Journal of Universal Mathematics
Vol.4 No.2 pp.140-156 (2021)
ISSN-2618-5660
DOI: 10.33773/jum.956862

SMARANDACHE CURVES ACCORDING TO ALTERNATIVE FRAME IN $\mathbb{E}^3$

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Abstract. In this study, we focus on Smarandache curves which are a special class of curves. These curves have previously been studied by many authors in different spaces. We will re-characterize these curves with the help of an alternative frame different from Frenet frame. Also, we will obtain frame vectors curvature and torsion of these curves.

1. Introduction

Curves, which have an important position in differential geometry, have enabled many studies. Many theories have been developed by establishing relations between Frenet frame. One of the special curves studied in differential geometry is Smarandache curve. Smarandache curve is defined as the regular curve drawn by these vectors, when the Frenet vectors of the unit speed regular curve are taken as position vectors [2]. A.T. Ali introduce special Smarandache curves in the Euclidean space. Some special Smarandache curves are expressed in 3-dimensional Euclidean space and introduced the Serret-Frenet elements of a special case [3]. NC-Smarandache curve with Frenet vectors {T,N,B} and unit Darboux vector C of the curve α is defined in the study titled "An application of Smarandache curves" [4]. In [5], authors obtain results about the characterization of Smarandache curves according to the Sabban frame formed on the S^2 unit sphere. In [7], authors classify general results of Smarandache curves with respect to the causal character of the curve. In her master's thesis named "Smarandache Curves of Bertrand Curve Pair According to Frenet Frame", she define Smarandache curves according to the Frenet vectors of the Bertrand partner curve and found some characterizations belonging to these curves [8]. In the study titled "Smarandache Curves According to Bishop Frame in Euclidean 3-Space", Smarandache curves belonging to Bishop frame are examined and they give some characterizations of these curves [6].
In this present paper, we introduce Smarandache curves according to the alternate frame defined by Uzunoglu et al. of a unit speed curve in Euclidean 3-Space.

Date: **Received:** 2021-06-24; **Accepted:** 2021-07-27.
2000 *Mathematics Subject Classification.* 53A04.
Key words and phrases. Euclidean Space, Frenet frame, Smarandache Curves, Alternative frame.

Firstly, we give Frenet frame, alternative frame and its properties. After that we mentione the relationship with alternative frame and Frenet frame. Then we define the special Smarandache curves according to alternative frame and we calculate the curvature, torsion, Frenet frame elements and alternative frame elements of this curves.

2. Preliminaries

In this section, basic definitions and theories about the Frenet frame and the Serret-Frenet formulas and the alternative frame will be given.

Definition 2.1. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve. The vectors {T,N,B} Frenet frame along the α can be defined as follows

$$T(s) = \alpha^{'}(s), \quad N(s) = \frac{T^{'}(s)}{||T^{'}(s)||}, \quad B(s) = T(s) \times N(s) \tag{2.1}$$

where T is the unit tangent vector field, N is the principal normal vector field, B is the binormal vector field. Frenet derivative formulas can be given as follows

$$\begin{bmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.2}$$

where κ is the curvature and τ is the torsion of the curve α [1]. The curvature and the torsion of the curve α are calculated as follows

$$\begin{cases} \kappa(s) = ||\alpha''(s)|| \\ \tau(s) = \frac{<\alpha^{'} \wedge \alpha^{''}, \alpha^{'''}>}{\|\alpha^{'} \wedge \alpha^{''}\|^2} \end{cases} . \tag{2.3}$$

Definition 2.2. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve. Each unit speed curve has at least four continuous derivatives one can associate three orthogonal unit vector field. T, N and B are tangent, the principal normal and the binormal vector fields, respectively. Uzunoğlu et al. [9] defined the alternative moving frame denote by $\{N, C, W\}$ along the curve α in Euclidean 3-space as

$$N(s) = N(s), \quad C(s) = \frac{N^{'}(s)}{\|N^{'}(s)\|}, \quad W(s) = N(s) \times C(s). \tag{2.4}$$

For the derivatives of the alternative moving frame, we have

$$\begin{bmatrix} N^{'}(s) \\ C^{'}(s) \\ W^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix} \tag{2.5}$$

where f and g are curvatures of the curve α as

$$\begin{cases} f = \sqrt{\kappa^2 + \tau^2} \\ g = \frac{(\tau/\kappa)^{'}}{1+\tau^2/\kappa^2} \end{cases} . \tag{2.6}$$

Definition 2.3. Let $\alpha : I \to E^3$ be a unit speed curve denote by {T,N,B} the moving Frenet frame. Smarandache curve is called the regular curve drawn by the vector whose position vector is

$$\beta(s) = \frac{a(s)T(s) + b(s)N(s) + c(s)B(s)}{\sqrt{a^2(s) + b^2(s) + c^2(s)}}$$

where a,b,c are real functions [4].

3. Smarandache Curves In Euclidean 3-Space

In this section, TN, TB, NB and TNB-Smarandache curves will be introduced and their curvature and torsion will be expressed in Euclidean 3-space.

Definition 3.1. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TN-Smarandache curve is defined by

$$\beta_{TN}(s) = \frac{1}{\sqrt{2}}(T + N). \tag{3.1}$$

Theorem 3.2. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TN-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{TN}} = \frac{\sqrt{2}}{(2\kappa^2+\tau^2)^2}\sqrt{\delta_1^2 + \mu_1^2 + \eta_1^2} \\ \\ \tau_{\beta_{TN}} = \frac{\sqrt{2}[(\tau^3+2\kappa^2\tau-\tau\kappa'+\kappa\tau')\bar{\delta}_1+(\kappa\tau'-\kappa'\tau)\bar{\mu}_1+(2\kappa^3+\kappa\tau^2)\bar{\eta}_1]}{(\tau^3+2\kappa^2\tau-\tau\kappa'+\kappa\tau')^2+(\kappa\tau'-\kappa'\tau)^2+(2\kappa^3+\kappa\tau^2)^2} \end{cases} \tag{3.2}$$

where

$$\begin{cases} \delta_1 = -[\kappa^2(2\kappa^2 + \tau^2) + \tau(\tau\kappa' - \kappa\tau')] \\ \mu_1 = -[\kappa^2(2\kappa^2 + 3\tau^2) - \tau(\tau^3 + \kappa\tau' - \tau\kappa')] \\ \eta_1 = \kappa[\tau(2\kappa^2 + \tau^2) - 2\tau\kappa' - \kappa\tau'] \end{cases} \tag{3.3}$$

$$\begin{cases} \bar{\delta}_1 = \kappa^3 + \kappa(\tau^2 - 3\kappa') - \kappa'' \\ \bar{\mu}_1 = -\kappa^3 - \kappa(\tau^2 + 3\kappa') - 3\tau\tau' + \kappa'' \\ \bar{\eta}_1 = -\kappa^2\tau - \tau^3 + 2\tau\kappa' + \kappa\tau' + \tau'' \end{cases} \tag{3.4}$$

