1, A_2 *are* $n\times n$ *complex matrices. We shall call the* $2n\times 2n$ *complex matrix*

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix},$$

uniquely determined by A, *the complex adjoint matrix or adjoint of the quaternion matrix* A [Lee(1949)].

Now we give some preliminaries related to our study. Let A be any $n\times n$ matrix. The ℓ_p norms of the matrix A are defined as

$$\|A\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p} \quad (1\le p<\infty).$$

If $p=\infty$, then

$$\|A\|_\infty = \lim_{n\to\infty}\|A\|_p = \max_{i,j}|a_{ij}|.$$

The well-known Euclidean norm of matrix A is

$$\|A\|_E = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$

and also the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\le i\le n}\lambda_i(A^H A)}$$

where A is $m\times n$ and A^H is the conjugate transpose of the matrix A. The following inequality holds:

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E$$

[Zielke(1988)]. A function Ψ is called a psi (or digamma) function if

$$(1.2) \qquad \Psi(x) = \frac{d}{dx}\{\ln|\Gamma(x)|\}$$

where

$$\Gamma(x) = \int_0^x e^{-t}t^{x-1}dt.$$

The n th derivatives of a Ψ function is called a polygamma function

$$(1.3) \qquad \Psi(n,x) = \frac{d}{dx^n}\Psi(x) = \frac{d}{dx^n}\left\{\frac{d}{dx}\ln|\Gamma(x)|\right\}.$$

If $n=0$ then $\Psi(0,x) = \Psi(x) = \left\{\frac{d}{dx}\ln|\Gamma(x)|\right\}$. On the other hand, if $a>0$, b is any number and n is positive integer, then

$$(1.4) \qquad \lim_{n\to\infty}\Psi(a, n+b) = 0$$

[Moencko(1977)]. Throughout the paper $\mathbb{Z}^+$ and $\mathbb{R}^+$ will represent the sets of positive integers and positive real numbers, respectively.

2. Matrices of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel

Definition 3. *The matrices in* x *quaternion which definitions have gave from Definition 1, for* $2\le t,l,m\in\mathbb{Z}^+$ *and* $p=1,2,\cdots,n$, $r=1,2,\cdots,n$ *and* $x_0=0$, $x_1=\frac{1}{\frac{1}{t}+p-r}$, $x_2=\frac{1}{\frac{1}{l}+p-r}$, $x_3=\frac{1}{\frac{1}{m}+p-r}$ *defining as below*

$$T = \left[\frac{\boldsymbol{i}}{\frac{1}{t}+p-r} + \frac{\boldsymbol{j}}{\frac{1}{l}+p-r} + \frac{\boldsymbol{k}}{\frac{1}{m}+p-r}\right]$$

is called Quaternion Cauchy-Toeplitz matrix. By the similar way

$$H = \left[\frac{\boldsymbol{i}}{\frac{1}{t}+p+r} + \frac{\boldsymbol{j}}{\frac{1}{l}+p+r} + \frac{\boldsymbol{k}}{\frac{1}{m}+p+r}\right]$$

is called Quaternion Cauchy-Hankel matrix.

In this section we are going to find upper and lower bounds for the Euclidean norm and the ℓ_p norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel matrices in (2.1) and (2.2).

2.1. The ℓ_p Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

Lets give some upper and lower bounds following theorem, for ℓ_p norms, of in definition (2.1) and (2.2) Quaternion Cauchy-Toeplitz and Quaternion

Cauchy-Hankel matrices.

Theorem 1. *For* ℓ_p *norm of definition* (2.1) *Quaternion Cauchy-Toeplitz matrix* $2 \le t, l, m \in \mathbb{Z}^+$ *let* p *be positive even integer*

$$\frac{p}{2p-1}\left\{\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,1+\tfrac{1}{t}\right)\right]\right.$$
$$+\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,1+\tfrac{1}{l}\right)\right]$$
$$\left.+\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,1+\tfrac{1}{m}\right)\right]\right\}^{\frac{1}{p}} \le n^{-\frac{1}{p}}\|T\|_p$$

(2.3)

$$n^{-\frac{1}{p}}\|T\|_p \le \left\{(t^2+l^2+m^2)^{\frac{p}{2}}+\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,1+\tfrac{1}{t}\right)\right]\right.$$
$$+\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,1+\tfrac{1}{l}\right)\right]$$
$$\left.+\frac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,1+\tfrac{1}{m}\right)\right]\right\}^{\frac{1}{p}}$$

are valid as upper and lower bounds.

