Nevşehir Bilim ve Teknoloji Dergisi Cilt 2(2) 75-79 2013

Lim-3 Durumundaki 4. Mertebe Operatörlerin Dissipatif Genişlemeleri

Hüseyin TUNA[1,*]

[1]*Mehmet Akif Ersoy Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Burdur*

Özet

Bu çalışmada, Lim-3 durumundaki skaler 4. mertebeden difereasiyel operatörlerinin maksimal dissipatif, kendine eş ve diğer genişlemeleri verilmiştir.

Anahtar Kelimeler: Dissipatif genişlemeler, kendine eş genişlemeler, sınır değer uzayı, sınır koşulu

Dissipative Extensions of Fourth Order Differential Operators in the Lim -3 Case[2]

Abstract

In this article, we give a description of all maximal dissipative, self adjoint and other extensions of scalar fourth order differential operators in the lim 3 case.

Keywords: Dissipative extensions, self adjoint extensions, a boundary value space, boundary condition

[*] e-mail: hustuna@gmail.com

[2] This work was presented in "International Conference on Analysis and Applied Mathematics", Gümüşhane/TURKEY, on 18-21 October 2012 and only the abstract of this work was published in the "Abstract Book" of the "International Conference on Analysis and Applied Mathematics".

1. Introduction

The theory of extensions of symmetric operators developed orginally by J. Von Neumann [1]. The problem on the description of all self adjoint extensions of a symmetric operator in terms of abstract boundary conditions was put forward for the first time in Calkin [2]. Later, Rofe- Beketov [3] described self adjoint extensions of a symmetric operator in terms of abstract boundary conditions with aid of linear relations. Bruk [4] and Kochubei [5] are introduced the notion of a space of boundary values. They described all maximal dissipative, acretive, self adjoint extensions of symmetric operators. For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [6].

A description of self adjoint extensions of a second order operator on an infinite interval was obtained by Fulton [7] and Krein [8]. For a scalar fourth order equation and two term differential expressions of arbitrary even order, the same question was investigated by Khol'kin [9], Mirzoev [10]. Gorbachuk [11] obtained a description of self adjoint extensions of Sturm Liouville operators with an operator potential in the absolutely indeterminate case. In the case when the deficiency indices take indeterminate values, a description of self adjoint extensions of differential operators was given in the works of Allahverdiev [12], Guseinov and Pashaev [13], Maksudov and Allahverdiev [14], Malamud and Mogilevsky [15], Mogilevsky [16].

In this paper, a space of boundary value is constructed for scalar fourth order differential operators in the Lim-3 case. We describe all maximal dissipative, acretive, self adjoint and other extensions in terms of boundary conditions.

2. Extensions of Fourth Order Differential Operators in the Lim-3 Case

Let us consider the differential expression

$$l(y)=y^{(4)}+q(x)y,\ 0\leq x<+\infty, \quad (2.1)$$

where $q(x)$ is a real continuous function in $[0,\infty)$.

We denote by L_0 the closure of the minimal operator (see [17]) generated by (2.1) and by D_0 its domain. Further, we denote by the set of all functions $y(x)$ from $L_2(0,\infty)$ whose first three derivatives are locally absolutely continuous in $[0,\infty)$ and $l(y)\in L_2(0,\infty)$; D is the domain of the maximal operator L, and $L=\boldsymbol{L_0^*}$ (see [17]).

Assume that $q(x)$ be such that the operator L_0 has defect index (3,3). Let $v_1(x),v_2(x),v_3(x)$ denote the solutions of $l(y)=0$ satisfying the initial conditions

$$v_1(0)=1,v_1'(0)=0,v_1''(0)=0,v_1'''(0)=0,$$
$$v_2(0)=0,v_2'(0)=1,v_2''(0)=0,v_2'''(0)=0,$$
$$v_3(0)=0,v_3'(0)=0,v_3''(0)=1,v_3'''(0)=0,$$

$v_1(x),v_2(x),v_3(x)$ are linearly independent and their Wronskian equals one. Since L_0 has defect index (3,3), $v_1(x),v_2(x),v_3(x)\in L_2(0,\infty)$.

