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ON THE NON-ISOTROPIC DISTANCE

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ABSTRACT

In this study, we have generated the Gauss-Weierstrass integral and Abel-Poisson integral which are generated by the λ-Abel-Poisson kernel. These kernels depen on non-isotropic distance. We study relationships between Gauss-Weierstrass integral and Abel-Poisson integral with Riesz potential generated by the non-isotropic distance.

Key Words: Non-Isotropic Distance, Weierstrass Kernel, Abel-Poisson Kernel, Riesz Potential.

İZOTROPİK OLMAYAN UZAKLIKLAR ÜZERİNE

ÖZET

Bu çalışmada, λ-Abel Poisson çekirdeği yardımıyla Gauss-Weierstrass integralini ve Abel-Poisson integralini elde ettik. Bu çekirdekler izotropik olmayan uzaklık ile elde edilen Riesz potansiyalleri ile Gauss-Weierstrass integrali ve Abel-Poisson integrali arasındaki ilişkiler çalışıldı.

Anahtar Kelime: İzotropik Olmayan Uzaklık, Weierstrass Çekirdeği, Abel Poisson çekirdeği, Riesz Potansiyeli.

1. INTRODUCTION

The classical Weierstrass kernel , Poisson kernel, Abel Poisson integral and their properties are well known [1]. In the referance [2], relationships between the Riesz Potential generated by the generalized Shift operators and Gauss Weierstrass integrals were proved.

In this article we prove relationships between the Riesz potentials generated by the non-isotropic distance and Weierstrass integral, Abel Poisson integral which are generated by the non-isotropic distance.

Firstly, we give some notations and definitions.

Let

$$|x|_\lambda := \left(|x_1|^{\frac{1}{\lambda_1}} + |x_2|^{\frac{1}{\lambda_2}} \ldots + |x_n|^{\frac{1}{\lambda_n}} \right)^{\frac{|\lambda|}{n}}$$

be the non-isotropic λ-distance [3]. Where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_k > 0$, $k = 1,2,\ldots,n$, $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ and $x \in R^n$.

The λ-Weierstrass kernel was defined as follows [4]:

$$W_\lambda(x,t) = c_n t^{-|\lambda|} e^{\frac{|x|^{\frac{n}{|\lambda|}}}{4t}} \quad , t > 0$$

where $c_n = \dfrac{1}{w_n 2^{2|\lambda|-1} \Gamma(|\lambda|)}, w_n = \int\limits_{S^{n-1}} \Omega(\theta) d\theta.$

The λ-Abel Poisson kernel was defined as follows [4]:

$$P_\lambda(x,t) = c_n^* \frac{t}{(t^2 + |x|_\lambda^{\frac{n}{|\lambda|}})^{|\lambda| + \frac{1}{2}}}, \; t > 0$$

where $c^*_n = \dfrac{2\Gamma(|\lambda| + \frac{1}{2})}{w_n \Gamma(|\lambda|) \Gamma(\frac{1}{2})}, w_n = \int\limits_{S^{n-1}} \Omega(\theta) d\theta.$

Passing to spherical coordinates by the following formulas:

$$y_1 = (\rho\cos\theta_1)^{2\lambda_1}, ..., y_n = (\rho\sin\theta_1\sin\theta_2...\sin\theta_{n-1})^{2\lambda_n}$$

we obtained that $|x|_\lambda = \rho^{\frac{2|\lambda|}{n}}$. It can be seen that the Jacobian $J(\rho,\theta)$ of this transformation is $J(\rho,\theta) = \rho^{\frac{2|\lambda|}{n}}\Omega(\theta)$, where $\Omega(\theta)$ is the bounded function, which depend only on angles $\theta_1, \theta_2, ..., \theta_{n-1}$.

The Riesz Potentials generated by the non-isotropic λ-distance is defined as follows:

$$(I_\lambda^\alpha f)(x) = C(\alpha,\lambda) \int_{R^n} f(x-y)|y|_\lambda^{\alpha-n} dy, \quad 0 < \alpha < n \tag{1}$$

where $C(\alpha,\lambda) = \dfrac{\Gamma(|\lambda| - \frac{\alpha|\lambda|}{n})2^{1-\frac{\alpha|\lambda|}{n}}}{w_n\Gamma(|\lambda|)\Gamma(\frac{\alpha|\lambda|}{n})}$.