Proof From definition ℓ_p norm

$$\|T\|_p^p = n(t^2+l^2+m^2)^{\frac{p}{2}} + \sum_{s=1}^{n-1}(n-s)$$
$$\times\left\{\left[\frac{t^2}{(1-st)^2}+\frac{l^2}{(1-sl)^2}+\frac{m^2}{(1-sm)^2}\right]+\left[\frac{t^2}{(1+st)^2}+\frac{l^2}{(1+sl)^2}+\frac{m^2}{(1+sm)^2}\right]\right\}^{\frac{1}{p}}$$
$$= n(t^2+l^2+m^2)^{\frac{p}{2}}$$
$$+\frac{(tn-1)}{(p-1)!t}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,n-\tfrac{1}{t}\right)\right]$$
$$+\frac{(tn+1)}{(p-1)!t}\left[\Psi\left(p-1,1+\tfrac{1}{t}\right)+\Psi\left(p-1,n+\tfrac{1}{t}\right)\right]$$
$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{t}\right)-\Psi\left(p-2,n-\tfrac{1}{t}\right)+\Psi\left(p-2,1+\tfrac{1}{t}\right)-\Psi\left(p-2,n+\tfrac{1}{t}\right)\right]$$
$$+\frac{(nl-1)}{(p-1)!l}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,n-\tfrac{1}{l}\right)\right]$$
$$+\frac{(nl+1)}{(p-1)!l}\left[\Psi\left(p-1,1+\tfrac{1}{t}\right)+\Psi\left(p-1,n+\tfrac{1}{t}\right)\right]$$
$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{l}\right)-\Psi\left(p-2,n-\tfrac{1}{l}\right)+\Psi\left(p-2,1+\tfrac{1}{l}\right)-\Psi\left(p-2,n+\tfrac{1}{l}\right)\right]$$
$$+\frac{(nm-1)}{(p-1)!m}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,n-\tfrac{1}{m}\right)\right]$$
$$+\frac{(nm+1)}{(p-1)!m}\left[\Psi\left(p-1,1+\tfrac{1}{m}\right)+\Psi\left(p-1,n+\tfrac{1}{m}\right)\right]$$
$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{m}\right)-\Psi\left(p-2,n-\tfrac{1}{m}\right)+\Psi\left(p-2,1+\tfrac{1}{m}\right)-\Psi\left(p-2,n+\tfrac{1}{m}\right)\right]$$

(2.4)

is written. If we divide both of side of equality with n and if take a limit for $n \to \infty$.

$$\lim_{n\to\infty} \tfrac{1}{n}\|T\|_p^p = \lim_{n\to\infty}\Big\{(t^2+l^2+m^2)^{\frac{p}{2}} + \tfrac{(tn-1)}{n(p-1)!t}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,n-\tfrac{1}{t}\right)\right]$$

$$+\tfrac{(tn+1)}{n(p-1)!t}\left[\Psi\left(p-1,1+\tfrac{1}{t}\right)+\Psi\left(p-1,n+\tfrac{1}{t}\right)\right]$$

$$+\tfrac{1}{n(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{t}\right)-\Psi\left(p-2,n-\tfrac{1}{t}\right)+\Psi\left(p-2,1+\tfrac{1}{t}\right)-\Psi\left(p-2,n+\tfrac{1}{t}\right)\right]$$

$$+\tfrac{(nl-1)}{n(p-1)!l}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,n-\tfrac{1}{l}\right)\right]$$

$$+\tfrac{(nl+1)}{n(p-1)!l}\left[\Psi\left(p-1,1+\tfrac{1}{t}\right)+\Psi\left(p-1,n+\tfrac{1}{t}\right)\right]$$

$$+\tfrac{1}{n(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{l}\right)-\Psi\left(p-2,n-\tfrac{1}{l}\right)+\Psi\left(p-2,1+\tfrac{1}{l}\right)-\Psi\left(p-2,n+\tfrac{1}{l}\right)\right]$$

$$+\tfrac{(nm-1)}{n(p-1)!m}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,n-\tfrac{1}{m}\right)\right]$$

$$+\tfrac{(nm+1)}{n(p-1)!m}\left[\Psi\left(p-1,1+\tfrac{1}{m}\right)+\Psi\left(p-1,n+\tfrac{1}{m}\right)\right]$$

$$+\tfrac{1}{n(p-2)!}\left[\Psi\left(p-2,1-\tfrac{1}{m}\right)-\Psi\left(p-2,n-\tfrac{1}{m}\right)+\Psi\left(p-2,1+\tfrac{1}{m}\right)-\Psi\left(p-2,n+\tfrac{1}{m}\right)\right]\Big\}$$

is written and we obtain,

$$\tfrac{1}{n}\|T\|_p^p \le \Big\{(t^2+l^2+m^2)^{\frac{p}{2}} + \tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,1+\tfrac{1}{t}\right)\right]$$

$$(2.5) \qquad +\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,1+\tfrac{1}{l}\right)\right]$$

$$+\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,1+\tfrac{1}{m}\right)\right]\Big\}$$

in this case, if we take a root of *p.* degree both of side of (2.6) inequality. We obtain,

$$n^{-\frac{1}{p}}\|T\|_p \le \Big\{(t^2+l^2+m^2)^{\frac{p}{2}} + \tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,1+\tfrac{1}{t}\right)\right]$$

$$(2.6) \qquad +\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,1+\tfrac{1}{l}\right)\right]$$

$$+\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,1+\tfrac{1}{m}\right)\right]\Big\}^{\frac{1}{p}}$$

it's already upper bound for $n^{-\frac{1}{p}}\|T\|_p$.