Definition 3.3. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TB-Smarandache curve is defined by

$$\beta_{TB}(s) = \frac{1}{\sqrt{2}}(T + B). \tag{3.5}$$

Theorem 3.4. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TB-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{TB}} = \frac{\sqrt{2(\delta_2^2+\mu_2^2)}}{(\kappa-\tau)^4} \\ \\ \tau_{\beta_{TB}} = \frac{\sqrt{2}[\kappa^2\tau\bar{\delta}_2-2\kappa\tau^2\bar{\delta}_2+\tau^3\bar{\delta}_2+\kappa^3\bar{\eta}_2-2\kappa^2\tau\bar{\eta}_2+\kappa\tau^2\bar{\eta}_2]}{(\tau(\kappa-\tau)^2)^2+(\kappa(\kappa-\tau)^2)^2} \end{cases} \tag{3.6}$$

where

$$\begin{cases} \delta_2 = -\kappa^4 + 3\kappa^3\tau - 3\kappa^2\tau^2 + \kappa\tau^3 \\ \mu_2 = 0 \\ \eta_2 = \kappa^3\tau - 3\kappa^2\tau^2 + 3\kappa\tau^3 - \tau^4 \end{cases} \tag{3.7}$$

$$\begin{cases} \bar{\delta}_2 = -3\kappa\kappa' + 2\kappa\tau' + \kappa^{'}\tau \\ \bar{\mu}_2 = (\tau-\kappa)(\tau^2+\kappa^2) + \kappa^{''} - \tau'' \\ \bar{\eta}_2 = -3\tau\tau' + 2\tau\kappa' + \kappa\tau' \end{cases} \tag{3.8}$$

Definition 3.5. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. NB-Smarandache curve is defined by

$$\beta_{NB}(s) = \frac{1}{\sqrt{2}}(N+B). \tag{3.9}$$

Theorem 3.6. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the NB-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{NB}} = \frac{\sqrt{2}}{(\kappa^2+2\tau^2)^2}\sqrt{\delta_3^2+\mu_3^2+\eta_3^2} \\ \\ \tau_{\beta_{NB}} = \frac{\sqrt{2}[(2\tau^3+\tau\kappa^2)\bar{\delta}_3+(\tau^{'}\kappa-\tau\kappa')\bar{\mu}_3+(\kappa^3+2\kappa\tau^2+\kappa\tau'-\tau\kappa')\bar{\eta}_3]}{(2\tau^3+\tau\kappa^2)^2+(\tau'\kappa-\tau\kappa')^2+(\kappa^3+2\kappa\tau^2+\kappa\tau'-\tau\kappa')^2} \end{cases} \tag{3.10}$$

where

$$\begin{cases} \delta_3 = (\kappa^2+2\tau^2)\kappa\tau + 2\tau(\kappa\tau^{'} - \tau\kappa^{'}) \\ \mu_3 = -(\kappa^2+2\tau^2)(\kappa^2+\tau^2) + \kappa(\kappa'\tau - \tau^{'}\kappa) \\ \eta_3 = (\kappa^2+2\tau^2)(-\tau^2) + \kappa(\kappa\tau' - \kappa'\tau) \end{cases} \tag{3.11}$$

$$\begin{cases} \bar{\delta}_3 = \kappa^3 + \kappa(\tau^2 - 3\kappa^{'}) - \kappa^{''} \\ \bar{\mu}_3 = -\kappa^3 - \kappa(\tau^2+3\kappa^{'}) - 4\tau\tau' + \kappa^{''} \\ \bar{\eta}_3 = -\kappa^2\tau - \tau^3 + 2\tau\kappa^{'} + \kappa\tau^{'} + \tau'' \end{cases} \tag{3.12}$$

Definition 3.7. [3] Let $\alpha(s)$ be a unit speed regular curve in E^3 and {T,N,B} be its moving Frenet-Serret frame. TNB-Smarandache curve is defined by

$$\beta_{TNB}(s) = \frac{1}{\sqrt{3}}(T+N+B). \tag{3.13}$$

Theorem 3.8. [3] *Let $\alpha(s)$ be a unit speed regular curve in E^3. The curvature and torsion of the TNB-Smarandache curve are as follows, respectively.*

$$\begin{cases} \kappa_{\beta_{TNB}} = \frac{\sqrt{3}}{(2\kappa^2+2\tau^2-2\kappa\tau)^2}\sqrt{\delta_4^2+\mu_4^2+\eta_4^2} \\ \\ \tau_{\beta_{TNB}} = \frac{\sqrt{3}[(\kappa^2\tau+\kappa\tau'-2\kappa\tau^2-\tau\tau^{'}+2\tau^3-\tau\kappa^{'}+\tau\kappa^2)\bar{\delta}_4+(\kappa\tau^{'}-\tau\kappa')\bar{\mu}_4+(2\kappa^3-\tau\kappa^{'})\bar{\eta}_4]}{(\kappa^2\tau+\kappa\tau'-2\kappa\tau^2-\tau\tau^{'}+2\tau^3-\tau\kappa^{'}+\tau\kappa^2)^2+(\kappa\tau^{'}-\tau\kappa')^2+(2\kappa^3-\tau\kappa^{'})^2} \end{cases} \tag{3.14}$$

where

$$\begin{cases} \delta_4 = \kappa\tau[4\kappa(\kappa-\tau) + 2(\tau'+\tau^2) + \kappa^{'}] - \kappa^2(2\kappa^2+\tau^{'}) - 2\kappa'\tau^2 \\ \delta_4 = 2\kappa\tau[(\kappa-\tau)^2 + 2\tau - 2\tau'] - 2(\kappa^4+\tau^4) + \kappa^{'}\tau^2 \; - \kappa^2\tau' \\ \delta_4 = \tau[2\kappa(\kappa^2+4\tau^2-\kappa'-2\kappa\tau) + (\tau\kappa'+\tau'-2\tau^3)] \end{cases} \tag{3.15}$$

$$\begin{cases} \bar{\delta}_4 = \kappa^3 + \kappa(\tau^2 - 3\kappa^{'}) - \kappa^{''} \\ \bar{\mu}_4 = -\kappa^3 - \kappa(\tau^2+3\kappa^{'}) - 3\tau\tau' + \kappa^{''} \\ \bar{\eta}_4 = -\kappa^2\tau - \tau^3 + 2\tau\kappa^{'} + \kappa\tau^{'} + \tau'' \end{cases} \tag{3.16}$$

4. Smarandache Curves According to Alternative Frame In E^3

In this section, these special curves will be re-examined on an alternative frame inspired by Smarandache curves defined according to the Frenet frame in Euclidean 3-space.

