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GLOBAL NONEXISTENCE OF SOLUTIONS FOR THE HIGHER ORDER KIRCHHOFF TYPE SYSTEM WITH LOGARITHMIC NONLINEARITIES

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Abstract. This paper deals with the system of a class of nonlinear higher-order Kirchhoff-type equations with logarithmic nonlinearities. Under the appropriate assumptions, the theorem of global nonexistence is established at positive initial energy levels.

1. Introduction

In this paper, we study the following initial-boundary value problem
(1.1)
$$\begin{cases} u_{tt}+M\left(\|D^m u\|^2+\|D^m v\|^2\right)(-\triangle)^m u+(-\triangle)^m u_t=|u|^{r-2}u\ln|u|, & x\in\Omega,\ t>0,\\ v_{tt}+M\left(\|D^m u\|^2+\|D^m v\|^2\right)(-\triangle)^m v+(-\triangle)^m v_t=|v|^{r-2}v\ln|v|, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x), & x\in\Omega,\\ \frac{\partial^i}{\partial\nu^i}u(x,t)=0,\quad \frac{\partial^i}{\partial\nu^i}v(x,t)=0,\ i=0,1,2,...m-1, & x\in\partial\Omega,\ t\geq 0,\end{cases}$$
where $Du=\nabla u=\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...\frac{\partial u}{\partial x_n}\right)$ and $r\geq 2\gamma+2$ are real numbers and $m\geq 1$ are positive integers. The Kirchhoff term $M(s)=\beta_1+\beta_2 s^\gamma$, $\gamma>0$, $\beta_1\geq 1$, $\beta_2\geq 0$. We will take $\beta_1=\beta_2=1$ for simplify. $\Omega\subset R^n$ is a regular and bounded domain with smooth boundary $\partial\Omega$. And v denotes the outer normal.

Problem (1.1) is a generalization of a model considered by Kirchhoff [9]. Kirchhoff type equation has in the mathematical description of small amplitude vibrations of an elastic string. In the case $M(s)=1$, $m=1$ and $p\geq 2$, a problem of the single wave equation of the (1.1) form becomes

$$u_{tt}-\triangle u+f(u_t)=|u|^{p-2}u\ln|u|. \tag{1.2}$$

Several results of the problem (1.2) concerning local or global existence and qualitative theory have been studied by many mathematicians(see [1, 2, 4, 5, 6, 7, 10, 13, 19]). In the case $M(s)\neq 1$, $m=1$ and $p\geq 2$, a problem of the single wave

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equation of (1.1) becomes the Kirchhoff-type equation which has been investigated by many authors [3, 14, 18].

In the case $M(s) \neq 1$, $m > 1$ the single form of the problem (1.1) without logarithmic source terms have been discussed by many authors (see [12, 16, 15, 11]).

Let us finally mention that wave equation system with logarithmic nonlinearies was studied by Wang et al [17].They proved global existence and finite time blow up under the different conditions by employing the potential well method and concavity method. In [8], the authors studied (1.1) problem with nonlinear damping terms. They established global existence and decay estimates.

The rest of this work is organized as follows. In Section 3, our aim is to prove the blow up of solution for $E(0) > 0$. In section 2, we give some lemmas which will be useful.

2. Preliminaries

Now we define the potential energy functional of problem (1.1)

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left(\|D^m u\|^2 + \|D^m v\|^2\right) + \frac{1}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad -\frac{1}{r}\left(\int_\Omega |u|^r \ln|u|\,dx + \int_\Omega |v|^r \ln|v|\,dx\right) + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right) \end{aligned} \tag{2.1}$$

and the Nehari functional

$$\begin{aligned} I(u,v) &= \left(\|D^m u\|^2 + \|D^m v\|^2\right) + \left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad -\left(\int_\Omega |u|^r \ln|u|\,dx + \int_\Omega |v|^r \ln|v|\,dx\right). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$\begin{aligned} J(u,v) &= \frac{I(u,v)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\ &\quad + \frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\ &\quad + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right). \end{aligned} \tag{2.3}$$

Then we can introduce the stable set

$$W = \left\{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) : I(u,v) > 0\right\} \cup \{0\},$$

the outer space of the potential well

$$V = \left\{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega) : I(u,v) < 0\right\}.$$

