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Stability of a nonlinear fractional pseudo-parabolic equation system regarding fractional order of the time

Nguyen Duc Phuong[a], Long Le Dinh*[b], Tuan Nguyen Anh[b], Ho Duy Binh*[b]

[a] *Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Ho Chi Minh, Vietnam*
[b] *Division of Applied Mathematics, Thu Dau Mot University, Binh Duong province, Viet Nam.*

Abstract

In this work, we investigate an issue of fractional order continuity for a system of pseudo-parabolic equations. Specifically, we focus on investigating the stability of the derivative index, the solution w_a is continuously with respect to fractional order a in the appropriate sense.

Keywords: Pseudo-parabolic; System of equations; Initial value; Regularization; Caputo derivative; Fractional order.
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1. Introduction

Fractional PDEs are important in a variety of domains, including physics and engineering of the memory effect, viscoelasticity, porous media, and other fields [1–14]. Viscosity has an essential role in the research of the material characteristics of constructions and biological materials. Many researchers have recently used fractional calculations to investigate the Viscosity of such materials accurately. The primary tool for solving that phenomenon model is fraction PDEs. Binh *et al.* [15] studied the dependence of the fractional order of

Email addresses: nguyenducphuong@iuh.edu.vn (Nguyen Duc Phuong), nguyenanhtuan@tdmu.edu.vn (Tuan Nguyen Anh)
*Corresponding author: Le Dinh Long (ledinhlong@tdmu.edu.vn)
*Corresponding author: Ho Duy Binh (hoduybinh@tdmu.edu.vn)

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derivatives with respect to the time variable for the pseudo-parabolic equation. To expand on the previous result, we consider the coupled nonlinear fractional pseudo-parabolic equations which shown as following

$$\begin{cases} \partial_t^a\big(u(t,x) + (1+k)\mathcal{B}u(t,x)\big) = \mathcal{F}((t,x,u(t,x)), & (t,x) \in (0,T] \times \Omega, \\ \partial_t^a\big(v(t,x) + (1+l)\mathcal{B}v(t,x)\big) = \mathcal{G}((t,x,v(t,x)), & (t,x) \in (0,T] \times \Omega, \end{cases} \tag{1}$$

with the boundary conditions and initial value conditions as followed.

$$u(t,x) = 0, u(t,x) \in (0,T] \times \partial\Omega, \qquad u(0,x) = f_0(x), x \in \Omega, \qquad u_t(0,x) = 0, x \in \Omega,$$

and

$$v(t,x) = 0, (t,x) \in (0,T] \times \partial\Omega, \qquad v(0,x) = g_0(x), x \in \Omega, \qquad v_t(0,x) = 0, x \in \Omega,$$

In (1), the operator $\mathcal{B}$ is defined in (11). The function $\mathcal{F}, \mathcal{G}, f_0, g_0$ will define later. The constant $a \in (0,1)$ is called the order of fractional. Furthermore, the Caputo fractional derivative ∂_t^a is defined as follows.

$$\partial_t^a u(x,t) = \frac{1}{\Gamma(1-a)} \int_0^t \frac{u_s'(x,s)ds,}{\partial_t^a u(x,t) = [\Gamma(1-a)]^{-1} \int_0^t (t-s)^{-a} \frac{\partial u(x,s)}{ds} ds,} \tag{2}$$

where u is a definitely continuous function with respect to time, $\Gamma(w) = \int\limits_0^\infty t^{w-1}e^{-t}dt$ is the Gamma function and u_s' is regarded as an ordinary first derivative of function u. Let Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. The case $0 < a < 1$, the time-fractional diffusion equation plays an crucial part in Brownian motion for normal diffusion, the macroscopic version of continuous time random walk model (1) becomes the well known pseudo-parabolic with derivative of integer order ($a = 1$)

$$u_t + (k+1)\mathcal{B}u_t = \mathcal{F}(u). \tag{3}$$

The equation (3) has a wide range of real world applications, it is called the pseudo-parabolic (see [17–19]). In addition, there are numerous works on the well-posedness of the pseudo-parabolic equation with classical derivative, as evidenced by [20–43, 60–68] and the references therein. Investigating the existence, uniqueness, and stability of fractional differential equations, has been the important goal in the scientific community, especially in fractional calculus. Until now, we hardly find many articles related to the fractional order of the pseudo-parabolic partial differential equation. Recently, the authors [46] researched and generalized Ulam-Hyers. In [50, 51, 59], authors were recognized a boundary value problem for fractional pseudo-parabolic equations. Benchohra [56–58] specialize in research about Hilfer fractional derivative. The regularity of the mild solutions for fractional pseudo-parabolic equations, on the other hand, has not been investigated.

As a result, the solutions' stability concerning these parameters is crucial for modeling purposes. The inspiration for this idea came from the article [54]. Experiments are used to define or compute the parameters of a practical problem. As a result, we only see its value incorrectly, and we can only get an approximate value even if the parameters are known precisely. Problem (1) becomes complicated when we have to analyze the upper and lower quantities utilizing terms of fractional order a. Therefore, we must choose an effective method to provide an appropriate estimate due to the non-local and nonlinear nature of our problem.

The rest of the paper is divided as follows: In Section 2, we present the Mittag-Leffler function's preliminaries and the mild solution. In the last section, the uniqueness of the solution and continuity of the solution with respect to fractional order will be present.