Definition 4.1. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NC-Smarandache curve is defined by

$$\beta_{NC}(s) = \frac{1}{\sqrt{2}}(N + C). \tag{4.1}$$

Theorem 4.2. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NC-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{\left[\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}\right]^2 + \left[\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5}^2+\tilde{\mu_5}^2+\tilde{\eta_5}^2}\right]^2} \\ g = \dfrac{\left[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5}^2+\tilde{\mu_5}^2+\tilde{\eta_5}^2}}{\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}}\right]'}{1+\left[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_5}\widehat{\delta_5}+\tilde{\mu_5}\widehat{\mu_5}+\tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5}^2+\tilde{\mu_5}^2+\tilde{\eta_5}^2}}{\frac{\sqrt{2}\cdot\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}{(2f^2+g^2)^2}}\right]^2} \end{cases} \tag{4.2}$$

where

$$\begin{cases} \delta_5 = -[f^2(2f^2+g^2) + g(gf' - fg')] \\ \mu_5 = -[f^2(2f^2+3g^2) - g(g^3 + fg' - gf')] \\ \eta_5 = f[g(2f^2+g^2) - 2(gf' - fg')] \end{cases} \tag{4.3}$$

$$\begin{cases} \bar{\delta_5} = [(\delta_5' - f\mu_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \delta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')] \\ \bar{\mu_5} = [(f\delta_5 + \mu_5' - g\eta_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \mu_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')] \\ \bar{\eta_5} = [(g\mu_5 + \eta_5')(\delta_5^2+\mu_5^2+\eta_5^2) - \eta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')] \end{cases} \tag{4.4}$$

$$\begin{cases} \widehat{\delta_5} = (-2ff' - f'' + f^3 - ff' + fg^2) \\ \widehat{\mu_5} = (-f^3 - ff' - 2ff' + f'' - 2gg' - fg^2 - gg') \\ \widehat{\eta_5} = (-f^2g - g^3 + 2gf' + fg' + g'') \end{cases} \tag{4.5}$$

$$\begin{cases} \tilde{\delta_5} = (g^3 + 2f^2g - gf' + fg'), \\ \tilde{\mu_5} = (fg' - f'g), \\ \tilde{\eta_5} = (-f^2g - g^3 + 2gf' + fg' + g'') \end{cases} \tag{4.6}$$

Proof. Let $\beta(s)$ be a unit speed regular NC-Smarandache curve as in (4.1). If we take the derivative of the Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{NC}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-fN + fC + gW), \tag{4.7}$$

and since

$$\left\|\frac{d\beta_{NC}}{ds_\beta}\right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{1}{2}(f^2+f^2+g^2)} = \sqrt{\frac{2f^2+g^2}{2}}. \tag{4.8}$$

From the equations (4.7) and (4.8), the tangent vector of β_{NC} is

$$T_{\beta_{NC}} = \frac{-fN+fC+gW}{\sqrt{2f^2+g^2}}. \tag{4.9}$$

If we take derivate this expression is again, we can see that

$$T'_{\beta_{NC}} = \frac{\sqrt{2}}{(2f^2+g^2)^2}(\delta_5 N + \mu_5 C + \eta_5 W) \tag{4.10}$$

where

$$\begin{cases} \delta_5 = -[f^2(2f^2+g^2) + g(gf' - fg')], \\ \mu_5 = -[f^2(2f^2+3g^2) - g(g^3 + fg' - gf')], \\ \eta_5 = f[g(2f^2+g^2) - 2(gf' - fg')]. \end{cases}$$

The curvature of the β_{NC} is indicated by the $\kappa_{\beta_{NC}}$ taking the norm of equation (4.10).

$$\kappa_{\beta_{NC}} = \frac{\sqrt{2}}{(2f^2+g^2)^2}\sqrt{\delta_5^2+\mu_5^2+\eta_5^2} \tag{4.11}$$

If the principal normal of β_{NC} is indicated by $N_{\beta_{NC}}$, it is found in the form of

$$N_{\beta_{NC}} = \frac{\delta_5 N + \mu_5 C + \eta_5 W}{\sqrt{\delta_5^2+\mu_5^2+\eta_5^2}}. \tag{4.12}$$

If we take the derivative of the equation (4.12), we obtain

$$N' = \frac{\sqrt{2}}{\sqrt{2f^2+g^2}} \frac{\bar{\delta}_5 N + \bar{\mu}_5 C + \bar{\eta}_5 W}{(2f^2+g^2)^{\frac{3}{2}}}. \tag{4.13}$$

where

$$\begin{cases} \bar{\delta_5} = [(\delta_5' - f\mu_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \delta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')], \\ \bar{\mu_5} = [(f\delta_5 + \mu_5' - g\eta_5)(\delta_5^2+\mu_5^2+\eta_5^2) - \mu_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')], \\ \bar{\eta_5} = [(g\mu_5 + \eta_5')(\delta_5^2+\mu_5^2+\eta_5^2) - \eta_5(\delta_5\delta_5' + \mu_5\mu_5' + \eta_5\eta_5')]. \end{cases}$$

If we take the norm of the equation (4.13), we get

$$\|N'_{\beta_{NC}}\| = \frac{\sqrt{2}}{\sqrt{2f^2+g^2}} \frac{\sqrt{\bar{\delta}_5^2+\bar{\mu}_5^2+\bar{\eta}_5^2}}{(\bar{\delta}_5^2+\bar{\mu}_5^2+\bar{\eta}_5^2)^{\frac{3}{2}}}. \tag{4.14}$$