We denote by Γ_1,Γ_2 the linear maps from D to C^3 defined by the formula

$$\boldsymbol{\Gamma_1 f}=\begin{pmatrix}\boldsymbol{f(0)}\\ \boldsymbol{f'(0)}\\ [\boldsymbol{f},\boldsymbol{v}_3]_\infty\end{pmatrix},\ \boldsymbol{\Gamma_2 f}=\begin{pmatrix}\boldsymbol{f'''(0)}\\ \boldsymbol{f''(0)}\\ [\boldsymbol{f},\boldsymbol{v}_2]_\infty\end{pmatrix}, \quad (2.2)$$

where

$[\mathbf{y},\mathbf{z}]_x$=[y‴(x)z(x)-y(x)z‴(x)]-[y″(x)z′(x)-y′(x)z″(x)] (0≤x<∞).

Lemma 1. For arbitrary y, z ∈D

$$(\boldsymbol{Ly},\boldsymbol{z})_{\mathbf{L}^2}-(\boldsymbol{y},\boldsymbol{Lz})_{\mathbf{L}^2} = (\boldsymbol{\Gamma}_1\boldsymbol{y},\boldsymbol{\Gamma}_2\boldsymbol{z})_{\mathbf{C}^3} - (\boldsymbol{\Gamma}_2\boldsymbol{y},\boldsymbol{\Gamma}_1\boldsymbol{z})_{\mathbf{C}^3}\,.$$

Proof. For every y,z∈D we have Green's formula

$$(\boldsymbol{Ly},\boldsymbol{z})_{\mathbf{L}^2}-(\boldsymbol{y},\boldsymbol{Lz})_{\mathbf{L}^2}=[\boldsymbol{y},\bar{\boldsymbol{z}}]_\infty-[\boldsymbol{y},\bar{\boldsymbol{z}}]_{\mathbf{0}}.$$

Then

$(\boldsymbol{\Gamma}_1\boldsymbol{y},\boldsymbol{\Gamma}_2\boldsymbol{z})_{\mathbf{C}^3}-(\boldsymbol{\Gamma}_2\boldsymbol{y},\boldsymbol{\Gamma}_1\boldsymbol{z})_{\mathbf{C}^3}$=y(0)z‴(0)-z(0)y‴(0) +y″(0)z′(0)-z″(0)y′(0) $+[\boldsymbol{y},\boldsymbol{v}_2]_\infty\,[\bar{\boldsymbol{z}},\boldsymbol{v}_3]_\infty$ $-[\bar{\boldsymbol{z}},\boldsymbol{v}_2]_\infty\,[\boldsymbol{y},\boldsymbol{v}_3]_\infty$.

We know that every y, z∈D

$$[\boldsymbol{y},\boldsymbol{v}_2]_\infty[\bar{\boldsymbol{z}},\boldsymbol{v}_3]_\infty-[\bar{\boldsymbol{z}},\boldsymbol{v}_2]_\infty[\mathbf{y},\mathbf{v}_3]_\infty=[\boldsymbol{y},\bar{\boldsymbol{z}}]_\infty \text{(see [9])}.$$

Hence

$$(\boldsymbol{\Gamma}_1\boldsymbol{y},\boldsymbol{\Gamma}_2\boldsymbol{z})_{\mathbf{C}^3} - (\boldsymbol{\Gamma}_2\boldsymbol{y},\boldsymbol{\Gamma}_1\boldsymbol{z})_{\mathbf{C}^3} =[\boldsymbol{y},\bar{\boldsymbol{z}}]_\infty-[\boldsymbol{y},\bar{\boldsymbol{z}}]_{\mathbf{0}}.$$

Then we have

$$(\boldsymbol{Ly},\boldsymbol{z})_{\mathbf{L}^2}-(\boldsymbol{y},\boldsymbol{Lz})_{\mathbf{L}^2} = (\boldsymbol{\Gamma}_1\boldsymbol{y},\boldsymbol{\Gamma}_2\boldsymbol{z})_{\mathbf{C}^3} - (\boldsymbol{\Gamma}_2\boldsymbol{y},\boldsymbol{\Gamma}_1\boldsymbol{z})_{\mathbf{C}^3}\,.$$

Lemma 2. For any complex numbers $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\beta_0,\beta_1$, there is a function y ∈D satisfying

$$y(0)=\alpha_0,\ y'(0)=\alpha_1,\ y''(0)=\alpha_2,\ y'''(0)=\alpha_3, \quad (2.3)$$

$$[\boldsymbol{y},\boldsymbol{v}_2]_\infty=\beta_0,\ [\boldsymbol{y},\boldsymbol{v}_3]_\infty=\beta_1.$$

