Research Article GU J Sci 36(3): 1369-1380 (2023) DOI: 10.35378/gujs.957499

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Synchronization Analysis of a Master-Slave BEC System via Active Control

Eren TOSYALI[1,*], Fatma AYDOGMUS[2]

[1] *Opticianry, Vocational School of Health Services, Istanbul Bilgi University, Inonu Cad. No:72, Kustepe, 34387, Istanbul, Turkey*

[2] *Department of Physics, Faculty of Science, Istanbul University, Balabanaga, Sehzadebasi Cad, Fatih, 34134, Istanbul, Turkey*

Highlights

- This paper focuses on the dynamic of BEC systems contained different external trapping potentials.
- The phase space diagrams are constructed for master-slave systems depend on system parameters.
- The synchronization is obtained in master-slave scheme for different initial values.
- The Lyapunov characteristic exponents of the master and slave system are calculated numerically.

Article Info

Received: 26 June 2021
Accepted: 01 Aug 2022

Keywords

Bose-Einstein condensate
Gross-Pitaevskii equation
Synchronization
Optical lattice potential

Abstract

This paper will focus on theoretical treatment of the dynamic of the Bose-Einstein Condensate (BEC) systems contained different external trapping potentials. We construct the phase space diagrams and Lyapunov Characteristic Exponents (LCEs) for master and slave systems depended on the system parameters and propose a nonlinear control for the synchronization of systems in their chaotic states. The synchronization is obtained in master-slave scheme for different initial values. Numerical results are also given to show the efficiency of the used control technique.

1. INTRODUCTION

It is well known that a chaotic system has finite and random behaviors of a deterministic dynamic system. It has sensitivity to initial conditions which makes them inherently unpredictable in the long term and does not predict the evolution of physical parameters such as amplitude and period over time. Most processes in nature show nonlinear behaviors and chaos help us understand many events in nature [1]. A BEC system with an external potential exhibits many rich results such a nonlinear system [2,3]. The chaotic dynamics of a BEC system depends on the applied external potential and interaction parameters between the atoms that create the condensation [4-7]. The BEC is given by $\Psi(x,t)$ macroscopic wave function and it is governed the Gross-Pitaevskii equation (GPE). This equation has a nonlinear structure and dependent on time and space [2,7]. The nonlinear term of GPE represents particle-particle interactions [8,9].

The studies to synchronize chaotic systems are very attractive because of the importance of applications in science and technology. After Carroll and Pecora's idea of synchronize certain subsystems of chaotic systems by coupling them with a signal in 1990 [10,11], have emerged different ways of synchronization between identical and distinct chaotic systems [12-15]. Because of the critical connection between control and synchronization, many control techniques are used to synchronize the chaotic systems [15-19]. Especially, active control technique is one of the most adequate technique widely accepted for chaos synchronization [14,20,21]. Hanie et.al. found the coupling between different modes of oscillations in BEC with large nonlinearity [22]. This shows us each mode of feedback carry the energy from one mode of oscillations to others. In particular, quantum synchronization of identical and nonidentical BECs could be

*Corresponding author, e-mail: eren.tosyali@bilgi.edu.tr

used to transfer information such as quantum signal transmission, quantum logic control, quantum information processing and quantum networks [23]. After that, control problems of the BEC systems have been attracted too much attention due to their rich dynamics. In order to improve and make more efficient the control of BEC systems, we study the synchronization of BEC under tilted bichromatical optical lattice and another external optical lattice is related to harmonic potential with constant amplitude in this paper. We examined behaviors of systems under these optical lattice potentials by Poincare sections of phase space and Lyapunov exponents. Active control technique is presented and applied to synchronize the given master-slave BEC systems. We think that the studies presented in this article will contribute positively to the works in this research area.

2. SYSTEM DESCRIPTION

A BEC is well described by classical GPE with a $\Psi(\mathrm{x,t})$ [8,9] which is macroscopic wave function. GPE equation explains the quantum effects of BEC very well. One dimensional GPE is given following

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + [V_{ext}(x) + g_{1D}|\Psi(x,t)|^2]\Psi(x,t)\,, \tag{1}$$

here m is boson's mass, $\hbar$ is Planck constant, V_{ext} is an external optical lattice potential trapping boson in the BEC. Also, g_{1D} define the interaction between atoms in the BEC and given by

$$g_{1D} = \frac{g_{3D}}{2\pi\, a_r^2} = 2a_s\, \hbar\, \omega_r$$

where a_s is the s-wave scattering length, this length is negative for attractive interaction between atoms, and it's positive for repulsive interaction. We take g_{1D} negative in our study. $a_r = \sqrt{\frac{\hbar}{m\omega_r}}$ and ω_r is ground state of a harmonic oscillator's frequency.