Since $C_{\beta_{NC}} = \frac{N'_{\beta_{NC}}}{\|N'_{\beta_{NC}}\|}$, if necessary calculations are made from the equations (4.13) and (4.14)

$$C_{\beta_{NC}} = \frac{\bar{\delta}_5 N + \bar{\mu}_5 C + \bar{\eta}_5 W}{\sqrt{\bar{\delta}_5^2+\bar{\mu}_5^2+\bar{\eta}_5^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NC}} = N_{\beta_{NC}} \times C_{\beta_{NC}}$. So we have

$$W_{\beta_{NC}} = \frac{1}{\sqrt{\delta_5^2+\mu_5^2+\eta_5^2} \cdot \sqrt{\bar{\delta_5}^2+\bar{\mu_5}^2+\bar{\eta_5}^2}} \begin{vmatrix} N & C & W \\ \delta_5 & \mu_5 & \eta_5 \\ \bar{\delta_5} & \bar{\mu_5} & \bar{\eta_5} \end{vmatrix}$$

and so on

$$W_{\beta_{NC}} = \frac{(\mu_5\bar{\eta_5} - \eta_5\bar{\mu_5})N - (\delta_5\bar{\eta_5} - \eta_5\bar{\delta_5})C + (\delta_5\bar{\mu_5} - \mu_5\delta_5)W}{\sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2} \cdot \sqrt{\bar{\delta_5}^2 + \bar{\mu_5}^2 + \bar{\eta_5}^2}}. \tag{4.15}$$

To find the torsion, we need to find the second and third derivates of the β_{NC} curve. These derivates are available below.

$$\beta_{NC}(s) = \frac{1}{\sqrt{2}}(N + C), \tag{4.16}$$

$$\beta_{NC}^{'} = \frac{1}{\sqrt{2}}(fC - fN + gW), \tag{4.17}$$

$$\beta_{NC}^{''} = \frac{1}{\sqrt{2}}(-(f^2 + f^{'})N + (-f^2 + f^{'} - g^2)C + (fg + g^{'})W), \tag{4.18}$$

$$\beta_{NC}^{'''} = \frac{1}{\sqrt{2}}(\widehat{\delta_5}N + \widehat{\mu_5}C + \widehat{\eta_5}W) \tag{4.19}$$

where

$$\begin{cases} \widehat{\delta_5} = (-2ff^{'} - f^{''} + f^3 - ff^{'} + fg^2), \\ \widehat{\mu_5} = (-f^3 - ff^{'} - 2ff^{'} + f^{''} - 2gg^{'} - fg^2 - gg^{'}), \\ \widehat{\eta_5} = (-f^2g - g^3 + 2gf^{'} + fg^{'} + g^{''}). \end{cases}$$

In equation (2.3), if the expressions (4.17), (4.18) and (4.19) are written in their places and the necessary calculations are made, torsion is found as

(4.20)

$$\tau_{\beta_{NC}} = \frac{\sqrt{2} \cdot \left[\ (g^3 + 2f^2g - gf^{'} + fg^{'})\widehat{\delta_5} + (fg^{'} - f^{'}g)\widehat{\mu_5} + (2f^3 + fg^2)\widehat{\eta_5}\ \right]}{(g^3 + 2f^2g - gf^{'} + fg^{'})^2 + (fg^{'} - f^{'}g)^2 + (2f^3 + fg^2)^2}$$

In equation (2.6), if the expressions (4.11) and (4.20) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5}^2 + \tilde{\eta_5}^2}]^2} \tag{4.21}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5^2} + \tilde{\eta_5^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_5}\widehat{\delta_5} + \tilde{\mu_5}\widehat{\mu_5} + \tilde{\eta_5}\widehat{\eta_5})}{\tilde{\delta_5^2} + \tilde{\mu_5^2} + \tilde{\eta_5^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_5^2 + \mu_5^2 + \eta_5^2}}{(2f^2 + g^2)^2}}]^2} \tag{4.22}$$

where

$$\begin{cases} \tilde{\delta_5} = (g^3 + 2f^2g - gf^{'} + fg^{'}), \\ \tilde{\mu_5} = (fg^{'} - f^{'}g), \\ \tilde{\eta_5} = (-f^2g - g^3 + 2gf^{'} + fg^{'} + g^{''}). \end{cases}$$

$\square$

Definition 4.3. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NW-Smarandache curve is defined by

$$\beta_{NW}(s) = \frac{1}{\sqrt{2}}(N + W). \tag{4.23}$$

Theorem 4.4. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}]^2 + [\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6}^2}]^2} \\ g = \dfrac{[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]'}{1+[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]^2} \end{cases} \tag{4.24}$$

where

$$\begin{cases} \bar{\delta}_6 = (-f^{'}f^2 - f^{'}g^2 + f^2f^{'} + fgg^{'}) \\ \bar{\mu}_6 = (-f^4 - f^2g^2 - g^2f^2 - g^4) \\ \bar{\eta}_6 = (g^{'}f^2 + g^{'}g^2 - gff^{'} - g^2g^{'}) \end{cases} \tag{4.25}$$

$$\begin{cases} \widehat{\delta_6} = (-3ff^{'} + 2fg^{'} + gf^{'}) \\ \widehat{\mu_6} = (f^2+g^2)(-f+g) + f^{''} - g^{''} \\ \widehat{\eta_6} = (-3gg^{'} + 2gf^{'} + fg^{'}) \end{cases} \tag{4.26}$$

$$\begin{cases} \tilde{\delta_6} = (f^2g - 2fg^2 + g^3), \\ \tilde{\mu_6} = 0, \\ \tilde{\eta_6} = (f^3 - 2f^2g + fg^2). \end{cases} \tag{4.27}$$

Proof. Let $\beta(s)$ be a unit speed regular NW-Smarandache curve as in (4.23). If we take the derivative of Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{NW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{(f-g)C}{\sqrt{2}}, \tag{4.28}$$

and since

$$\left\|\frac{d\beta_{NW}}{ds_\beta}\right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{(f-g)^2}{2}} = \frac{|f-g|}{\sqrt{2}}. \tag{4.29}$$

From the equations (4.28) and (4.29), tangent vector of β_{NW} is

$$T_{\beta_{NW}} = \begin{cases} C & f > g \\ -C & f < g \end{cases}. \tag{4.30}$$

If we take derivate this expression is again, we can see that

$$T^{'}_{\beta_{NW}} = \frac{\sqrt{2}(-fN + gW)}{|f-g|} \tag{4.31}$$

The curvature of the β_{NW} is indicated by the $\kappa_{\beta_{NW}}$ taking the norm of equation (4.31).

