Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi

Afyon Kocatepe University Journal of Science and Engineering

AKÜ FEMÜBİD **14** (2014) 011301 (1-3) AKU J. Sci. Eng.**14** (2014) 011301 (1-3)

DOİ: 10.5578/fmbd.6907 ***Araştırma Makalesi / Research Article***

Some Newly Defined Sequence Spaces Using Regular Matrix of Fibonacci Numbers

ShyamalDebnath, SubrataSaha

Tripura University, Department of Mathematics, Agartala, India.

e-mail:shyamalnitamath@gmail.com

Arrival date:31.10.2013; Accepted date:06.01.2014

Key words

Fibonacci Number; Regular Matrix; Sequence Space. 11B39, 46B45

Abstract

The main purpose of this paper is to introduce the new sequence spaces $c_0(F)$, $c(F)$ and $l_\infty(F)$ based on the newly defined regular matrix *F* of Fibonacci numbers. We study some basic topological and algebraic properties of these spaces. Also we investigate the relations related to these spaces.

1. Introduction

Let *w* be the space of all real sequences. Any vector subspace of *w* is called a sequence space. We shall write *c*, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences.

Let *X, Y* be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in N$.Then, A defines a matrix mapping (Debnath and Debnath, communicated; Malkowsky and Rakocevic, 2007) from *X* into *Y* and we denote it by $A : X \to Y$, if for every sequence $x= (x_k) \in X$, the sequence $Ax = \{ A_n(x)\}_{n=1}^{\infty}$, the *A*-transform of *x*, is in *Y;* where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k,\ (n \in N)$$

By *(X,Y),* we denote the class of all matrices *A* such that $A : X \to Y$. Thus $A \in (X,Y)$ if and only if the series on the right hand side above converges for each $n \in N$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$. The matrix domain *X(A)* of an infinite matrix A in a sequence space X is defined by

$$X(A) = \{x = (x_k) \in w : Ax \in X\},$$

which is a sequence space (Altay, Basar and Mursaleen, 2006; Kara and Basarir, 2012; Mursaleen and Noman, 2010; Tripathy and Sen, 2002).

A sequence space *X* is called *FK* space if it is a complete linear metric space with continuous coordinates $p_n : X \to R$ $(n \in N)$, where *R* denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in N$. A *BK* space is a normed *FK* space, i.e, a *BK* space is a Banach space with continuous coordinates. The spaces *c*, c_0 and l_∞ are *BK* spaces with $\|x\| = sup_k|x_k|$.

The following lemma (Known as The Toeplitz Theorem) contains necessary and sufficient condition for regularity of a matrix.

Lemma 1.1(Wilansky, 1984)**:** Matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is regular if and only if the following three conditions hold:

(1) There exists $M > 0$ such that for every $n = 1, 2, \ldots$ the following inequality holds:

$$\sum_{k=1}^{\infty}|a_{nk}| \le M;$$

(2) $lim_{n\to\infty}\, a_{nk} = 0$ for every $k = 1, 2, \ldots$

(3) $lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$

Let (p_k) be a sequence of positive numbers and $P_n = \sum_{k=1}^{n} p_k$.

Then the matrix $R^p = (r_{nk}^p)$ of the Riesz mean is given by

$$r_{nk}^p = \begin{cases} \frac{p_k}{P_n}, & if\ 1 \le k \le n; \\ 0, & otherwise \end{cases}$$

It is known that the Riesz matrix is a Toeplitz matrix if and only if $P_n \to \infty$ as $n \to \infty$ (Basar, 2011).

The Fibonacci numbers (Kara and Basarir, 2012; Koshy, 2001) are the sequence of numbers

$\{f_n\}_{n=1}^{\infty}$ defined by the linear recurrence equations

$f_0 = 0$ and $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$; $n \geq 2$.

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are given as follows (Kalman and Mena, 2003; Vajda, 1989):

$\sum_{k=1}^{n} f_n = f_{n+2} - 1$; $n \geq 1$,

$\sum_{1}^{n} f_n^{\;2} = f_n f_{n+1}$; $n \geq 1$,

$\sum_{k=1}^{\infty} \frac{1}{f_k}$ converges.

In this paper, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$, which differs from existing Fibonacci matrix by using Fibonacci numbers f_n (Kara and Basarir, 2012) and introduce some new sequence spaces related to matrix domain of F in the sequence spaces c_0, c and l_∞.

2. Main Result

Now, we define the Fibonacci matrix $F = (f_{nk})_{n,k=1}^{\infty}$, by

$$f_{n,k} = \begin{cases} \frac{f_k}{f_{n+2}-1} & (1 \leq k \leq n); \\ 0, & otherwise \end{cases}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0 & \cdots \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is obvious that the matrix F is triangular matrix i.e, $f_{nn} \neq 0$ for $k \leq n$ and $f_{nk} = 0$ for $k > n$ *(n=1,2,3,…)*. Also it follows from the lemma 1.1 that the method F is regular.

Now, we introduce the following sequence spaces based on the infinite matrix F:

$c(F) = \{ x = (x_k) \in w : Fx \in c\}$

$c_0(F) = \{ x = (x_k) \in w : Fx \in c_0\}$

$l_\infty(F) = \{ x = (x_k) \in w : Fx \in l_\infty\}$

where $Fx = \{F_n(x)\}_{n=1}^{\infty}$ and $F_n(x) = \sum_{k=1}^{\infty} f_{nk}\, x_k = \frac{1}{f_{n+2}-1} \sum_{k=1}^{n} f_{nk}\, x_k$, $(n \in N)$.

Theorem 2.1: The spaces $c(F)$, $c_0(F)$ and $l_\infty(F)$ are *BK* spaces with the same norm given by

$\|x\|_{X(F)} = \|Fx\|_X = sup_n |F_n(x)|$

where $X \in \{c, c_0, l_\infty\}$.