Lets obtain lower bound now. Let p be positive even integer, if we throw out first term of (2.5), right hand has been gotten smaller, so

$$n^{-\frac{1}{p}}\|T\|_p \geq \frac{p}{2p-1}\left\{\sum_{s=1}^{n-1}(1-\frac{s}{n})\left\{\left[\frac{t^2}{(1-st)^2}+\frac{l^2}{(1-sl)^2}+\frac{m^2}{(1-sm)^2}\right]\right.\right.$$
$$\left.\left.+\left[\frac{t^2}{(1+st)^2}+\frac{l^2}{(1+sl)^2}+\frac{m^2}{(1+sm)^2}\right]\right\}^{\frac{p}{2}}\right\}^{\frac{1}{p}}$$

If we take a limit for $n \to \infty$.

$$n^{-\frac{1}{p}}\|T\|_p \geq \tfrac{p}{2p-1}\left\{\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{t}\right)+\Psi\left(p-1,1+\tfrac{1}{t}\right)\right]\right.$$

(2.7)
$$+\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{l}\right)+\Psi\left(p-1,1+\tfrac{1}{l}\right)\right]$$
$$\left.+\tfrac{1}{(p-1)!}\left[\Psi\left(p-1,1-\tfrac{1}{m}\right)+\Psi\left(p-1,1+\tfrac{1}{m}\right)\right]\right\}^{\frac{1}{p}}$$

is obtained. It's already lower bound for $n^{-\frac{1}{p}}\|T\|_p$. Hence, proof has been completed from (2.6) and (2.7).

Theorem2. *For ℓ_p norm of definition(2.2) Quaternion Cauchy-Hankel matrix $2 \leq t,l,m \in \mathbb{Z}^+$ let p be positive even integer*

$$\|H\|_p \leq \left\{\frac{(-1)^{p-1}}{(p-1)!}\left[\frac{1+t}{t}\Psi\left(p-1,1+\frac{1+t}{t}\right)+\frac{1+l}{l}\Psi\left(p-1,1+\frac{1+l}{l}\right)+\frac{1+m}{m}\Psi\left(p-1,1+\frac{1+m}{m}\right)\right]\right.$$
$$\left.+\frac{(-1)^{p-1}}{(p-2)!}\left[\Psi\left(p-2,1+\frac{1+t}{t}\right)+\Psi\left(p-2,1+\frac{1+l}{l}\right)+\Psi\left(p-2,1+\frac{1+m}{m}\right)\right]\right\}^{\frac{1}{p}}$$

is valid for upper bound.

Proof. From definition of ℓ_p norm

(2.8)
$$\|H\|_p^p = \sum_{s=1}^{n} s\left[\frac{t^2}{(1+(s+1)t)^2}+\frac{l^2}{(1+(s+1)l)^2}+\frac{m^2}{(1+(s+1)m)^2}\right]^{\frac{p}{2}}$$
$$+\sum_{s=1}^{n-1}(n-s)\left[\frac{t^2}{(1+(n+s+1)t)^2}+\frac{l^2}{(1+(n+s+1)l)^2}+\frac{m^2}{(1+(n+s+1)m)^2}\right]^{\frac{p}{2}}$$

if we compute sum where both of side of equality. We obtain

$$\sum_{s=1}^{n} s\left[\frac{t^2}{(1+(s+1)t)^2}+\frac{l^2}{(1+(s+1)l)^2}+\frac{m^2}{(1+(s+1)m)^2}\right]^{\frac{p}{2}}$$

$$=\frac{1}{(p-1)!}\frac{1+t}{t}\left[\Psi\left(p-1,n+1+\frac{1+t}{t}\right)-\Psi\left(p-1,1+\frac{1+t}{t}\right)\right]$$

$$+\frac{1+l}{l}\left[\Psi\left(p-1,n+1+\frac{1+l}{l}\right)-\Psi\left(p-1,1+\frac{1+l}{l}\right)\right]$$

$$+\frac{1+m}{m}\left[\Psi\left(p-1,n+1+\frac{1+m}{m}\right)-\Psi\left(p-1,1+\frac{1+m}{m}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,n+1+\frac{1+t}{t}\right)-\Psi\left(p-2,1+\frac{1+t}{t}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,n+1+\frac{1+l}{l}\right)-\Psi\left(p-2,1+\frac{1+l}{l}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,n+1+\frac{1+m}{m}\right)-\Psi\left(p-2,1+\frac{1+m}{m}\right)\right]$$