Lemma 1: Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ and $A(\lambda) = \dfrac{\lambda_{\max}}{\lambda_{\min}}$ for all $\lambda_k > 0$, $k = 1,2,...,n$, then we have

$$\frac{1}{A(\lambda)}\frac{|\lambda|}{n} \le \lambda_k \le A(\lambda)\frac{|\lambda|}{n}. \tag{2}$$

Proof: If $\lambda_{\min} = \min\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\lambda_{\max} = \max\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then, it is clear that

$$n\lambda_{\min} \le |\lambda| \le n\lambda_{\max}. \tag{3}$$

In this case, there are following inequalies :

$$\lambda_k \ge \lambda_{\min} = \frac{\lambda_{\min}}{\lambda_{\max}}\lambda_{\max} = \frac{1}{A(\lambda)}\lambda_{\max}$$

$$= \frac{1}{A(\lambda)} \frac{n\lambda_{\max}}{n} \geq \frac{1}{A(\lambda)} \frac{|\lambda|}{n} \quad (from(3))$$

and

$$\lambda_k \leq \lambda_{\max} = \frac{\lambda_{\max}}{\lambda_{\min}} \lambda_{\min} = A(\lambda)\lambda_{\min\,x}$$

$$= A(\lambda)\frac{n\lambda_{\min}}{n} \leq A(\lambda)\frac{|\lambda|}{n} \quad (from(3))$$

Lemma 2: For all $k = 1,2,...,n$

$$|y_k| \leq \begin{cases} |y|_\lambda^{A(\lambda)} & , if\ |y|_\lambda \geq 1 \\ |y|_\lambda^{\frac{1}{A(\lambda)}} & , if\ |y|_\lambda \leq 1 \end{cases}$$

Proof: From the non-isotropic λ-distance definition, we have the following inequality;

$$|y_k| = \left(|y_k|^{\frac{1}{\lambda_k}}\right)^{\lambda_k} \leq \left(|y_1|^{\frac{1}{\lambda_k}} + ... + |y_k|^{\frac{1}{\lambda_k}} + ... + |y_n|^{\frac{1}{\lambda_n}}\right)^{\lambda_k} \quad (4)$$

If $|y|_\lambda \geq 1$, then $\left(|y_1|^{\frac{1}{\lambda_1}} + ... + |y_n|^{\frac{1}{\lambda_n}}\right) \geq 1$. From Lemma 1 and formula (4) we have

$$|y_k| \leq \left(|y_1|^{\frac{1}{\lambda_1}} + ... + |y_n|^{\frac{1}{\lambda_n}}\right)^{A(\lambda)\frac{|\lambda|}{n}} = |y|_\lambda^{A(\lambda)}.$$

On the other hand, for $|y|_\lambda \leq 1$ we have

$$|y_k| \leq \left(|y_1|^{\frac{1}{\lambda_1}} + \ldots + |y_n|^{\frac{1}{\lambda_n}} \right)^{\frac{1}{A(\lambda)} \frac{|\lambda|}{n}} = |y|_\lambda^{\frac{1}{A(\lambda)}}.$$

For non-isotropic λ-distance, there is the following equality

$$\left(|t^{\lambda_1} y_1|^{\frac{1}{\lambda_1}} + \ldots + |t^{\lambda_n} y_n|^{\frac{1}{\lambda_n}} \right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}} |y|_\lambda, \; t > 0.$$

This equality give us that non-isotropic λ-distance is at the order of a homogenous function $\frac{|\lambda|}{n}$. So the non-isotropic λ-distance has following properties.

a. $|y|_\lambda = 0 \Leftrightarrow y = 0$

b. $|t^\lambda y|_\lambda = |t|^{\frac{|\lambda|}{n}} |y|_\lambda$

c. $|x + y|_\lambda \leq 2^{\left(1 + \frac{1}{\lambda_{\min}}\right)^{\frac{|\lambda|}{n}}} \left(|x|_\lambda + |y|_\lambda \right)$

By $V_t f$ we define the Gauss-Weierstrass integral generated by the non-isotropic λ-distance such that.

$$(V_t f)(x) = \int_{R^n} f(y) W_\lambda(x - y, t) dy \;, \; t > 0. \tag{5}$$

Furthermore we define Abel-Poisson integral generated by the non-isotropic λ-distance with the following equality.