2.1. Master System

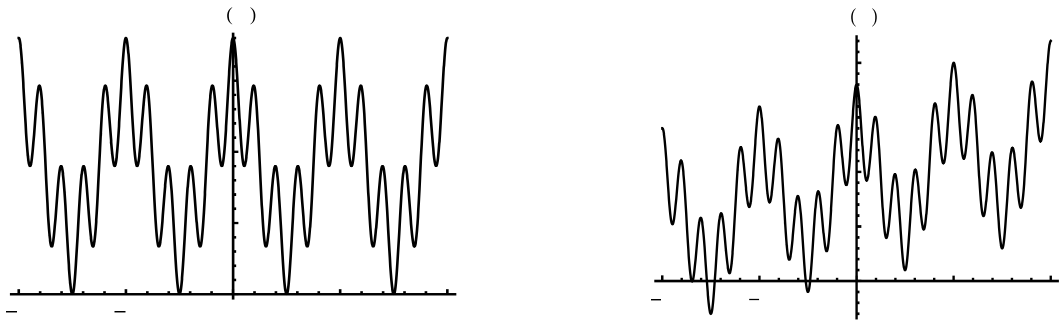
We choose trapping potential as

$$V_{ext}(x) = V(x) + Fx \tag{2}$$

where F is the internal force, and Fx produce a tilted potential, accelerates the bosons in the x axis and causes bosons tunnelling out of the traps [4,5,24]. Also $V(x)$ is the optical lattice potential and given by

$$\mathrm{V}(x) = V_1 \mathrm{Cos}^2(w_1 x) + V_2 \mathrm{Cos}^2(w_2 x), \tag{3}$$

here V_1 and V_2 are amplitudes of the external potential. The imbrication laser beams of different frequencies are able to produce bichromatic optical lattice potential. Therefore, bichromatic optical lattice potential can be called as double well potential. In this paper, we put in an additional tilted force in to bichromatic potential. That leads to accelerate bosons in the bichromatic potential. Also, bosons are able to be permitted jumping in one trap to another trap. Figure 1 shows the bichromatic optical laser form along the x-axis.



(a) *(b)*

Figure 1. *Display of Bichromatic potential for parameter sets* $\nu_1 = 1, \nu_2 = 0.8, w_1 = 2\pi, w_2 = 5\pi$, *(a)* $\Gamma = 0$, *(b)* $\Gamma = 0.1$

We consider a simple the solution of Equation (1) as below [24,25]. This solution allows us understanding of BEC dynamics. All atoms in the BEC at $T = 0$ are in one state and energy. The occupancy of states is described by $f(E) = 1/\left(e^{\frac{E-\mu}{KT}} - 1\right)$. The ground chemical potential μ is equal to ground state energy E_0. Therefore, $f(E_0) = \infty$ that means all atoms in BEC fill the same lowest energy level. Hence, the phase factor of the eigenstate of GPE is $e^{\frac{-iE_0 t}{\hbar}} = e^{\frac{-i\mu t}{\hbar}}$.

$$\Psi(x,t) = \Phi(x)e^{\frac{-i\mu t}{\hbar}}, \tag{4}$$

here $\Phi(x)$ is a time independent function and μ is the chemical potential. We fixed the value of the chemical potential as a parameter in GPE after making dimensionless. Normalized $\Phi(x)$ gives the total number of bosons in the BEC, as given,

$$\int |\Phi(x)|^2 \, dx = N, \tag{5}$$

here N is total number of bosons in the condensate. If we substitute Equations (3) and (4) into Equation (1), yields

$$\mu\Phi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi(x) + [V_1 Cos^2(w_1 x) + V_2 Cos^2(w_2 x) + Fx + g_{1D}|\Phi(x)|^2]\Phi(x), \tag{6}$$

we can rewrite in the form such as

$$\frac{d^2\Phi}{dx^2} = [\upsilon_1 \text{Cos}^2(w_1 x) + \upsilon_2 \text{Cos}^2(w_2 x) + \Gamma x - \gamma + \eta|\Phi|^2]\Phi. \tag{7}$$

In order to simplify the wave function we rescale dimensionless parameters as $\upsilon_1 = \frac{2mV_1}{\hbar^2}, \upsilon_2 = \frac{2mV_2}{\hbar^2}, \gamma = \frac{2m\mu}{\hbar^2}, \Gamma = \frac{2mF}{\hbar^2}, \eta = \frac{2mg_{1D}}{\hbar^2}$.