We introduce the total energy

$$
\begin{aligned}
E(u,v) &= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\quad +\frac{1}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1} \\
&\quad -\frac{1}{r}\left(\int_\Omega |u|^r \ln|u|\,dx+\int_\Omega |v|^r \ln|v|\,dx\right)+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right). \qquad (2.4)
\end{aligned}
$$

For $(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)\,,\ t\geq 0$

$$
\begin{aligned}
E(0) &= \frac{1}{2}\left(\|u_1\|^2+\|v_1\|^2\right)+\frac{1}{2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right) \\
&\quad +\frac{1}{2\gamma+2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)^{\gamma+1} \\
&\quad -\frac{1}{r}\left(\int_\Omega |u_0|^r \ln|u_0|\,dx+\int_\Omega |v_0|^r \ln|v_0|\,dx\right)+\frac{1}{r^2}\left(\|u_0\|_r^r+\|v_0\|_r^r\right). \qquad (2.5)
\end{aligned}
$$

is the initial total energy. We introduce by (2.4) and (2.3)

$$
E(u,v)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+J\left(u,v\right), \qquad (2.6)
$$

Lemma 2.1. *Let k be a number with $2\leq k<\infty$ if $n\leq 2s$ and $2\leq k\leq \frac{2n}{n-2k}$ if $n>2s$. Then there is a constant such that*

$$
\|u\|_k\leq C\left\|D^m u\right\|,\forall(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega).
$$

Lemma 2.2. *$E(t)$ is a nonincreasing function for $t\geq 0$ and*

$$
E'(t)=-\left(\|D^m u_t\|^2+\|D^m v_t\|^2\right)\leq 0. \qquad (2.7)
$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation of (1.1) by v_t, and integrating on Ω, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\frac{d}{dt}\left(\frac{1}{r^2}\|u\|_r^r-\frac{1}{r}\int_\Omega |u|^r\ln|u|\,dx\right) \\
&\frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma}\right)\frac{d}{dt}\|D^m u\|^2 \\
&= -\int_\Omega |D^m u_t|^2\,dx, \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_t\|^2+\frac{d}{dt}\left(\frac{1}{r^2}\|v\|_r^r-\frac{1}{r}\int_\Omega |v|^r\ln|v|\,dx\right) \\
&\frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma}\right)\frac{d}{dt}\|D^m v\|^2 \\
&= -\int_\Omega |D^m v_t|^2\,dx. \qquad (2.9)
\end{aligned}
$$

A summarization of (2.8) and (2.9) hence gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2+\|v_t\|^2\right)\\
&+\frac{1}{2}\left(1+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma}\right)\frac{d}{dt}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&\frac{d}{dt}\left(-\frac{1}{r}\left(\int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx\right)+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\right)\\
(2.10)\quad =\ & -\left(\int_\Omega |D^m u_t|^2\,dx+\int_\Omega |D^m v_t|^2\,dx\right).
\end{aligned}$$

Integrating (2.10) with respect to t on $[0,t]$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&+\frac{1}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\\
&-\frac{1}{r}\left(\int_\Omega |u|^r\ln|u|\,dx+\int_\Omega |v|^r\ln|v|\,dx\right)\\
&+\left(\int_0^t \|D^m u_\tau\|^2\,d\tau+\int_\Omega \|D^m v_\tau\|^2\,d\tau\right)\\
=\ &\frac{1}{2}\left(\|u_1\|^2+\|v_1\|^2\right)+\frac{1}{2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)\\
&+\frac{1}{2\gamma+2}\left(\|D^m u_0\|^2+\|D^m v_0\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u_0\|_r^r+\|v_0\|_r^r\right)\\
(2.11)\quad &-\frac{1}{r}\left(\int_\Omega |u_0|^r\ln|u_0|\,dx+\int_\Omega |v_0|^r\ln|v_0|\,dx\right).
\end{aligned}$$

By using the definition of total energy and initial total energy, we restate (2.11) as

$$E(t)+\left(\int_0^t \|D^m u_\tau\|^2\,d\tau+\int_\Omega \|D^m v_\tau\|^2\,d\tau\right)=E(0). \tag{2.12}$$

□

Now, we give some properties related with $J(u,v)$ and $I(u,v)$, respectively.