$$\kappa_{\beta_{NW}} = \frac{\sqrt{2}}{(f-g)}\sqrt{f^2+g^2} \tag{4.32}$$

If the β_{NW} is indicated by principal normal $N_{\beta_{NW}}$, it is found in the form of

$$N_{\beta_{NW}} = \frac{1}{\sqrt{f^2+g^2}}(-fN+gW) \tag{4.33}$$

If we take the derivative of the equation (4.33), we obtain that

$$N^{'} = \frac{\sqrt{2}}{|f-g|}\cdot\frac{\bar{\delta}_6 N+\bar{\mu}_6 C+\bar{\eta}_6 W}{(f^2+g^2)^{\frac{3}{2}}}. \tag{4.34}$$

where

$$\begin{cases} \bar{\delta}_6 = (-f^{'}f^2 - f^{'}g^2 + f^2 f^{'} + fgg^{'}) \\ \bar{\mu}_6 = (-f^4 - f^2g^2 - g^2f^2 - g^4) \\ \bar{\eta}_6 = (g^{'}f^2 + g^{'}g^2 - gff^{'} - g^2g^{'}) \end{cases}$$

If we take the norm of the equation (4.34), we get

$$\|N^{'}_{\beta_{NW}}\| = \frac{\sqrt{2}}{(f^2+g^2)^{\frac{3}{2}}|f-g|}\cdot\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}. \tag{4.35}$$

Since $C_{\beta_{NW}} = \frac{N^{'}_{\beta_{NW}}}{\|N^{'}_{\beta_{NW}}\|}$, if necessary calculations are made from the equations (4.34) and (4.35),

$$C_{\beta_{NW}} = \frac{N^{'}_{\beta_{NW}}}{\|N^{'}_{\beta_{NW}}\|} = \frac{\bar{\delta}_6 N+\bar{\mu}_6 C+\bar{\eta}_6 W}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NW}} = N_{\beta_{NW}} \times C_{\beta_{NW}}$,

$$W_{\beta_{NW}} = \frac{1}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}\cdot\sqrt{f^2+g^2}}\begin{vmatrix} N & C & W \\ -f & 0 & g \\ \bar{\delta}_6 & \bar{\mu}_6 & \bar{\eta}_6 \end{vmatrix}$$

and so on

$$W_{\beta_{NW}} = \frac{-g\bar{\mu}_6 N + (f\bar{\eta}_6 + g\bar{\delta}_6)C - f\bar{\mu}_6 W}{\sqrt{\bar{\delta}_6^2+\bar{\mu}_6^2+\bar{\eta}_6^2}\cdot\sqrt{f^2+g^2}}.$$

To find the torsion, we need to find the second and third derivatives of the β_{NW} curve. The derivates are available below.

$$\beta_{NW}(s) = \frac{1}{\sqrt{2}}(N+W), \tag{4.36}$$

$$\beta^{'}_{NW} = \frac{1}{\sqrt{2}}(fC-gC), \tag{4.37}$$

$$\beta^{''}_{NW} = \frac{1}{\sqrt{2}}(-f^2+gf)N+(f^{'}-g^{'})C+(fg-g^2)W), \tag{4.38}$$

$$\beta^{'''}_{NW} = \frac{1}{\sqrt{2}}(\widehat{\delta_6}N+\widehat{\mu_6}C+\widehat{\eta_6}W) \tag{4.39}$$

where

$$\begin{cases} \widehat{\delta_6} = (-3ff' + 2fg' + gf') \\ \widehat{\mu_6} = (f^2+g^2)(-f+g) + f'' - g'' \\ \widehat{\eta_6} = (-3gg' + 2gf' + fg') \end{cases}$$

In equation (2.3), if the expressions (4.37), (4.38) and (4.39) are written in their places and the necessary calculations are made, torsion of β_{NW} is found as

$$\tau_{\beta_{NW}} = \frac{\sqrt{2}\cdot\left[\ (f^2g - 2fg^2 + g^3)\bar{\delta_5} + 0 + (f^3 - 2f^2g + fg^2)\bar{\eta_5},\ \right]}{(f^2g - 2fg^2 + g^3)^2 + (f^3 - 2f^2g + fg^2)^2} \tag{4.40}$$

In equation (2.6), if the expressions (4.32) and (4.40) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}]^2 + [\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6} + \tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2} + \tilde{\eta_6}^2}]^2} \tag{4.41}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]'}{1+[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_6}\widehat{\delta_6}+\tilde{\eta_6}\widehat{\eta_6})}{\tilde{\delta_6^2}+\tilde{\eta_6^2}}}{\frac{\sqrt{2}\cdot\sqrt{f^2+g^2}}{(f-g)}}]^2} \tag{4.42}$$

where

$$\begin{cases} \tilde{\delta_6} = (f^2g - 2fg^2 + g^3), \\ \tilde{\mu_6} = 0, \\ \tilde{\eta_6} = (f^3 - 2f^2g + fg^2). \end{cases}$$

□

Definition 4.5. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. CW-Smarandache curve is defined by

$$\beta_{CW}(s) = \frac{1}{\sqrt{2}}(C+W). \tag{4.43}$$

Theorem 4.6. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of CW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{2}\cdot\sqrt{\delta_7^2+\mu_7^2+\eta_7^2}}{(f^2+2g^2)^2}]^2 + [\frac{\sqrt{2}\cdot(\tilde{\delta_7}\widehat{\delta_7}+\tilde{\mu_7}\widehat{\mu_7}+\tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2}+\tilde{\mu_7}^2+\tilde{\eta_7}^2}]^2} \\ \\ g = \frac{[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_7}\widehat{\delta_7}+\tilde{\mu_7}\widehat{\mu_7}+\tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2}+\tilde{\mu_7^2}+\tilde{\eta_7^2}}}{\frac{\sqrt{2}\cdot\sqrt{\delta_7^2+\mu_7^2+\eta_7^2}}{(f^2+2g^2)^2}}]'}{1+[\frac{\frac{\sqrt{2}\cdot(\tilde{\delta_7}\widehat{\delta_7}+\tilde{\mu_7}\widehat{\mu_7}+\tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2}+\tilde{\mu_7^2}+\tilde{\eta_7^2}}}{\frac{\sqrt{2}\cdot\sqrt{\delta_7^2+\mu_7^2+\eta_7^2}}{(f^2+2g^2)^2}}]^2} \end{cases} \tag{4.44}$$

where

$$\begin{cases} \delta_7 = (fg(f^2+2g^2)) + 2g(fg^{'} - gf^{'}) \\ \mu_7 = -(f^2+2g^2)(f^2+g^2) + f(f^{'}g - g^{'}f) \\ \eta_7 = -g^2(f^2+2g^2) + f(fg^{'} - gf^{'}) \end{cases} \tag{4.45}$$