We consider a simple solution of Equation (7) as

$$\Phi(x) = \phi(x)e^{i\theta(x)}. \tag{8}$$

Substituting Equation (8) into Equation (7) leads to two coupled equations,

$$\frac{d^2\phi}{dx^2} = \phi\left(\frac{d\theta}{dx}\right)^2 + [\upsilon_1 \text{Cos}^2(w_1 x) + \upsilon_2 \text{Cos}^2(w_2 x) + \Gamma x - \gamma + \eta|\phi|^2]\phi, \tag{9a}$$

$$\frac{d}{dx}\left(2\phi^2\frac{d\theta}{dx}\right)=0. \tag{9b}$$

Equation (9b) express the flow density,

$$J=2\phi^2\frac{d\theta}{dx}. \tag{10}$$

If we substitute $\frac{J}{2\phi^2}=\frac{d\theta}{dx}$ into Equation (9a), we obtain second order nonlinear differential equation as follows,

$$\frac{d^2\phi}{dx^2}=\frac{J^2}{4\phi^3}++[\upsilon_1 Cos^2(w_1x)+\upsilon_2 Cos^2(w_2x)+\Gamma x-\gamma+\eta|\phi|^2]\phi. \tag{11}$$

To obtain analytical solution of Equation (11) is hard due to its nonlinearity and complexity. Therefore, we use numerical algorithm to solve the equation. To simplify calculation, we transform the system to first-order by changing variable to $x_1=\phi$ and $y_1=\frac{d\phi}{dx}$,

$$\frac{dx_1}{dx}=y_1, \tag{12a}$$

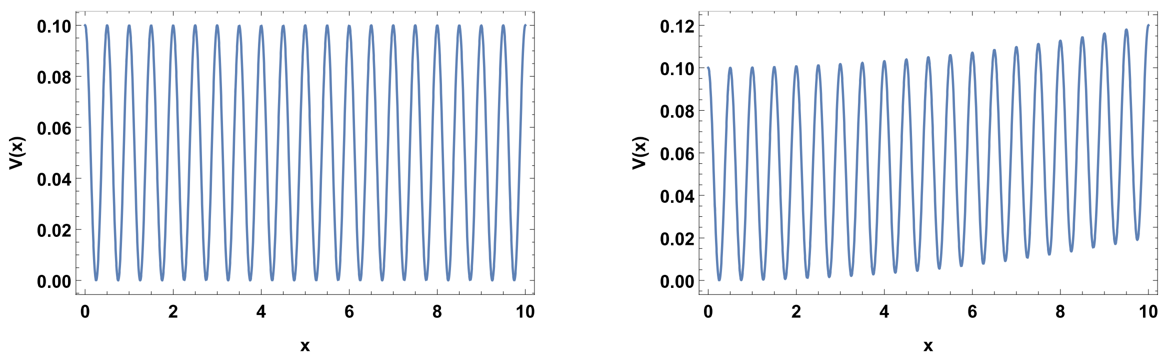
$$\frac{dy_1}{dx}=\frac{J^2}{4x_1^3}+[\upsilon_1 Cos^2(w_1x)+\upsilon_2 Cos^2(w_2x)+\Gamma x-\gamma+\eta|x_1|^2]x_1. \tag{12b}$$

2.2. Slave System

We describe the external trap potential as,

$$V_{ext}(x)=V_1 Cos^2(w_1x)+\frac{1}{2}\zeta\omega_x x^2. \tag{13}$$

The external potential V_{ext} consists of two parts. While the first part of external potential is related to optical lattice potential and the second part is related to harmonic potential. Here V_1 is the amplitude, ζ is a numerical quantity and ω_x is the wavenumber of lattice. In Figure 2, we present the evolution of the external potential for parameters set $V_1=0.1$, $w_1=2\pi$ (a) $\beta=0$ (b) $\beta=4\times10^{-4}$.



(a) (b)

Figure 2. *Display of the slave system potential for parameter sets* $\upsilon_1=0.1$, $w_1=2\pi$ *(a)* $\beta=0$ *(b)* $\beta=4\times10^{-4}$

We consider simple the solution of Equation (1) as below same with master system to obtain a better understanding of BEC dynamics

$$\Psi(x,t) = \Phi(x)e^{\frac{-i\mu t}{\hbar}} \tag{14}$$

where μ is the chemical potential and $\Phi(x)$ is a time independent real function. Normalized $\Phi(x)$ gives the total number of bosons in the BEC, as given,

$$\int |\Phi(\mathrm{x})|^2\, \mathrm{dx} = \mathrm{N}, \tag{15}$$

here N is total number of bosons in the condensate. Substitution of Equations (13) and (14) into Equation (1) give

$$\mu\Phi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi(x,t) + \left[V_1 Cos^2(\mathrm{w}_1 x) + \frac{1}{2}\zeta\omega_x x^2 + g_0|\Phi(x,t)|^2\right]\Phi(x,t), \tag{16}$$

which can also be rewritten in the form such as

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = \left[\upsilon_1 Cos^2(\mathrm{w}_1 x) + \frac{1}{2}\beta x^2 - \gamma + \eta|\Phi|^2\right]\Phi, \tag{17}$$

where $\left(\upsilon_1 = \frac{2mV_1}{\hbar^2}, \gamma = \frac{2m\mu}{\hbar^2}, \beta = \frac{2m\zeta\omega_x}{\hbar^2}, \eta = \frac{2mg_0}{\hbar^2}\right)$. We consider the solution of Equation (17) of the form

$$\Phi(\mathrm{x}) = \phi(\mathrm{x})\mathrm{e}^{\mathrm{i}\theta(\mathrm{x})}. \tag{18}$$