Proof. Let f be an arbitrary element of $L_2(0,\infty)$ satisfying

$$(\boldsymbol{f},\boldsymbol{v}_2)_{\mathbf{L}^2}=\beta_0+\alpha_2,\ (\boldsymbol{f},\boldsymbol{v}_3)_{\mathbf{L}^2}=\beta_1-\alpha_1. \quad (2.4)$$

There is such an f, even among the linear combinations of v_1,v_2, and v_3. If we set $f=c_1v_1+c_2v_2+c_3v_3$ then conditions (2.4) are a system of equations in the constants c_1,c_2,c_3 whose determinant is the Gram determinant of the linearly independent functions v_1,v_2,v_3 and is therefore nonzero. Let y(x) denote the soulution of l(y)=f satisfying the initial conditions $y(0)=\alpha_0$, $y'(0)=\alpha_1$, $y''(0)=\alpha_2$, $y'''(0)=\alpha_3$. We claim that y(x) is the desired element. Applying Green' formula to y(x) and $\boldsymbol{v_j}$ we obtain

$$(\boldsymbol{f},\boldsymbol{v_j})_{\mathbf{L}^2}=(\boldsymbol{l}(\boldsymbol{y}),\boldsymbol{v_j})_{\mathbf{L}^2}=[\boldsymbol{y},\boldsymbol{v_j}]_\infty-[\boldsymbol{y},\boldsymbol{v_j}]_{\mathbf{0}},\ \mathbf{j}=\mathbf{2},\mathbf{3}.$$

But $l(\boldsymbol{v_j})=0$ (j=2,3). Since $y(0)=\alpha_0$, $y'(0)=\alpha_1$, $y''(0)=\alpha_2$, $y'''(0)=\alpha_3$, we have

$$[\boldsymbol{y},\boldsymbol{v_j}]_{\mathbf{0}}=\begin{cases}-\boldsymbol{\alpha}_2, \boldsymbol{j}=\mathbf{2}\ \boldsymbol{ise}\\ \boldsymbol{\alpha}_1, \boldsymbol{j}=\mathbf{3}\ \boldsymbol{ise}\end{cases}$$

Therefore,

$$(\boldsymbol{f},\boldsymbol{v}_2)_{\mathbf{L}^2}=[\boldsymbol{y},\boldsymbol{v}_2]_\infty+\alpha_2,$$

$$(\boldsymbol{f},\boldsymbol{v}_3)_{\mathbf{L}^2}=[\mathbf{y},\mathbf{v}_3]_\infty-\alpha_1.$$

Hence and from the conditions (2.4), we have

$$[\boldsymbol{y},\boldsymbol{v}_2]_\infty=\beta_0,\ [\boldsymbol{y},\boldsymbol{v}_3]_\infty=\beta_1.$$

We recall that a triple (H, Γ_1, Γ_2) is called a space of boundary values of a closed symmetric operator A on a Hilbert space H if Γ_1 and Γ_2 are linear maps from D ($\boldsymbol{A}^*$) to H with equal deficiency numbers and such that:

i) for every f, g ∈ D($\boldsymbol{A}^*$) ,

$$(\boldsymbol{A}^*\mathbf{f},\mathbf{g})_{\boldsymbol{H}}-(\mathbf{f},\boldsymbol{A}^*\mathbf{g})_{\boldsymbol{H}}=(\boldsymbol{\Gamma}_1\boldsymbol{f},\boldsymbol{\Gamma}_2\boldsymbol{g})_{\boldsymbol{H}}-(\boldsymbol{\Gamma}_2\boldsymbol{f},\boldsymbol{\Gamma}_1\boldsymbol{g})_{\boldsymbol{H}};$$

ii) any F_1,F_2∈H there is a vector f∈D($\boldsymbol{A}^*$) such that $\Gamma_1 f=F_1$, $\Gamma_2 f=F_2$ ([5], [18]).

Theorem 1. The triple (C^3,Γ_1,Γ_2) defined by (2.2) is a boundary spaces of the operator L_0.

Proof. First condition of the definition of a space of boundary value follows from Lemma 1 and second condition follows from Lemma 2.

Corollary 1. For any contraction K in C^3 the restriction of the operator L to the set of functions $y \in D$ satisfying either

$$(K-I)\Gamma_1 y+i(K+I)\Gamma_2 y=0 \quad (2.5)$$

or

$$(K-I)\Gamma_1 y-i(K+I)\Gamma_2 y=0 \quad (2.6)$$

is respectively the maximal dissipative and accretive extension of the operator L_0. Conversely, every maximal dissipative (accretive) extension of the operator L_0 is the restriction of L to the set of functions $y \in D$ satisfying (2.5) ((2.6)), and the contraction K is uniquely determined by the extension. The maximal symmetric extensions of L_0 in $L_2(0,\infty)$ are described by conditions (2.5) ((2.6)), in which K is an isometry. These conditions define selfadjoint extensions if K is unitary.