$$\begin{cases} \bar{\delta_7} = [(\delta_7^{'} - f\mu_7)(\delta_7^2+\mu_7^2+\eta_7^2) - \delta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\mu_7} = [(f\delta_7 + \mu_7^{'} - g\eta_7)(\delta_7^2+\mu_7^2+\eta_7^2) - \mu_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\eta_7} = [(g\mu_7 + \eta_7^{'})(\delta_7^2+\mu_7^2+\eta_7^2) - \eta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \end{cases} \tag{4.46}$$

$$\begin{cases} \widehat{\delta_7} = (-f^{''} + f(2g^{'} + f^2) + g(f^{'} + gf)) \\ \widehat{\mu_7} = (f(-3f^{'} + gf) + g(-3g^{'} + g^2) - g^{''}) \\ \widehat{\eta_7} = -g(f^2+g^2+3g^{'}) + g^{''} \end{cases} \tag{4.47}$$

$$\begin{cases} \tilde{\delta_7} = (2g^3 + gf^2) \\ \tilde{\mu_7} = (g^{'}f - gf^{'}) \\ \tilde{\eta_7} = (f^3 + 2fg^2 + fg^{'} - gf^{'}) \end{cases} \tag{4.48}$$

Proof. Let $\beta(s)$ be a unit speed regular CW-Smarandache curve as in (4.43). If we take the derivative of Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{CW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-fN + gW - gC), \tag{4.49}$$

and since

$$\left\|\frac{d\beta_{CW}}{ds_\beta}\right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{1}{2}(f^2+g^2+g^2)} = \sqrt{\frac{f^2+2g^2}{2}} \tag{4.50}$$

From the equations (4.49) and (4.50), tangent vector of β_{CW} is

$$T_{\beta_{CW}} = \frac{-fN + gW - gC}{\sqrt{f^2+2g^2}}. \tag{4.51}$$

If we take derivate this expression is again, we can see that

$$T_{\beta_{CW}}^{'} = \frac{\delta_7 N + \mu_7 C + \eta_7 W}{(f^2+2g^2)^{\frac{3}{2}}} \cdot \frac{\sqrt{2}}{\sqrt{(f^2+2g^2)}} \tag{4.52}$$

where

$$\begin{cases} \delta_7 = (fg(f^2+2g^2)) + 2g(fg^{'} - gf^{'}) \\ \mu_7 = -(f^2+2g^2)(f^2+g^2) + f(f^{'}g - g^{'}f) \\ \eta_7 = -g^2(f^2+2g^2) + f(fg^{'} - gf^{'}) \end{cases}$$

The curvature of the β_{CW} is indicated by the $\kappa_{\beta_{CW}}$ taking the norm of equation (4.52).

$$\kappa_{\beta_{CW}} = \frac{\sqrt{2}}{(f^2+2g^2)^2}\sqrt{\delta_7^2+\mu_7^2+\eta_7^2} \tag{4.53}$$

If the principal normal of β_{CW} is indicated by $N_{\beta_{CW}}$, it is found in the form of

$$N_{\beta_{CW}} = \frac{\delta_7 N + \mu_7 C + \eta_7 W}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}} \tag{4.54}$$

If we take the derivative of the equation (4.54), we obtain

$$N^{'} = \frac{\sqrt{2}}{\sqrt{f^2 + 2g^2}} \frac{\bar{\delta_7} N + \bar{\mu_7} C + \bar{\eta_7} W}{(2f^2 + g^2)^{\frac{3}{2}}} \tag{4.55}$$

where

$$\begin{cases} \bar{\delta_7} = [(\delta_7^{'} - f\mu_7)(\delta_7^2 + \mu_7^2 + \eta_7^2) - \delta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\mu_7} = [(f\delta_7 + \mu_7^{'} - g\eta_7)(\delta_7^2 + \mu_7^2 + \eta_7^2) - \mu_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \\ \bar{\eta_7} = [(g\mu_7 + \eta_7^{'})(\delta_7^2 + \mu_7^2 + \eta_7^2) - \eta_7(\delta_7\delta_7^{'} + \mu_7\mu_7^{'} + \eta_7\eta_7^{'})] \end{cases}$$

If we take the norm of the equation (4.55), we get

$$\|N^{'}_{\beta_{CW}}\| = \frac{\sqrt{2}}{\sqrt{f^2 + 2g^2}} \frac{\sqrt{\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2}}{(\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2)^{\frac{3}{2}}} \tag{4.56}$$

Since $C_{\beta_{CW}} = \frac{N^{'}_{\beta_{CW}}}{\|N^{'}_{\beta_{CW}}\|}$, if necessary calculations are made from the equations (4.55) and (4.56)

$$C_{\beta_{CW}} = \frac{\bar{\delta_7} N + \bar{\mu_7} C + \bar{\eta_7} W}{\sqrt{\bar{\delta}_7^2 + \bar{\mu}_7^2 + \bar{\eta}_7^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{CW}} = N_{\beta_{CW}} \times C_{\beta_{CW}}$. So we have

$$W_{\beta_{CW}} = \frac{1}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2} \cdot \sqrt{\bar{\delta_7}^2 + \bar{\mu_7}^2 + \bar{\eta_7}^2}} \begin{vmatrix} N & C & W \\ \delta_7 & \mu_5 & \eta_7 \\ \bar{\delta_7} & \bar{\mu_7} & \bar{\eta_7} \end{vmatrix}$$

and so on

$$W_{\beta_{CW}} = \frac{(\mu_7\bar{\eta_7} - \eta_5\bar{\mu_7})N - (\delta_7\bar{\eta_7} - \eta_7\bar{\delta_7})C + (\delta_7\bar{\mu_7} - \mu_7\delta_7)W}{\sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2} \cdot \sqrt{\bar{\delta_7}^2 + \bar{\mu_7}^2 + \bar{\eta_7}^2}}. \tag{4.57}$$

To find the torsion, we need to find the second and third derivates of the β_{CW} curve. These derivates are available below.

$$\beta_{CW}(s) = \frac{1}{\sqrt{2}}(C + W), \tag{4.58}$$

$$\beta^{'}_{CW} = \frac{1}{\sqrt{2}}(-fN + gW - gC), \tag{4.59}$$

$$\beta^{''}_{CW} = \frac{1}{\sqrt{2}}(-f^{'} + gf)N + (-f^2 - g^2 - g^{'})C + (g^{'} - g^2)W), \tag{4.60}$$

$$\beta^{'''}_{CW} = \frac{1}{\sqrt{2}}(\widehat{\delta_7} N + \widehat{\mu_7} C + \widehat{\eta_7} W) \tag{4.61}$$

where

$$\begin{cases} \widehat{\delta_7} = (-f^{''} + f(2g^{'} + f^2) + g(f^{'} + gf)) \\ \widehat{\mu_7} = (f(-3f^{'} + gf) + g(-3g^{'} + g^2) - g^{''}) \\ \widehat{\eta_7} = -g(f^2 + g^2 + 3g^{'}) + g^{''} \end{cases}$$