Substituting Equation (18) into Equation (17) leads to coupled differential equations,

$$\frac{\mathrm{d}^2\phi}{\mathrm{dx}^2} = \phi\left(\frac{\mathrm{d}\theta}{\mathrm{dx}}\right)^2 + \left[\upsilon_1 \mathrm{Cos}^2(\mathrm{w}_1\mathrm{x}) + \frac{1}{2}\beta\mathrm{x}^2 - \gamma + \eta|\phi|^2\right]\phi, \tag{19a}$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(2\phi^2\frac{\mathrm{d}\theta}{\mathrm{dx}}\right) = 0. \tag{19b}$$

Equation (19b) express the flow density,

$$J = 2\phi^2\frac{d\theta}{dx}. \tag{20}$$

If we substitute $\frac{J}{2\phi^2} = \frac{d\theta}{dx}$ into Equation (9a), we obtain second order nonlinear differential equation as follows,

$$\frac{d^2\phi}{dx^2} = \frac{J^2}{\phi^3} + \left[\upsilon_1 Cos^2(\mathrm{w}_1 x) + \frac{1}{2}\beta x^2 - \gamma + \eta|\phi|^2\right]\phi. \tag{21}$$

To simplify the numerical calculation, we rewrite the Equation (21) as first-order couple equation system by the transformation $x_2 = \phi$ and $y_2 = \frac{d\phi}{dx}$,

$$\frac{dx_2}{dx} = y_2, \tag{22a}$$

$$\frac{dy_2}{dx} = \frac{J^2}{4x_2^3} + \left[\upsilon_1 Cos^2(w_1 x) + \frac{1}{2}\beta x^2 - \gamma + \eta|x_2|^2\right] x_2. \tag{22b}$$

3. SYNCHRONIZATION AND ACTIVE CONTROL TECHNIQUE

Active control technique is an efficient control technique for synchronization in master-slave systems [20]. A nonlinear active controller drives the controlled chaotic slave system's states to achieve synchronize the master system's states. [14,16,21].

If we consider a master and slave systems described by the following

$$\frac{da}{dx} = Aa + g(a) \tag{23}$$

and

$$\frac{db}{dx} = Bb + f(b) + u(x) \tag{24}$$

where a and b are the state vectors, A, B are constant system matrices, $g(a)$ and $g(b)$ are nonlinear functions, and $u(x)$ is an active controller. The synchronization error in master-slave systems can be expressed as

$$\frac{de}{dx} = \frac{db}{dx} - \frac{da}{dx} = Ce + G(a,b) + u(x) \tag{25}$$

where $e = b - a$, $C = \bar{B} - \bar{A}$ is the common parts of the system matrices and $G(a,b) = f(a) - g(b) + (B - \bar{B})b - (A - \bar{A})a$. The aim is to design the proper active controller $u(x)$ to synchronize systems. The active controller should be designed to reach the system asymptotically stable at the origin. The active control function is described by

$$u(x) = -G(a,b) + \tau(x) \tag{26}$$

where $\tau(x) = -Ke$ is a linear controller and K is a linear gain matrix. Inserting of Equation (26) into (25) we obtain

$$\frac{de}{dx} = Ce + \tau(x), \tag{27}$$

we substitute $\tau(x)$ into Equation (27)

$$\frac{de}{dx} = (C - K)e. \tag{28}$$

If the dynamics of error function are asymptotically stable at the origin, the real parts of all eigenvalues of matrix C – K should be negative, [26,27] and also the controlled chaotic master and slave systems are exponentially synchronized for any initial conditions [26-28]. Using this synchronization principle, we obtain nonlinear active controller to synchronize the master and slave BEC chaotic systems in the next section.