$$(A_t f)(x) = \int_{R^n} f(y) P_\lambda(x - y, t) dy \;, \; t > 0. \tag{6}$$

Theorem 1: There is the following equality for $W_\lambda(x,t),\, t > 0$

$$\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1} W_\lambda(x,t)dt = \frac{2^{1-\frac{\alpha|\lambda|}{n}}\Gamma(|\lambda|-\frac{\alpha|\lambda|}{n})}{w_n\Gamma(|\lambda|)}|x|_\lambda^{\alpha-n}$$

where $0<\alpha<n$ and $w_n = \int_{S^{n-1}}\Omega(\theta)d\theta.$

Proof: Proof of Theorem 1 is easy by Gamma function.

Theorem 2: Let $f\in L_p, 1<p<\infty, 0<\alpha<n.$ Then following equality holds for almost every $x\in R^n$.

$$(I_\lambda^\alpha f)(x) = \frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}(V_t f)(x)dt \tag{7}$$

Proof: Firstly we consider the right side of (7) then we have the following equality by Theorem 1 and formula (1).

$$\begin{aligned}\frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}(V_t f)(x)dt &= \frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}\left\{\int_{R^n} f(y)W_\lambda(x-y,t)dy\right\}dt \\ &= \frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty f(x-y)\left\{\int_{R^n} t^{\frac{\alpha|\lambda|}{n}-1}W_\lambda(y,t)dy\right\}dt \\ &= C(\alpha,\lambda)\int_{R^n} f(x-y)|y|_\lambda^{\alpha-n}dy \\ &= (I_\lambda^\alpha f)(x).\end{aligned}$$

Theorem 3: If $f\in L_p, 1<p<\frac{n}{\alpha}, 0<\alpha<n$, then

$$(I_\lambda^\alpha(V_\lambda f))(x)=\frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}(V_t(V_\lambda f))(x)dt.$$

Proof: From the right hand side of the equality (8), we have

$$\frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}(V_t(V_\lambda f))(x)dt=$$

$$\frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty t^{\frac{\alpha|\lambda|}{n}-1}\left\{\int_{R^n}(V_t f)(y)W_\lambda(x-y,t)dy\right\}dt$$

$$=\frac{1}{\Gamma(\frac{\alpha|\lambda|}{n})}\int_0^\infty (V_t f)(x-y)\left\{\int_{R^n} t^{\frac{\alpha|\lambda|}{n}-1}W_\lambda(y,t)dy\right\}dt$$

$$=C(\alpha,\lambda)\int_{R^n}(V_t f)(x-y)|y|_\lambda^{\alpha-n}dy$$

$$=(I_\lambda^\alpha(V_\lambda f))(x).$$

Theorem 4: For the $P_\lambda(x,t)$-Abel kernel, we have the equality of

$$\int_0^\infty t^{\frac{2\alpha|\lambda|}{n}-1}P_\lambda(x,t)dt=\frac{\Gamma(\frac{\alpha|\lambda|}{n}+\frac{1}{2})\Gamma(|\lambda|-\frac{\alpha|\lambda|}{n})}{w_n\Gamma(|\lambda|)\Gamma(\frac{1}{2})}|x|_\lambda^{\alpha-n}.$$

Proof: For the proof of this theorem, the following equality will be enough.

$$\int_0^\infty x^{m-1}(1+x)^{-m-n}dx=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ m,n>0.$$

Theorem 5: Let $f\in L_p$, $0<\alpha<n$. In this case there is the following inequality for almost every $x\in R^n$.

$$(I_{\lambda}^{\alpha} f)(x) = \frac{1}{\Gamma(\frac{2\alpha|\lambda|}{n})} \int_{0}^{\infty} t^{\frac{2\alpha|\lambda|}{n}-1} (A_t f)(x) dt.$$

Theorem 6: Let $f \in L_p$, $0 < \alpha < n$ and $1 \leq p < \frac{\alpha}{n}$. Then we have

$$(I_{\lambda}^{\alpha} (A_{\lambda} f))(x) = \frac{1}{\Gamma(\frac{2\alpha|\lambda|}{n})} \int_{0}^{\infty} t^{\frac{2\alpha|\lambda|}{n}-1} (A_t (A_{\lambda} f))(x) dt.$$

Theorem 5 and Theorem 6 have similar proofs with Theorem 2 and Theorem 3.

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