Lemma 2.3. *For any* $(u,v)\in H_0^m(\Omega)\times H_0^m(\Omega)$, $\|D^m u\|\neq 0$ *and* $\|D^m u\|\neq 0$,*let* $g(\lambda)=J(\lambda u,\lambda v)$. *Then we have*

i) $\lim\limits_{\lambda\to 0} g(\lambda)=0$, $\lim\limits_{\lambda\to\infty} g(\lambda)=-\infty$,
ii) There is a unique λ^* such that $g'(\lambda)=0$,

iii) Then we have

$$I(\lambda u,\lambda v)=\lambda g'(\lambda)\begin{cases} >0, & 0\le\lambda<\lambda^{*},\\ =0, & \lambda=\lambda^{*},\\ <0, & \lambda^{*}<\lambda. \end{cases}$$

Proof. By the definition of $J(u,v)$, we obtain

$$\begin{aligned} g(\lambda) &= J(\lambda u,\lambda v)\\ &= \frac{1}{2}\lambda^{2}\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)+\frac{1}{r^{2}}\lambda^{r}\left(\|u\|_{r}^{r}+\|v\|_{r}^{r}\right)\\ &\quad -\frac{1}{r}\ln|\lambda|\,\lambda^{r}\left(\|u\|_{r}^{r}+\|v\|_{r}^{r}\right)-\frac{1}{r}\lambda^{r}\left(\int_{\Omega}u^{r}\ln|u|\,dx+\int_{\Omega}v^{r}\ln|v|\,dx\right)\\ &\quad +\frac{1}{2\gamma+2}\lambda^{2\gamma+2}\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)^{2\gamma+2}. \end{aligned}\tag{2.13}$$

Since $\|D^{m}u\|\neq0$, and $\|D^{m}v\|\neq0$, $\lim\limits_{\lambda\to0}g(\lambda)=0$, $\lim\limits_{\lambda\to\infty}g(\lambda)=-\infty$. Now, differentiating $g(\lambda)$ with respect to λ, we have

$$\begin{aligned} g'(\lambda) &= \lambda\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)+\lambda^{2\gamma+1}\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)^{2\gamma+2}\\ &\quad -\lambda^{r-1}\left(\int_{\Omega}u^{r}\ln|u|\,dx+\int_{\Omega}v^{r}\ln|v|\,dx\right)-\lambda^{r-1}\ln|\lambda|\left(\|u\|_{r}^{r}+\|v\|_{r}^{r}\right)\\ &= \lambda\Bigg(\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)+\beta_{2}\lambda^{2\gamma}\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)^{2\gamma+2}\\ &\quad -\lambda^{r-2}\left(\int_{\Omega}u^{r}\ln|u|\,dx+\int_{\Omega}v^{r}\ln|v|\,dx\right)-\lambda^{r-2}\ln|\lambda|\left(\|u\|_{r}^{r}+\|v\|_{r}^{r}\right)\Bigg). \end{aligned}\tag{2.14}$$

Let

$$\begin{aligned} \psi(\lambda) &= \lambda^{2\gamma}\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)^{2\gamma+2}\\ &\quad -\lambda^{r-2}\left(\int_{\Omega}u^{r}\ln|u|\,dx+\int_{\Omega}v^{r}\ln|v|\,dx\right)\\ &\quad -\lambda^{r-2}\ln|\lambda|\left(\|u\|_{r}^{r}+\|v\|_{r}^{r}\right). \end{aligned}$$

Then from $2\gamma\le r-2$ we can deduce that $\lim\limits_{\lambda\to\infty}\psi(\lambda)=-\infty$, $\psi(\lambda)$ is monotone decreasing when $\lambda>\lambda^{1}$ and there exists a unique λ^{1} such that $\psi\left(\lambda^{1}\right)=0$. Then we obtain there is a $\lambda^{*}>\lambda^{1}$ such that $\lambda\left[\left(\|D^{m}u\|^{2}+\|D^{m}v\|^{2}\right)+\psi(\lambda)\right]=0$, which means $g'(\lambda)=0$.