4. APPLICATION OF ACTIVE CONTROL TECHNIQUE TO BEC

Active control technique is applied to synchronize the two chaotic BEC master and slave systems in this section. The master and slave systems are given as,

$$\frac{dx_1}{dx} = y_1, \tag{29a}$$

$$\frac{dy_1}{dx} = \frac{J^2}{4x_1^3} + [\upsilon_1 Cos^2(w_1 x) + \upsilon_2 Cos^2(w_2 x) + \Gamma x - \gamma + \eta|x_1|^2]x_1 \tag{29b}$$

and

$$\frac{dx_2}{dx} = y_2 + u_1, \tag{30a}$$

$$\frac{dy_2}{dx} = \frac{J^2}{4x_2^3} + \left[\upsilon_1 Cos^2(w_1 x) + \frac{1}{2}\beta x^2 - \gamma + \eta|x_2|^2\right]x_2 + u_2. \tag{30b}$$

u_1 and u_2 are controllers which synchronize two chaotic systems. The error dynamics are given by $e_1 = x_2 - x_1$ and $e_2 = y_2 - y_1$. We obtain error dynamics by subtracting Equation (30a), (30b) from Equation (29a), (29b)

$$\frac{de_1}{dx} = e_2 + u_1(x), \tag{31a}$$

$$\frac{de_2}{dx} = \frac{J^2}{4x_2^3} - \frac{J^2}{4x_1^3} - \upsilon_2 Cos^2(w_2 x)x_1 + [\upsilon_1 Cos^2(w_1 x) - \gamma]e_1 + \frac{1}{2}\beta x^2 x_2 - \Gamma x x_1 + \eta(x_2^3 - x_1^3) + u_2(x). \tag{31b}$$

We consider the error functions as follows,

$$u_1(x) = \tau_1(x), \tag{32a}$$

$$u_2(x) = -\frac{J^2}{4x_2^3} + \frac{J^2}{4x_1^3} - \eta(x_2^3 - x_1^3) + \upsilon_2 Cos^2(w_2 x)x_1 - \frac{1}{2}\beta x^2 x_2 + \Gamma x x_1 + \tau_2(x). \tag{32b}$$

The Substitution of Equation (32a) and Equation (32b) into Equation (31a) and Equation (31b) leads,

$$\frac{de_1}{dx} = e_2 + \tau_1(x), \tag{33a}$$

$$\frac{de_2}{dx} = +[\upsilon_1 Cos^2(w_1 x) - \gamma]e_1 + \tau_2(x). \tag{33b}$$

τ_1 and τ_2 are control inputs, and they are also as function of e_1 and e_2. The control inputs τ_1 and τ_2 are given as

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = D\begin{bmatrix}e_1\\ e_2\end{bmatrix} \tag{34}$$

where $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2 × 2 constant feedback matrix. The system as can be written as,

$$\begin{bmatrix} \frac{de_1}{dx} \\ \frac{de_2}{dx} \end{bmatrix} = C \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (35)$$

here C is is the coefficient matrix as below

$$C = \begin{bmatrix} 0 + a & 1 + b \\ \upsilon_1 Cos^2(w_1 x) - \gamma + c & 0 + d \end{bmatrix}. \quad (36)$$

Because of the Routh-Hurwitz criterion [29], Lyapunov stability theory and active control technique, master- slave BEC systems are synchronized. The particular choice of controller gains $a = -1,\ b = -1,\ d = -1, c = -(\upsilon_1 Cos^2(w_1 x) - \gamma)$,