Proof: By Theorem 4.3.12 of Wilanksy, 1984 [p.63] and as the matrix F is triangular, we have the result**.**

Remark 2.2: It can be easily seen that the absolute property does not hold on the spaces $c(F)$, $c_0(F)$, $l_\infty(F)$ i.e., $\|x\|_{X(F)} \neq \||x|\|_{X(F)}$ for at least one sequence x in each of these spaces, where $|x| = (|x_k|)$. Thus the spaces $c(F)$, $c_0(F)$ and $l_\infty(F)$ are *BK* spaces of non-absolute type.

Theorem 2.3: The sequence spaces $c(F)$, $c_0(F)$ and $l_\infty(F)$ are norm isomorphic to the spaces c, c_0 and l_∞, respectively i.e, $c(F) \cong c$, $c_0(F) \cong c_0$ *and* $l_\infty(F) \cong l_\infty$.

Proof: X denotes any of the spaces c, c_0 or l_∞ and $X(F)$ be the respective one of the *spaces* $c(F)$, $c_0(F)$ or $l_\infty(F)$. Since the matrix F is triangular, it has a unique inverse, which is also triangular (Wilansky, 1984, proposition 1.1). Therefore the linear operator L_F : X (F) $\rightarrow$ X, defined by $L_F(x) = F(x)$ for all $x \in X(F)$, is bijective and is norm preserving by above norm in theorem 2.1. Hence $X(F) \cong X$**.**

Theorem 2.4: The inclusions $c_0(F) \subset c(F) \subset l_\infty(F)$ strictly hold.

Proof: It is clear that the inclusion $c_0(F) \subset c(F) \subset l_\infty(F)$ hold.

Consider the sequence $x = (x_k)$ defined by $x_k = 1$, for all $k \in N$. Then we have for every $n \in N$,

$$F_n(x) = \frac{1}{f_{n+2}-1}\sum_{k=1}^{n} f_k = 1$$

This shows that $Fx \in c$ but not in c_0. Thus the sequence x is in $c(F)$ but not in $c_0(F)$. Hence the inclusion $c_0(F) \subset c(F)$ strictly holds.

Again, consider the sequence $x = (x_k)$ defined by $x_k = \frac{(-1)^k(f_{k+2}+f_{k+1}-1)}{f_k}$, for all $k \in N$.

Then we have for every $n \in N$,

$$F_n(x) = \frac{1}{f_{n+2}-1}\sum_{k=1}^{n} f_k\, x_k = (-1)^n$$

This shows that $Fx \in l_\infty$ but not in c. Thus the sequence x is in $l_\infty(F)$ but not in $c(F)$. Hence the inclusion $c(F) \subset l_\infty(F)$ strictly holds.

Theorem 2.5: The inclusion $c_0 \subset c_0(F)$, $c \subset c(F)$ and $l_\infty \subset l_\infty(F)$ holds.

Proof: As F is a regular matrix, so the inclusion $c_0 \subset c_0(F)$ and $c \subset c(F)$ are obvious.

Now, let $x = (x_k) \in l_\infty$. Then there is a constant $M > 0$ such that $|x_k| \leq M$ for all $k \in N$. Thus for each $n \in N$

$$|F_n(x)| \leq \frac{1}{f_{n+2}-1}\sum_{k=1}^{n} f_k\, |x_k|$$

$$\leq \frac{M}{f_{n+2}-1}\sum_{k=1}^{n} f_k = M$$

which shows that $Fx \in l_\infty$ i.e., $x \in l_\infty(F)$. Thus we conclude that $l_\infty \subset l_\infty(F)$.

Example: Consider the sequence $x = (x_k) = (1, 0, 1, 0, 1, 0, \ldots\ldots\ldots\ldots)$. Then we have for every

$n \in N$,

$$F_n(x) = \frac{1}{f_{n+2}-1}\sum_{k=1}^{n} f_k\, x_k = \frac{1}{f_{n+2}-1}(f_1 + f_3 + \ldots + f_n)$$

which is convergent.

This shows that $Fx \in c$ but x is not in c. Thus the sequence x is in $c(F)$. Hence the inclusion $c \subset c(F)$ strictly holds.

Similarly, we can show the other inclusions are strict.

References

Altay B., Basar F. and Mursaleen M., 2006. On the Euler sequence spaces which include the spaces l_p and l_∞ I, *Informations Science,* **176***,* 1450-1462.

Basar F., 2011. Summability Theory and Its Applications, Bentham Science Publishers, Istanbul.

Debnath S. and Debnath J., On I-statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation (Communicated).

Kara E. E. and Basarir M., 2012. An application of Fibonacci numbers into infinite Toeplitz matrices, *CJMS.* **1(1)**, 43-47.

Kalman D. and Mena R., June 2003. The Fibonacci numbers-Exposed, *Mathematics Magazine*. **76(3)**.

Koshy T., 2001. Fibonacci and Lucas Numbers with Applications, Wiley.

Mursaleen M. and Noman A. K., 2010. On the space of λ-convergent and bounded sequences, *Thai J. Math*. **8(2)**, 311-329.

Malkowsky E. and Rakocevic V., 2007. On matrix domains of triangles, *Appl. Math.Comput*., **189(2)**, 1146-1163

Tripathy B. C. and Sen M., 2002. On a new class of sequences related to the space l_p, *Tamkang J. Math.* **33(2)**, 167-171.

Vajda S., 1989. Fibonacci and Lucas Numbers, and Golden Section: Theory and Applications, Chichester: Ellis Horword.

Wilansky A., 1984. Summability through functional analysis, North-Holland mathematics Studies 85, Elsevier Science Publishers, Amsterdam: New York: Oxford.