In equation (2.3), if the expressions (4.59), (4.60) and (4.61) are written in their places and the necessary calculations are made, torsion is found as

(4.62)

$$\tau_{\beta_{CW}} = \frac{\sqrt{2} \cdot \left[\ (2g^3 + gf^2)\widehat{\delta_7} + (g^{'}f - gf^{'})\widehat{\mu_7} + (f^3 + 2fg^2 + fg^{'} - gf^{'})\widehat{\eta_7} \ \right]}{(2g^3 + gf^2)^2 + (g^{'}f - gf^{'})^2 + (f^3 + 2fg^2 + fg^{'} - gf^{'})^2}$$

In equation (2.6), if the expressions (4.53) and (4.62) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{(f^2 + 2g^2)^2}]^2 + [\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7}^2 + \tilde{\eta_7}^2}]^2} \tag{4.63}$$

and

$$g = \frac{[\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7^2} + \tilde{\eta_7^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{f^2 + 2g^2}}]'}{1 + [\frac{\frac{\sqrt{2} \cdot (\tilde{\delta_7}\widehat{\delta_7} + \tilde{\mu_7}\widehat{\mu_7} + \tilde{\eta_7}\widehat{\eta_7})}{\tilde{\delta_7^2} + \tilde{\mu_7^2} + \tilde{\eta_7^2}}}{\frac{\sqrt{2} \cdot \sqrt{\delta_7^2 + \mu_7^2 + \eta_7^2}}{f^2 + 2g^2}}]^2} \tag{4.64}$$

where

$$\begin{cases} \tilde{\delta_7} = (2g^3 + gf^2), \\ \tilde{\mu_7} = (g^{'}f - gf^{'}), \\ \tilde{\eta_7} = (f^3 + 2fg^2 + fg^{'} - gf^{'}). \end{cases}$$

□

Definition 4.7. Let $\beta(s)$ be a unit speed regular curve in E^3 and {N,C,W} be its moving alternative frame. NCW-Smarandache curve is defined by

$$\beta_{NCW}(s) = \frac{1}{\sqrt{3}}(N + C + W). \tag{4.65}$$

Theorem 4.8. *Let $\beta(s)$ be a unit speed regular curve in E^3. The curvature and torsion of NCW-Smarandache curve are as follows, respectively.*

$$\begin{cases} f = \sqrt{[\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}]^2 + [\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}]^2} \\ \\ g = \frac{[\frac{\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8^2} + \tilde{\eta_8^2}}}{\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}}]'}{1 + [\frac{\frac{\sqrt{3} \cdot (\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8^2} + \tilde{\eta_5^2}}}{\frac{\sqrt{3} \cdot \sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}}]^2} \end{cases} \tag{4.66}$$

where

(4.67)

$$\begin{cases} \delta_8 = gff' - 2f'g^2 - 2f^4 - 4f^2g^2 + 4f^3g + 2g^3f + 2fgg' - f^2g' \\ \mu_8 = f^2(-2f^2 - 4g^2 - 2fg - g') + g^2(-2g^4 + 2fg - g' + fg(f' - g')) \\ \eta_8 = 2f^2(fg - 2g^2 + g') + g^2(4fg - 2g^2 + f') - fg(g' + 2f') \end{cases}$$

(4.68)
$$\begin{cases} \bar{\delta_8} = [(\delta_8' - f\mu_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \delta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\mu_8} = [(f\delta_8 + \mu_8' - g\eta_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \mu_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\eta_8} = [(g\mu_8 + \eta_8')(\delta_8^2 + \mu_8^2 + \eta_8^2) - \eta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \end{cases}$$

(4.69)
$$\begin{cases} \widehat{\delta_8} = (f^3 + fg^2 - 3ff' - f'' + 2g'f + gf') \\ \widehat{\mu_8} = (g^3 - f^3 - 3(ff' + gg') - (-f'' + g'') + fg(f - g)) \\ \widehat{\eta_8} = (g'' - f^2g - 3gg' - g^3 + 2gf' + fg') \end{cases}$$

(4.70)
$$\begin{cases} \tilde{\delta_8} = (2f^2g - 2fg^2 + fg' - gf') \\ \tilde{\mu_8} = (fg' - f'g) \\ \tilde{\eta_8} = (2f^3 + 2fg^2 - 2gf^2 - gf' + fg') \end{cases}$$

Proof. Let $\beta(s)$ be a unit speed reguler NCW-Smarandache curve as in (4.65). If we take the derivative of the Smarandache curve according to arclenght parameter, we have

$$\frac{d\beta_{NCW}}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{3}}(fC - fN + gW - gC), \tag{4.71}$$

and since

$$\left\|\frac{d\beta_{NCW}}{ds_\beta}\right\| = 1,$$

we can see

$$\frac{ds_\beta}{ds} = \sqrt{\frac{2}{3}(f^2 + g^2 - gf)}. \tag{4.72}$$

From the equations (4.71) and (4.72) tangent vector of β_{NCW} is

$$T_{\beta_{NCW}} = \frac{fC - fN + gW - gC}{\sqrt{2(f^2 + g^2 - gf)}} \tag{4.73}$$

If we take derivate this expression is again, we can see that

$$T'_{\beta_{NCW}} = \frac{\delta_8 N + \mu_8 C + \eta_8 W}{(2f^2 + 2g^2 - 2gf)^{\frac{3}{2}}}\frac{\sqrt{3}}{\sqrt{(2f^2 + 2g^2 - 2gf)}} \tag{4.74}$$

where

$$\begin{cases} \delta_8 = gff' - 2f'g^2 - 2f^4 - 4f^2g^2 + 4f^3g + 2g^3f + 2fgg' - f^2g' \\ \mu_8 = f^2(-2f^2 - 4g^2 - 2fg - g') + g^2(-2g^4 + 2fg - g' + fg(f' - g')) \\ \eta_8 = 2f^2(fg - 2g^2 + g') + g^2(4fg - 2g^2 + f') - fg(g' + 2f') \end{cases}$$

The curvature of the β_{NCW} is indicated by the $\kappa_{\beta_{NCW}}$ taking the norm of equation (4.74)

$$\kappa_{\beta_{NCW}} = \frac{\sqrt{3}}{(2f^2 + 2g^2 - 2gf)^2}\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}. \tag{4.75}$$