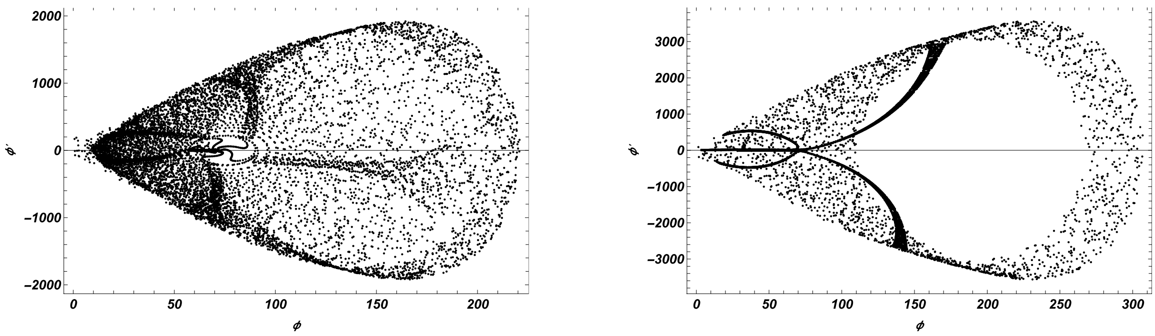
$$\tau_1(x) = -e_1 - e_2, \quad (37a)$$

$$\tau_2(x) = -(\upsilon_1 Cos^2(w_1 x) - \gamma)e_1 - e_2. \quad (37b)$$

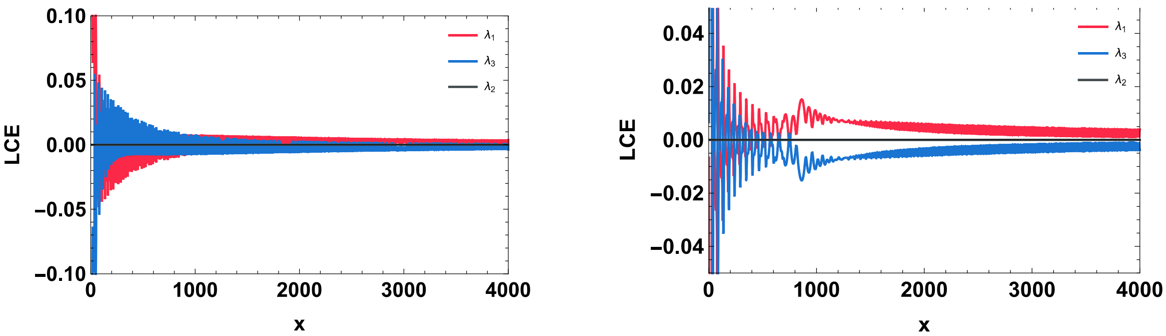
5. NUMERICAL RESULTS FOR CHAOS SYNCHRONIZATION IN BEC

Analytical solution of GPE is quite difficult and often unlikely because of the integrability of it can be easily broken by different forms of external potentials. The nonlinearity of interactions which makes the systems complex, so we use the numerical methods to have a view on the dynamic of the system. In Section 5, we give the simulation results for chaotic synchronization of the master (Equations (29a), (29b)) and slave systems (Equations (30a), (30b)) by using the fourth order Runge-Kutta algorithm. The master and slave systems given in Equation (29) and (30) are solved with step size 0.1. and length 4000. Parameters are $\upsilon_1 = 0.2,\ \upsilon_2 = 1,\ \beta = 0.0001,\ w_1 = 2\pi,\ w_2 = 5\pi,\ \eta = -0.015,\ J = 0.4,\ \gamma = 0.5$ and $\Gamma = 0.1$. Possible Initial conditions are chosen randomly as $(x_1(0), y_1(0)) = (-0.56, 0.3)$ and $(x_2(0), y_2(0)) = (0.6985, 0.2668)$.

In Figure 3, we construct the Poincare sections for the master and slave systems for the fixed parameters; $\upsilon_1 = 0.2, \upsilon_2 = 1, \beta = 0.0001, w_1 = 2\pi, w_2 = 5\pi, \eta = -0.015, J = 0.4, \gamma = 0.5, \Gamma = 0.1$ and randomly selected possible initial contiditions $(x_1(0), y_1(0)) = (-0.56, 0.3)$ and $(x_2(0), y_2(0)) = (0.6985, 0.2668)$. The systems show chaotic behavior depending on system parameters for both master and slave systems. We also calculated LCEs for master and slave systems. We use algorithm in order to calculate Lyapunov exponents which is given in references [30]. LCEs are $\lambda_1 = 0.00369066, \lambda_2 = 0, \lambda_3 = -0.00369086$ for master systems and $\lambda_1 = 0.00394237, \lambda_2 = 0, \lambda_3 = -0.003942239$ for slave systems.



(a) *(b)*



(c) *(d)*

***Figure 3.** (a) Chaotic dynamics of master system in phase (b) Chaotic dynamics of slave system in phase (c) Lyapunov Characteristic Exponents of master system (d) Lyapunov Characteristic Exponents of slave system*

Figure 4 displays that the error signals rapidly go to zero and this causes the synchronization of the master system with the slave system for different initial values. The controller is activated at $x = 0$ and solved for the initial conditions are $\left(e_1(0), e_2(0)\right) = (0.1385, -0.0332)$ obtained from intial conditions are $\left(x_1(0), y_1(0)\right) = (-0.56, 0.3)$ and $\left(x_2(0), y_2(0)\right) = (0.6985, 0.2668)$.