and

$$\sum_{s=1}^{n-1}(n-s)\left[\frac{t^2}{(1+(n+s+1)t)^2}+\frac{l^2}{(1+(n+s+1)l)^2}+\frac{m^2}{(1+(n+s+1)m)^2}\right]^{\frac{p}{2}}$$

$$=\frac{1}{(p-1)!}\frac{1+t+2tn}{t}\left[\Psi\left(p-1,1+\frac{1+t+tn}{t}\right)-\Psi\left(p-1,1+\frac{1+t+tn}{t}\right)\right]$$

$$+\frac{1+l+2nl}{l}\left[\Psi\left(p-1,1+\frac{1+l+nl}{l}\right)-\Psi\left(p-1,1+\frac{1+l+nl}{l}\right)\right]$$

$$+\frac{1+m+2mn}{m}\left[\Psi\left(p-1,1+\frac{1+m+mn}{m}\right)-\Psi\left(p-1,1+\frac{1+m+mn}{m}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1+\frac{1+t+tn}{t}\right)-\Psi\left(p-2,1+\frac{1+t+tn}{t}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1+\frac{1+l+nl}{l}\right)-\Psi\left(p-2,1+\frac{1+l+nl}{l}\right)\right]$$

$$+\frac{1}{(p-2)!}\left[\Psi\left(p-2,1+\frac{1+m+mn}{m}\right)-\Psi\left(p-2,1+\frac{1+m+mn}{m}\right)\right]$$

if we take limit of (2.8) for $n\to\infty$. We use properties of polygamma function.

$$\lim_{n\to\infty}\sum_{s=1}^{n-1}(n-s)\left[\frac{t^2}{(1+(n+s+1)t)^2}+\frac{l^2}{(1+(n+s+1)l)^2}+\frac{m^2}{(1+(n+s+1)m)^2}\right]^{\frac{p}{2}}=0 \tag{2.9}$$

and

$$\lim_{n\to\infty}\sum_{s=1}^{n} s\left[\frac{t^2}{(1+(s+1)t)^2}+\frac{l^2}{(1+(s+1)l)^2}+\frac{m^2}{(1+(s+1)m)^2}\right]^{\frac{p}{2}}$$

$$=\tfrac{(-1)^{p-1}}{(p-1)!}\left[\tfrac{1+t}{t}\Psi\left(p-1,1+\tfrac{1+t}{t}\right)+\tfrac{1+l}{l}\Psi\left(p-1,1+\tfrac{1+l}{l}\right)+\tfrac{1+m}{m}\Psi\left(p-1,1+\tfrac{1+m}{m}\right)\right] \tag{2.10}$$

$$+\tfrac{(-1)^{p-1}}{(p-2)!}\left[\Psi\left(p-2,1+\tfrac{1+t}{t}\right)+\Psi\left(p-2,1+\tfrac{1+l}{l}\right)+\Psi\left(p-2,1+\tfrac{1+m}{m}\right)\right]$$

Hence, we obtain

$$\|H\|_p^p\leq\tfrac{(-1)^{p-1}}{(p-1)!}\left[\tfrac{1+t}{t}\Psi\left(p-1,1+\tfrac{1+t}{t}\right)+\tfrac{1+l}{l}\Psi\left(p-1,1+\tfrac{1+l}{l}\right)+\tfrac{1+m}{m}\Psi\left(p-1,1+\tfrac{1+m}{m}\right)\right]$$

$$+\tfrac{(-1)^{p-1}}{(p-2)!}\left[\Psi\left(p-2,1+\tfrac{1+t}{t}\right)+\Psi\left(p-2,1+\tfrac{1+l}{l}\right)+\Psi\left(p-2,1+\tfrac{1+m}{m}\right)\right]$$

from equalities of (2.9) and (2.10). If we root of p. both of side of inequality, we obtain

$$\|H\|_p\leq\left\{\tfrac{(-1)^{p-1}}{(p-1)!}\left[\tfrac{1+t}{t}\Psi\left(p-1,1+\tfrac{1+t}{t}\right)+\tfrac{1+l}{l}\Psi\left(p-1,1+\tfrac{1+l}{l}\right)+\tfrac{1+m}{m}\Psi\left(p-1,1+\tfrac{1+m}{m}\right)\right]\right.$$

$$\left.+\tfrac{(-1)^{p-1}}{(p-2)!}\left[\Psi\left(p-2,1+\tfrac{1+t}{t}\right)+\Psi\left(p-2,1+\tfrac{1+l}{l}\right)+\Psi\left(p-2,1+\tfrac{1+m}{m}\right)\right]\right\}^{\frac{1}{p}}$$

Hence, the proof is completed.

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