If the principal normal of β_{NCW} is indicated by $N_{\beta_{NCW}}$, it is found in the form of

$$N_{\beta_{NCW}} = \frac{\delta_8 N + \mu_8 C + \eta_8 W}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}. \tag{4.76}$$

If we take the derivative of the equation (4.76), we obtain

$$N' = \frac{\sqrt{3}}{\sqrt{2f^2 + 2g^2 - 2gf}} \frac{\bar{\delta}_8 N + \bar{\mu}_8 C + \bar{\eta}_8 W}{(\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2)^{\frac{3}{2}}} \tag{4.77}$$

where

$$\begin{cases} \bar{\delta}_8 = [(\delta_8' - f\mu_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \delta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\mu}_8 = [(f\delta_8 + \mu_8' - g\eta_8)(\delta_8^2 + \mu_8^2 + \eta_8^2) - \mu_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \\ \bar{\eta}_8 = [(g\mu_8 + \eta_8')(\delta_8^2 + \mu_8^2 + \eta_8^2) - \eta_8(\delta_8\delta_8' + \mu_8\mu_8' + \eta_8\eta_8')] \end{cases}$$

If we take the norm of the equation (4.77), we get

$$\|N'_{\beta_{NCW}}\| = \frac{\sqrt{3}}{\sqrt{2f^2 + 2g^2 - 2gf}} \frac{\sqrt{\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2}}{(\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2)^{\frac{3}{2}}} \tag{4.78}$$

Since $C_{\beta_{NCW}} = \frac{N'_{\beta_{NCW}}}{\|N'_{\beta_{NCW}}\|}$, if necessary calculations are made from the equations (4.77) and (4.78)

$$C_{\beta_{NCW}} = \frac{\bar{\delta}_8 N + \bar{\mu}_8 C + \bar{\eta}_8 W}{\sqrt{\bar{\delta}_8^2 + \bar{\mu}_8^2 + \bar{\eta}_8^2}}.$$

From the definition of Darboux vector, we know $W_{\beta_{NCW}} = N_{\beta_{NCW}} \times C_{\beta_{NCW}}$. So we have

$$W_{\beta_{NCW}} = \frac{1}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}\sqrt{\bar{\delta}_8^{\,2} + \bar{\mu}_8^{\,2} + \bar{\eta}_8^{\,2}}} \begin{vmatrix} N & C & W \\ \delta_8 & \mu_8 & \eta_8 \\ \bar{\delta}_8 & \bar{\mu}_8 & \bar{\eta}_8 \end{vmatrix}$$

and so on

$$W_{\beta_{NCW}} = \frac{(\mu_8\bar{\eta}_8 - \eta_8\bar{\mu}_8)N - (\delta_8\bar{\eta}_8 - \eta_8\bar{\delta}_8)C + (\delta_8\bar{\mu}_8 - \mu_8\delta_8)W}{\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2} \cdot \sqrt{\bar{\delta}_8^{\,2} + \bar{\mu}_8^{\,2} + \bar{\eta}_8^{\,2}}}$$

To find the torsion, we need to find the second and third derivates of the β_{NCW} curve. These derivates are available below.

$$\beta_{NCW}(s) = \frac{1}{\sqrt{3}}(N + C + W), \tag{4.79}$$

$$\beta'_{NCW} = \frac{1}{\sqrt{3}}(fC - fN + gW - gC), \tag{4.80}$$

$$\beta''_{NCW} = \frac{1}{\sqrt{3}}((-f' - f^2 + gf)N + (-f^2 + f' - g' - g^2)C + (fg - g^2 + g')W), \tag{4.81}$$

$$\beta'''_{NCW} = \frac{1}{\sqrt{3}}(\widehat{\delta_8} N + \widehat{\mu_8} C + \widehat{\eta_8} W) \tag{4.82}$$

where

$$\begin{cases} \widehat{\delta_8} = (f^3 + fg^2 - 3ff^{'} - f^{''} + 2g^{'}f + gf^{'}) \\ \widehat{\mu_8} = (g^3 - f^3 - 3(ff^{'} + gg^{'}) - (-f^{''} + g^{''}) + fg(f-g)) \\ \widehat{\eta_8} = (g^{''} - f^2 g - 3gg^{'} - g^3 + 2gf^{'} + fg^{'}) \end{cases}$$

In equation (2.3), if the expressions (4.80), (4.81) and (4.82) are written in their places and the necessary calculations are made, torsion is found as

$$\tau_{\beta_{NCW}} = \frac{\sqrt{3}\cdot\Big[\ (2f^2g - 2fg^2 + fg^{'} - gf^{'})\widehat{\delta_8} + (fg^{'} - f^{'}g)\widehat{\mu_8} + (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'})\widehat{\eta_8}\ \Big]}{(2f^2g - 2fg^2 + fg^{'} - gf^{'})^2 + (fg^{'} - f^{'}g)^2 + (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'})^2} \tag{4.83}$$

In equation (2.6), if the expressions (4.75) and (4.83) are written in their places and the necessary calculations are made, curvature and torsion according to alternative frame are obtained as

$$f = \sqrt{[\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{(2f^2 + 2g^2 - 2gf)^2}]^2 + [\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8}^2 + \tilde{\eta_8}^2}]^2} \tag{4.84}$$

and

$$g = \frac{[\frac{\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8^2} + \tilde{\eta_8^2}}}{\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{2f^2 + 2g^2 - 2gf}}]'}{1 + [\frac{\frac{\sqrt{3}\cdot(\tilde{\delta_8}\widehat{\delta_8} + \tilde{\mu_8}\widehat{\mu_8} + \tilde{\eta_8}\widehat{\eta_8})}{\tilde{\delta_8^2} + \tilde{\mu_8^2} + \tilde{\eta_8^2}}}{\frac{\sqrt{3}\cdot\sqrt{\delta_8^2 + \mu_8^2 + \eta_8^2}}{2f^2 + 2g^2 - 2gf}}]^2} \tag{4.85}$$

where

$$\begin{cases} \tilde{\delta_8} = (2f^2g - 2fg^2 + fg^{'} - gf^{'}), \\ \tilde{\mu_8} = (fg^{'} - f^{'}g), \\ \tilde{\eta_8} = (2f^3 + 2fg^2 - 2gf^2 - gf^{'} + fg^{'}). \end{cases}$$

□

5. Conclusion

Smarandache curves have been studied many times since they were defined. The importance of this study is that, unlike the studies in the literature, these curves are re-characterized with the help of an alternative frame different from Frenet frame.

6. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding

The author(s) declared that has no received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest

The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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