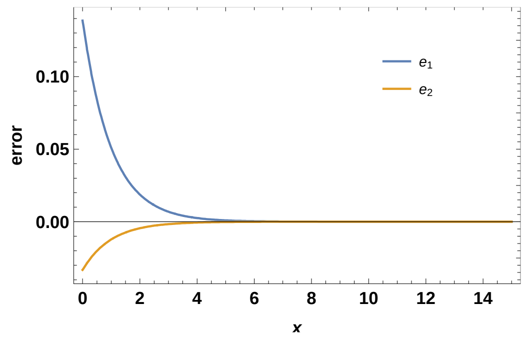
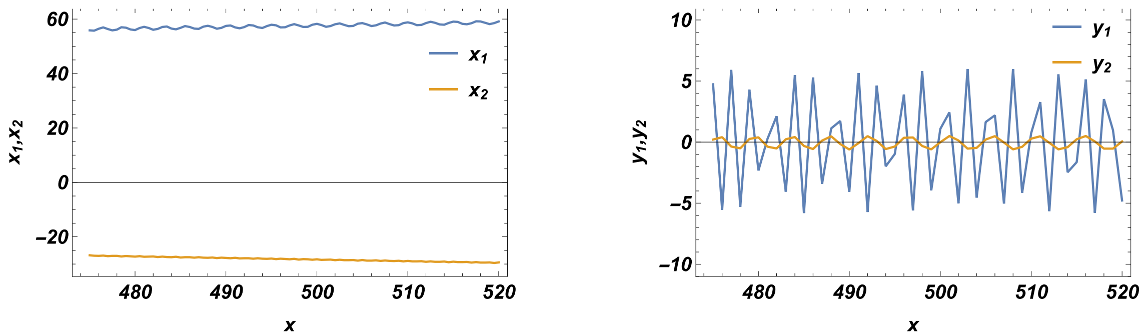
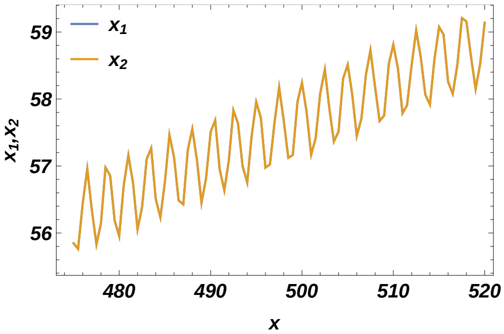


***Figure 4.** Error dynamics of (e_1, e_2) the controller activated at the $x = 0$ for $\left(e_1(0), e_2(0)\right) =$ $(0.1385, -0.0332)$.*

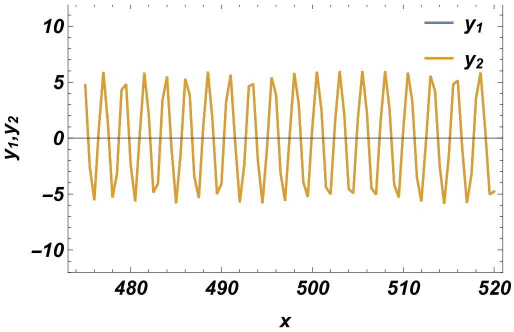
In Figure 5, we build the spatial evolution diagrams of unsynchronized and synchronized master and slave systems for the different initial conditions $\left(x_1(0), y_1(0)\right) = (-0.56, 0.3)$ and $\left(x_2(0), y_2(0)\right) =$ $(0.6985, 0.2668)$. The master and slave systems are solved numerically for 3500 steps with 0.1 step size. These results are in harmony with those in Figure 4.



(a) *(b)*



(c)



(d)

Figure 5. *(a) and (b) spatial evolution of unsynchronized master and slave systems (c) and (d) synchronized master and slave systems for* $470 < x < 520$.

6. CONCLUSION

Chaos in BEC systems can appear when a destructive process occurs in the system and undermines the stability of the condensates. So, knowing the structure of the system and the investigations on estimations and controlling chaos are great significance for the creation of BEC. In this study, we firstly present the BEC systems held two different external optical potential, after that we obtain chaotic synchronization for our BEC. Changing the parameters of external potentials can inspect chaotic behavior of BEC and it was observed that some system parameters defining BEC system cause the evolution of system dynamic. Also we calculate the LCEs, are an important tool in studying chaos, for master-slave systems. The results of LCEs confirm the presence of chaos in the systems. We apply master-slave active control technique to achieve synchronization of systems with different external lattice potentials in their chaotic states. As is seen from obtained numerical results, controlled BEC system approach synchronization for any initial condition in their chaotic states by proper controller design and the error of master-slave systems converge to zero rapidly. The ability to design synchronized systems in chaotic BEC systems can open effective opportunities to apply chaos to other areas. In this sense, we believe that this study will make a significant contribution in this subject.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Gutzwiller, M., “Chaos in classical and quantum mechanics”, First Edition, New York, Springer, 116-155, (1990).

[2] Bose, S., “Plancks gesetz und lichtquantenhypothese”, Zeitschrift für Physik, 26: 178–181, (1924).

[3] Einstein, A., “Quantentheorie des einatomigen idealen gases”, Akademie Vorträge: Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1: 245–247, (1924).

[4] Fang, J. S., Liao, X. P., “Stability of trapped bose einstein condensates in one dimensional tilted optical lattice potential”, Chinese Physics B, 20(4): 040310 1-5, (2011).

[5] Adhikari, S. K., “Localization of a bose einstein condensate vortex in a bichromatic optical lattice”, Physical Review A, 81: 043636 1-9, (2010).