The last property (iii), is only a simple corollary of the fact that

$$\lambda\frac{dJ(\lambda u,\lambda v)}{d\lambda}=\lambda g'(\lambda)=I(\lambda u,\lambda v). \tag{2.15}$$

$\square$

Lemma 2.4. *i) The definition of the potential well depth*

$$d = \inf_{u \in N} J(u, v), \tag{2.16}$$

where

$$N = \{(u, v) : (u, v) \in H_0^m(\Omega) \times H_0^m(\Omega) \setminus \{0\} : I(u, v) = 0\},$$

is equivalent to

(2.17)

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u, \lambda v) \mid (u, v) \in H_0^m(\Omega) \times H_0^m(\Omega), \|D^m u\|^2 \neq 0, \|D^m v\|^2 \neq 0 \right\}.$$

ii) The constant d in (2.16) satisfies

$$d = \frac{(r-2)}{2r} \left(\frac{1}{C_1^{r+1}} \right)^{\frac{2}{r-1}},$$

where C_1 is the optimal constant of Lemma 2.1 ($H_0^m(\Omega) \hookrightarrow L^{r+1}$) and

$$\begin{cases} 2\gamma + 2 \leq r \leq \frac{n+2m}{n-2m}, & n > 2m, \\ 2\gamma + 2 \leq r \leq \infty, & n \leq 2m. \end{cases} \tag{2.18}$$

Proof. i) The definition of d from (iii) of Lemma 2.3 it implies that for any $(u, v) \in H_0^m(\Omega) \times H_0^m(\Omega)$, there exist a λ^* such that $I(\lambda^* u, \lambda^* v) = 0$, that is $(\lambda^* u, \lambda^* v) \in N$. By the definition of d we obtain

$$J(\lambda^* u, \lambda^* v) \geq d \text{ for any } (u, v) \in H_0^m(\Omega) \times H_0^m(\Omega) / \{0\}. \tag{2.19}$$

And because of Lemma 2.3

$$\sup_{\lambda \geq 0} J(\lambda u, \lambda v) = J(\lambda^* u, \lambda^* v),$$

which by virtue of (2.19) means

$$\inf_{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega)} \sup_{\lambda \geq 0} J(\lambda u, \lambda v) = \inf_{(u,v) \in H_0^m(\Omega) \times H_0^m(\Omega)} J(\lambda^* u, \lambda^* v) \geq d, \tag{2.20}$$

As $(u, v) \in H_0^m(\Omega) \times H_0^m(\Omega) / \{0\}$, we obtain d is not equivalent to 0, which gives (2.17). On the other hand, from the definition of d given by (2.17) it implies that there exists λ^1 such that

$$\sup_{\lambda \geq 0} J(\lambda u, \lambda v) = \sup J\left(\lambda^1 u, \lambda^{1*} v\right).$$

Then from Lemma 2.3 we can deduce $\lambda^* = \lambda^1$. And it shows that

$$I\left(\lambda^1 u, \lambda^1 v\right) = I(\lambda^* u, \lambda^* v) = 0,$$

which means $\left(\lambda^1 u, \lambda^1 v\right) \in N$. By the definition of d, we get,

$$d = \inf_{(\lambda^* u, \lambda^* v) \in N} J\left(\lambda^1 u, \lambda^1 v\right),$$

that is

$$d = \inf_{(u,v) \in N} J(u, v). \tag{2.21}$$

This complete our proof for (i).

ii) By virtue of $I(u,v)=0$ and definition of $I(u,v)$ and the embedding theorems we obtain

$$\left(\|D^m u\|^2+\|D^m v\|^2\right)+\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}=\int_\Omega |u|^r \ln|u|\,dx+\int_\Omega |v|^r \ln|v|\,dx,$$

$$\begin{aligned}
\left(\|D^m u\|^2+\|D^m v\|^2\right) &\le \int_\Omega |u|^r \ln|u|\,dx+\int_\Omega |v|^r \ln|v|\,dx \\
&\le \|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1} \\
&\le C_1^{r+1}\left(\|D^m u\|^{r+1}+\|D^m v\|^{r+1}\right) \\
&\le C_1^{r+1}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\frac{r-1}{2}}\left(\|D^m u\|^2+\|D^m v\|^2\right), \qquad (2.22)
\end{aligned}$$

which means

$$\|D^m u\|^2+\|D^m v\|^2 \ge \left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}. \qquad (2.23)$$

From the definition of d, we have $(u,v)\in N$. By the definition of $J(u,v)$, (2.22), (2.3) and $I(u,v)=0$, we get

$$\begin{aligned}
J(u,v) &= \frac{I(u,v)}{r}+\frac{(r-2)}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\quad+\frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right) \\
&\ge \frac{(r-2)}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right) \\
&\ge \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}},
\end{aligned}$$

where $2\gamma\le r-2$. Combining of (2.21) and (2.23), we can see clearly that

$$d=\frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}.$$

□

Lemma 2.5. *Let (u,v) be a weak solution problem of (1.1) and $(u_0,v_0)\in H_0^{r_1}(\Omega)\times H_0^{r_2}(\Omega)$, $(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega)$. Suppose that $E(0)<d$*

i) if $(u_0,v_0)\in W$, then $(u,v)\in W$ for $0\le t\le T$;

ii) if $(u_0,v_0)\in V$, then $(u,v)\in V$ for $0\le t\le T$,

where T is the maximum existence time of $(u(t),v(t))$.