3. References

[1] Von Neumann J., “Allgemeine Eigenwertheorie Hermitischer Functionaloperatoren”, ***Math. Ann***. 102,49-131, 1929.

[2] Calkin J. W., “Abstract boundary conditions”, ***Trans. Amer. Math. Soc*.,**45, 3, 369-442, 1939.

[3] Rofe-Beketov F.S., “Self-adjoint extensions of differential operators in a space of vector valued functions"’, Dokl. Akad. Nauk SSSR 184,1034-1037, 1969 ; English transl. in ***Soviet Math. Dokl***. 10,188-192, 1969.

[4] Bruk V. M. “On a class of boundary --value problemswith a spectral parameter in the boundary conditions”, Mat. Sb., 100, 210-216. , 1976.

[5] Kochubei A. N., “Extensions of symmetric operators and symmetric binary relations”, Mat. Zametki 17, 41-48, 1975; English transl. in Math. Notes 17, 25-28, 1975.

[6] Gorbachuk M.L., “Gorbachuk V.I. and Kochubei A.N., The theory of extensions of symmetric operators and boundary-value problems for differential equations”, ***Ukrain. Mat***. Zh. 411299-1312, 1989; English transl. in Ukrainian Math. J. 41, 1117-1129, 1989.

[7] Fulton C.T., “Parametrization of Titchmarsh"s m (λ)- functions in the limit circle case”, ***Trans. Amer. Math. Soc***. 229, 51-63 , 1977.

[8] Krein M. G., “On the indeterminate case of the Sturm-Liouville boundaryvalue problem in the interval (0,∞)”, Akad. Nauk SSSR Ser. Mat. 16, 292-324, 1952.

[9] Khol'kin A. M., “Self-adjoint boundary conditions at-infinity for a quasi regular system of even-order differential equations”, 174-183 in: Theory of operators in function spaces and its applications, Naukova Dumka, Kiev, 1981.

[10] Mirzoev G. A., “Fourth order quasi regular differential operator” Dokl. Akad. Nauk SSSR 251, no.3, 550-553, 1980; English transl**. *Soviet Math. Dpkl***. 21, 480-483, 1980.

[11] Gorbachuk M. L., "On spectral functions of a second order differential operator with operator coefficients", Ukrain. Mat. Zh. 18, no.2, 3-21, 1966; English transl. Amer. Math. Soc. Transl. Ser. II 72, 177-202, 1968.

[12] Allahverdiev B. P., "On extensions of symmetric Schrödinger operators with a matrix potential", Izvest. Ross. Akad. Nauk. Ser . Math. 59, 19-54, 1995; English transl. Izv. Math. 59, 45-62, 1995.

[13] Guseĭnov I. M. and Pashaev R. T., "Description of self adjoint extensions of a class of differential operators of order 2n with defect indices (n+k,,n+k),0<k<n", Izv.Akad.Nauk Azerb. Ser. Fiz. Tekh. Mat. Nauk, No.2, 15-19 (in Russian) , 1983.

[14] Maksudov F.G. and Allahverdiev B.P., "On the extensions of Schrödinger operators with a matrix potentials", Dokl. Akad. Nauk 332, no.118-20, 1993,;English transl. Russian Acad. Sci. Dokl. Math. 48 no.2, 240-243, 1994.

[15] Malamud M. M. and Mogilevskiy V. I., "On extensions of dual pairs of operators, Dopov". Nats Akad. Nauk. Ukr. no. 1, 30-37, 1997.

[16] Mogilevskiy V. I., "On proper extensions of a singular differential operator in a space of vector functions", Dopov. Akad. Nauk. Ukraini, no.9, 29-33 (in Russian) , 1994.

[17] Naimark M. A., "Linear Differential Operators", 2nd edn.,1968, Nauka, Moscow, English transl. of 1st. edn., 1,2, New York, 1969.

[18] Gorbachuk M. L. and Gorbachuk V. I., "Boundary Value Problems for Operator Differential Equations", Naukova Dumka, Kiev, 1984; English transl., Birkhauser Verlag 1991.