[6] Chong, G., Hai, W., Xie, Q., “Spatial chaos of trapped bose–einstein condensate in one-dimensional weak optical lattice potential”, Chaos: An Interdisciplinary Journal of Nonlinear Science, 14: 217–223, (2004).

[7] Hai, W., Lu, G., Zhong, H., “Regular and chaotic bose-einstein condensate in an accelerated wannier-stark lattice”, Physical Review A, 79: 053610 1-6, (2009).

[8] Gross, E. P., “Structure of a quantized vortex in boson systems”, A Nuovo Cimento, 20: 454–477, (1961).

[9] Pitaevskii, L., Stringari, S., “Bose-Einstein condensation”, First Edition , Oxford, A Clarendon Press Publication, 184-213, (2016).

[10] Pecora, L. M., Carroll, T. L., “Synchronization in chaotic systems”, Physical Review Letter, 64: 821–824, (1990).

[11] Pecora, L., Carroll, T., “Synchronized chaotic signals and systems”, ICASSP-92: 1992 IEEE International Conference on Acoustics, Speech, and Signal Processing, San Francisco USA, 4: 137–140, (1992).

[12] Yang, Q., Wu, Z. M., Wu, J. G., Xia, G. Q., “Influence of injection patterns on chaos synchronization performance between a multimode laser diode and a single-mode laser”, Optics Communications, 281: 5025–5030, (2008).

[13] Robert, B., Feki, M., Iu, H. H. C., “Control of a pwm inverter using proportional plus extended time-delayed feedback”, International Journal of Bifurcation and Chaos, 16(01): 113–128, (2006).

[14] Hamidzadeh, S. M., Esmaelzadeh, R., “Control and synchronization chaotic satellite using active control”, International Journal of Computer Applications, 94(10): 29–33, (2014).

[15] Hsu, C. F., Tsai, J. Z., Chiu, C. J., “Chaos synchronization of nonlinear gyros using self-learning pid control approach”, Applied Soft Computing, 12(1): 430–439, (2012).

[16] Yalcın, M. E., Suykens, J. A. K., Vandewalle, J. P. L., “Cellular Neural Networks, Multi-Scroll Chaos and Synchronization”, First Edition, Singapore, World Scientific, 50: 109-113, (2005).

[17] Idowu, B. A., Hai, W., Vincent, U. E., “Synchronization and stabilization of chaotic dynamics in a quasi-1d Bose-Einstein condensate”, Journal of Chaos, 8: 723581 1-8, (2013).

[18] Zhang, Z. Y., Feng, X. Q., Yao, Z. H., Jia, H. Y., “Chaotic synchronization in Bose–Einstein condensate of moving optical lattices via linear coupling”, Chinese Physics B, 24(11): 110503 1-6, (2015).

[19] Park, J. H., “Chaos synchronization of a chaotic system via nonlinear control”, Chaos, Solitons and Fractals, 25(3): 579–584, (2005).

[20] Ucar, A., Lonngren, K. E., Bai, E. W., “Synchronization of the unified chaotic systems via active control”, Chaos, Solitons and Fractals, 27(5): 1292– 1297, (2006).

[21] Yassen, M., “Chaos synchronization between two different chaotic systems using active control”, Chaos, Solitons and Fractals, 23(1): 131–140, (2005).

[22] Hanie, S. A., Ferris, A. J., Close, J. D., Hope, J. J., “Control of an atom laser using feedback”, Physical Reviev A, 69(1): 013605 1-6, (2004).

[23] Li, W., Li. C., Song, H., “Quantum synchronization of chaotic oscillator behaviors among coupled BEC–optomechanical systems”, Quantum Information Processing, 16(80): 1-15, (2017).

[24] Dalfovo, F., Giorgini, S., Pitaevskii, L. P., Stringari, S., “Theory of Bose-Einstein condensation in trapped gases”, Review Modern Physics, 71: 463–512, (1999).

[25] Coen, S., Haelterman, M., “Domain wall solitons in binary mixtures of Bose- Einstein condensates”, Physical Review Letter, 87: 140401 1-4, (2001).

[26] Routh, E., “A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion”, First Edition, New York, Macmillan, 82-89, (1877).

[27] Hurwitz, A., “Ueber die bedingungen, unter welchen eine gleichung nur wurzeln mit negativen reellen theilen besitzt”, Mathematische Annalen, 46: 273– 284, (1895).

[28] Park, J. H., “Chaos synchronization of a chaotic system via nonlinear control”, Chaos, Solitons and Fractals, 25(3): 579–584, (2005).

[29] Khalil, H., “Nonlinear Systems”, Third Edition, New Jersey, Pearson Prentice Hall, 111-174, (2001).

[30] Sandri, M., “Numerical calculation of lyapunov exponents”, The Mathematica Journal, 6(3): 78-84, (1996).