Proof. We only prove case (i), case (ii) is similar. Let $(u(t),v(t))$ be a weak solution problem of (1.1) under the conditions and $(u_0,v_0)\in W$ and T can define of the maximum existence time of $(u(x,t),v(x,t))$. Then by (2.7) the energy functional is nonincreasing about t. So that, we have $E((u(t),v(t)))<E(0)<d$ which means $I((u(t),v(t)))>0$ for $0<t<T$. We will use contradiction and we suppose that; there is a $t_1\in(0,T)$ such that $I(u(t_1),v(t_1))<0$. In this way there is a $t^* \in (0,T)$ to make $I(u(t^*), v(t^*)) = 0$ because of continuity of $I(u(t), v(t))$ about time. Then by (2.16), we get

$$d > E(0) \geq E(u(t^*), v(t^*)) \geq J(u(t^*), v(t^*)) \geq d,$$

which is a contradiction. □

Lemma 2.6. *Under the condition of Lemma 2.5 (ii), we get*

$$d < \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right).$$

Proof. By using definition of the d, we get

$$d = \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}},$$

which together $I(u,v) < 0$. Then similar calculations at (2.22), we get

$$\|D^m u\|^2 + \|D^m v\|^2 \geq \left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}, \tag{2.24}$$

which means

$$d < \frac{(r-2)}{2r}\left(\frac{1}{C_1^{r+1}}\right)^{\frac{2}{r-1}}.$$

□

3. Finite time blow up of solutions for positive initial energy

In tis part we introduce the finite time blow up solution to problem (1.1) with $E(0) > 0$. Now we give some lemmas which will be used the proof of the Theorem 3.3.

Lemma 3.1. *Let (u,v) be a weak solution problem of (1.1) and $(u_0, v_0) \in H_0^m(\Omega) \times H_0^m(\Omega), (u_1, v_1) \in H_0^m(\Omega) \times H_0^m(\Omega)$. Suppose that $E(0) > 0$ and initial data supplies*

$$\|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1) > \frac{2r(C+2)}{(r-2)C} E(0) > 0, \tag{3.1}$$

where C is the best constant of Lemma 2.1.

By $(u,v) \in V$, the map

$$\left\{t \mapsto \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t)\right\}$$

is strictly increasing.

Proof. Defining the following auxiliary function

$$G(t) = \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t), \tag{3.2}$$

where

$$G(0) = \|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1). \tag{3.3}$$

By taking derivative of above function, we get

$$\begin{aligned}G'(t) &= 2(D^m u, D^m u_t) + 2(D^m v, D^m v_t) \\ &\quad +2\left(\|u_t\|^2 + \|v_t\|^2\right) + 2[(u, u_{tt}) + (v, v_{tt})] \\ &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) - 2I(u, v).\end{aligned}$$

By $I(u,v) < 0$, for all $t \in [0, \infty)$ it gives that

$$G'(t) > 0. \tag{3.4}$$

From (3.1), (3.3) and (3.4) we obtain

$$G(t) > G(0) > 0,$$

which gives that the map

$$\left\{t \mapsto \|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t)\right\}$$

is strictly increasing. □

Lemma 3.2. *Under the conditions of Lemma 3.1 (u, v) is the solution of problem (1.1) with the maximum existence time interval $[0, T)$ and $T \le \infty$. If $(u_0, v_0) \in V$, then the all solutions (u, v) belong to V.*

Proof. Our purpose is to show that $(u, v) \in V$. Arguing by contradiction, we consider that $t^* \in (0, T)$ is the first time which satisfies

$$I(u(t^*), v(t^*)) = 0,$$

and

$$I(u(t), v(t)) < 0 \text{ for } t \in [0, t^*).$$

Then from Lemma 3.1 and the continuity of (u, v) and (u_t, v_t) in t, for $t \in (0, t^*)$ we get

$$\begin{aligned}&\|D^m u\|^2 + \|D^m v\|^2 + 2(u, u_t) + 2(v, v_t) \\ > \; &\|D^m u_0\|^2 + \|D^m v_0\|^2 + 2(u_0, u_1) + 2(v_0, v_1) \\ > \; &\frac{2r(C+2)}{(r-2)C} E(0).\end{aligned} \tag{3.5}$$

By (2.3), (2.6) and (2.12) we arrive at

$$\begin{aligned}E(0) &= E(t) + \left(\int_0^t \|D^m u_\tau\|^2 d\tau + \int_\Omega \|D^m v_\tau\|^2 d\tau\right) \\ &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{I(t)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\ &\quad + \frac{(r-2\gamma-2)}{2\gamma+2}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} + \frac{1}{r^2}\left(\|u\|_r^r + \|v\|_r^r\right) \\ &\quad + \left(\int_0^t \|D^m u_\tau\|^2 d\tau + \int_\Omega \|D^m v_\tau\|^2 d\tau\right) \\ &\ge \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{I(t)}{r} + \frac{(r-2)}{2r}\left(\|D^m u\|^2 + \|D^m v\|^2\right).\end{aligned} \tag{3.6}$$

By using $r \geq 2\gamma + 2$, $I\left(u\left(t^*\right), v\left(t^*\right)\right) = 0$, Young's inequality and Lemma 211, we conclude that

$$\begin{aligned}
E(0) &\geq E(t^*) \\
&\geq \frac{1}{2}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{I(t^*)}{r} + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{r}\right)\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right) + \frac{(r-2)}{2r}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&= \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2 + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\quad + \frac{(r-2)}{r\,(C+2)}\left(\|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left(\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2 + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right) \\
&\quad + \frac{(r-2)\,C}{r\,(C+2)}\left(\|u(t^*)\|^2 + \|v(t^*)\|^2\right) \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left[\|u_t(t^*)\|^2 + \|v_t(t^*)\|^2\right. \\
&\quad \left. + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2 + \left(\|u(t^*)\|^2 + \|v(t^*)\|^2\right)\right] \\
&\geq \frac{(r-2)\,C}{2r\,(C+2)}\left\{\left[2\left(u_t(t^*), u(t^*)\right) + 2\left(v_t(t^*), v(t^*)\right)\right]\right. \\
&\quad \left. + \|D^m u(t^*)\|^2 + \|D^m v(t^*)\|^2\right\}.
\end{aligned} \tag{3.7}$$

Clearly, we show that (3.7) contradicts (3.5). This completes the proof of lemma. □

Theorem 3.3. *Let (u, v) be a weak solution of problem of (1.1) and $(u_0, v_0) \in H_0^m(\Omega) \times H_0^m(\Omega)$, $(u_1, v_1) \in H_0^m(\Omega) \times H_0^m(\Omega)$. Suppose that (3.1) holds. Therefore the solution of problem (1.1) blows up in finite time as long as $E(0) > 0$ and $(u_0, v_0) \in V$.*

Proof. We prove the finite time blow up of solution to (1.1). If it is not this case , we suppose existence time $T = \infty$. For any $T_0 > 0$, we define the auxiliary function

$$\begin{aligned}
\Phi(t) &= \|u\|^2 + \|v\|^2 + \int_0^t \left(\|D^m u\|^2 + \|D^m v\|^2\right) d\tau \\
&\qquad (T_0 - t)\left(\|D^m u\|^2 + \|D^m v\|^2\right).
\end{aligned} \tag{3.8}$$

It is clear that $\Phi(t) > 0$ for all $t \in [0, T_0]$. In view of continuity of $\Phi(t)$ in t, we obtain that there is a $\xi > 0$ which is independent on T_0 such that

$$\Phi(t) > \xi. \tag{3.9}$$

Then by $t \in [0, T_0]$,we derive

$$
\begin{aligned}
\Phi'(t) &= 2\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right) \\
&\quad + \left(\|D^m u\|^2 + \|D^m v\|^2\right) - \left(\|D^m u_0\|^2 + \|D^m v_0\|^2\right) \\
&= 2\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right) \\
&\quad +2\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right)\right),
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
\Phi''(t) &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) + 2(u, u_{tt}) + 2(v, v_{tt}) \\
&\quad 2(D^m u, D^m u_t) + 2(D^m v, D^m v_t) \\
&= 2\left(\|u_t\|^2 + \|v_t\|^2\right) - 2I(u, v).
\end{aligned} \tag{3.11}
$$

From (3.10) it implies

$$
\begin{aligned}
(B'(t))^2 &= 4\left((u, u_t)^2 + (v, v_t)^2\right) \\
&\quad +4\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right)\right)^2 \\
&\quad +8\left(\begin{array}{c} \left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right) \\ \left((u, u_t) + (v, v_t)\right) \end{array}\right).
\end{aligned} \tag{3.12}
$$

Our aim is to estimate each terms in (3.12) by Cauchy-Schwarz and Young's inequalities. We obtain the first and second terms as follow

$$
\begin{aligned}
(u, u_t)^2 + (v, v_t)^2 &\le \left(\|u\|\|u_t\| + \|v\|\|v_t\|\right)^2 \\
&\le \left(\|u\|^2 + \|v\|^2\right)\left(\|u_t\|^2 + \|v_t\|^2\right),
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
&\left(\int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right)^2 \\
\leq\ &\left(\int_0^t \|D^m u(\tau)\| \|D^m u_\tau(\tau)\| + \|D^m v(\tau)\| \|D^m v_\tau(\tau)\| d\tau\right)^2 \\
\leq\ &\left(\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right)^{\frac{1}{2}} + \left(\|D^m u_\tau(\tau)\|^2 \|D^m v_\tau(\tau)\|^2\right)^{\frac{1}{2}} d\tau\right)^2 \\
\leq\ &\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \\
&\int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau. \qquad (3.14)
\end{aligned}
$$

For the last term by using again Cauchy-Schwarz and Young's inequalities we obtain

$$
\begin{aligned}
&2\left(\left((u, u_t) + (v, v_t)\right) \int_0^t \left(D^m u(\tau), D^m u_\tau(\tau)\right) + \left(D^m v(\tau), D^m v_\tau(\tau)\right) d\tau\right) \\
\leq\ &2\left(\left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}} \left(\|u_t\|^2 + \|v_t\|^2\right)^{\frac{1}{2}}\right) \\
&\left(\int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau\right)^{\frac{1}{2}} \\
\leq\ &\left(\|u_t\|^2 + \|v_t\|^2\right) \int_0^t \left(\|D^m u(\tau)\|^2 + \|D^m v(\tau)\|^2\right) d\tau \\
&+ \left(\|u\|^2 + \|v\|^2\right) \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau. \qquad (3.15)
\end{aligned}
$$

Substituting (3.13)-(3.15) into (3.12) becomes

$$
(\Phi'(t))^2 \leq 4\Phi(t)\left(\left(\|u_t\|^2 + \|v_t\|^2\right) + \int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau\right). \qquad (3.16)
$$

Combining (3.11) and (3.16) we obtain

$$\begin{aligned}
&\Phi''(t)\Phi(t)-\frac{\zeta}{4}\left(\Phi'(t)\right)^2\\
\geq\ &\Phi(t)\left(\Phi''(t)-\zeta\left(\begin{array}{c}\left(\|u_t\|^2+\|v_t\|^2\right)\\ +\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\end{array}\right)\right)\\
\geq\ &\Phi(t)\left(2\left(\|u_t\|^2+\|v_t\|^2\right)-2I(u,v)\right.\\
&\left.-\zeta\left(\left(\|u_t\|^2+\|v_t\|^2\right)+\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\right)\right).
\end{aligned}\tag{3.17}$$

Let

$$\begin{aligned}
\eta(t) &= (2-\zeta)\left(\|u_t\|^2+\|v_t\|^2\right)-2I(u,v)\\
&\quad-\zeta\left(\int_0^t\left(\|D^m u_\tau(\tau)\|^2+\|D^m v_\tau(\tau)\|^2\right)d\tau\right).
\end{aligned}\tag{3.18}$$

By Lemma 2.2 we get

$$\begin{aligned}
E(0) &= E(t)+\left(\int_0^t\|D^m u_\tau\|^2 d\tau+\int_\Omega\|D^m v_\tau\|^2 d\tau\right)\\
&= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{r-2}{2r}\left(\|D^m u\|^2+\|D^m v\|^2\right)\\
&\quad+\frac{r-2\gamma-2}{2\gamma+2}\left(\|D^m u\|^2+\|D^m v\|^2\right)^{\gamma+1}\\
&\quad\frac{I(u,v)}{r}+\frac{1}{r^2}\left(\|u\|_r^r+\|v\|_r^r\right)\\
&\quad+\left(\int_0^t\|D^m u_\tau\|^2 d\tau+\int_\Omega\|D^m v_\tau\|^2 d\tau\right).
\end{aligned}\tag{3.19}$$

Then by combining (3.18) and (3.19), noting $\zeta = \frac{4C+2r+4}{C+2}$, which guarantees $2 < \zeta < r+2$, and using Lemma 2.1 again, it gives that

$$\begin{aligned}
\zeta(t) &= (r+2-\zeta)\left(\|u_t\|^2 + \|v_t\|^2\right) - 2rE(0) \\
&\quad + (2r-\zeta)\int_0^t \left(\|D^m u_\tau(\tau)\|^2 + \|D^m v_\tau(\tau)\|^2\right) d\tau \\
&\quad + (r-2)\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\
&\quad + \frac{r-2\gamma-2}{\gamma+1}\left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\gamma+1} \\
&\quad + \frac{2}{r}\left(\|u\|_r^r + \|v\|_r^r\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2 + \|v_t\|^2\right) - 2rE(0) \\
&\quad + (r-2)\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2 + \|v_t\|^2\right) - 2rE(0) \\
&\quad + \frac{2(r+2-\zeta)}{C}\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\
&\quad + \left((r-2) - \frac{2(r+2-\zeta)}{C}\right)\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\
&\geq (r+2-\zeta)\left(\|u_t\|^2 + \|v_t\|^2 + 2\left(\|u\|^2 + \|v\|^2\right)\right) - 2rE(0) \\
&\quad + \left((r-2) - \frac{2(r+2-\zeta)}{C}\right)\left(\|D^m u\|^2 + \|D^m v\|^2\right) \\
&\geq \frac{C(r-2)}{C+2}\left[\|u_t\|^2 + \|v_t\|^2 + 2\left(\|u\|^2 + \|v\|^2\right)\right. \\
&\quad \left. + \|D^m u\|^2 + \|D^m v\|^2\right] - 2rE(0) \\
&\geq \frac{C(r-2)}{C+2}\left[2(u,u_t) + 2(v,v_t) + \|D^m u\|^2 + \|D^m v\|^2\right] \\
&\quad -2rE(0). \qquad (3.20)
\end{aligned}$$

Therefore by Lemma 3.1 and Lemma 3.2, we conclude that

$$\begin{aligned}
\zeta(t) &\geq \frac{C(r-2)}{C+2}\left[2(u,u_t) + 2(v,v_t) + \|D^m u\|^2 + \|D^m v\|^2\right] - 2rE(0) \\
&= \frac{C(r-2)}{C+2}\left[2(u,u_t) + 2(v,v_t) + \|D^m u\|^2 + \|D^m v\|^2 - \frac{2r(C+2)}{C(r-2)}\right] \\
&\geq \frac{C(r-2)}{C+2}\left[2(u_0,u_1) + 2(v_0,v_1) + \|D^m u_0\|^2 + \|D^m v_0\|^2 - \frac{2r(C+2)}{C(r-2)}\right] \\
&> \sigma_2 > 0,
\end{aligned}$$

which shows that

$$\Phi''(t)\,\Phi(t) - \frac{\zeta}{4}\left(\Phi'(t)\right)^2 > \Phi(t)\,\sigma_2 > 0.$$

Let $y(t) = \Phi(t)^{-\frac{\zeta-4}{4}}$, then we obtain

$$y''(t) \leq -\frac{\zeta-4}{4}\sigma_2 y(t)^{\frac{\zeta}{\zeta-4}}, \ t \in [0, T_0],$$

where $\zeta = \frac{4C+2r+4}{C+2} \geq 4$.

That is

$$\lim_{t \to T^*} y(t) = 0,$$

where T^* is independent of initial choice of T_0 and $T^* < T_0$. Therefore, we can conclude that

$$\lim_{t \to T^*} \Phi(t) = \infty.$$

□

4. Conclusion

This paper has been able to prove the blow up result for a higher order Kirchhoff type system with logarithmic nonlinearities. This result is new for these types of systems, and it generalises many related problems in the literature.

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