2. Preliminaries

2.1. The Mittag-Leffler function

For $a > 0, b \in \mathbb{R}$ and $\varpi \in \mathbb{C}$, the Mittag-Leffler function is denoted by the symbol $E_{a,b}(\varpi)$ and has the following definition:

$$E_{a,b}(\varpi) = \sum_{n=1}^{\infty} \frac{\varpi^n}{\Gamma(na+b)}.$$

Remember the following lemmas (see for instance [9, 48, 53]), that would be useful for Section 3.

Lemma 2.1. *For all $\varpi > 0$, there exist $\mathcal{D} > 0$ depends on $a \in (0,1)$ such that*

$$|E_{a,1}(-\varpi)| \le \mathcal{D}, \qquad |E_{a,a}(-\varpi)| \le \mathcal{D}.$$

Lemma 2.2. *If $a \in (0,1), \lambda > 0$ and $\varpi \in \mathbb{R}^+$ then we get*

$$\begin{aligned} &\partial_\varpi E_{a,1}(-\lambda\varpi^a) = -\lambda\varpi^{a-1}E_{a,a}(-\lambda\varpi^a),\\ &\partial_\varpi(\varpi^{a-1}E_{a,a}(-\lambda\varpi^a)) = \varpi^{a-2}E_{a,a-1}(-\lambda\varpi^a). \end{aligned} \tag{4}$$

Proof. See [60], Lemma (2.2). □

Lemma 2.3. *If T is large enough and $0 < a < 1$ then there exist two constants $\mathcal{D}_a^1$ and $\mathcal{D}_a^2$ such that*

$$\frac{\mathcal{D}_a^1}{1+\lambda_j T^a} \le \left|E_{a,1}(-\lambda_j T^a)\right| \le \frac{\mathcal{D}_a^2}{1+\lambda_j T^a}, \quad \forall j \in \mathbb{N}\cdot \tag{5}$$

Proof. See [44]. □

Lemma 2.4 (See [47], Lemma 2.3)**.**

$$1+\varpi \le |E_{a,1}(-\varpi)| \le \frac{\mathcal{D}_2(a_1,a_2)}{1+\varpi}, \qquad |E_{a,a}(-\varpi)| \le \frac{\mathcal{D}_3(a_1,a_2)}{1+\varpi}. \tag{6}$$

Let $0 < a_1 < a_2 < 1$ and $a \in (a_1,a_2)$. Then there exists constants $\mathcal{D}_i(a_1,a_2) > 0$ ($i = \overline{1,3}$) depending only on a_1, a_2 such that

$$\frac{\mathcal{D}_1(a_1,a_2)}{1+\varpi} \le |E_{a,1}(-\varpi)| \le \frac{\mathcal{D}_2(a_1,a_2)}{1+\varpi}, \qquad |E_{a,a}(-\varpi)| \le \frac{\mathcal{D}_3(a_1,a_2)}{1+\varpi}. \tag{7}$$

for any $\varpi > 0$.

Lemma 2.5. *Let $0 < a_1 < a < a' < a_2$, $0 < \varpi \le T$. There exists $\mathcal{D}_\epsilon$ such that*

$$|\varpi^a - \varpi^{a'}| \le \max(T^{a_2+2\epsilon},1)\mathcal{D}_\epsilon(a'-a)^\epsilon T^{a-\epsilon}. \tag{8}$$

here ϵ is a number greater than zero and independent of a.

Proof. See [60], Lemma 3.2. □

Lemma 2.6. *Assume that $\epsilon > 0$ and $0 < a_1 < a < a' < a_2 < 1$. Then there is a positive constant $\mathcal{D}_1(a_1,a_2,\epsilon,\theta,T)$, such that*

$$|E_{a,1}(-\lambda_j\xi^a) - E_{a',1}(-\lambda_j\xi^{a'})| \le \mathcal{D}_1(a_1,a_2,\epsilon,\theta,T)\lambda_j^{\theta-1}t^{-a_2(1-\theta)-\epsilon}\Big[(a'-a)^\epsilon + (a'-a)\Big], \tag{9}$$

where $\theta \in [0,1]$ and $\xi \in (0,T]$.

Proof. See [60]. □

Lemma 2.7. *Suppose that $0 < a_1 < a < a' < a_2 < 1$. If $\theta \in [0,1]$ and $\epsilon > 0$ then there exist $\mathcal{D}_2(a_1,a_2,\epsilon,\theta,T) > 0$ which is independent on a and a' such that*

$$\begin{aligned} &\Big|\xi^{a'-1}E_{a',a'}(-\lambda_j\xi^{a'}) - \xi^{a-1}E_{a',a'}(-\lambda_j t^{a'})\Big|\\ &\qquad\qquad \le \mathcal{D}_2(a_1,a_2,\epsilon,\theta,T)\lambda_j^{\theta-1}t^{a_1\theta-\epsilon-1}\Big[(a'-a)^\epsilon + (a'-a)\Big]. \end{aligned} \tag{10}$$

Proof. See [60]. □

Lemma 2.8 (See [69], Lemma 8)**.** *Assume exist $\mu_1 > -1, \mu_2 > -1$ such that $\mu_1 + \mu_2 > -1$ and $\mu_3 > -1$ then the following estimates hold*

$$\mathscr{C}^{\mu_3}_{\mu_1,\mu_2}(\gamma) := \sup_{t\in[0,T]} t^{\mu_3}\int_0^1 s^{\mu_1}(1-s)^{\mu_2}e^{-\gamma t(1-s)}ds \xrightarrow{\gamma\to\infty} 0.$$

2.2. Some spaces and solution representation

Next, we need familiar Hilbert spaces $\mathbb{L}^2(\Omega), H_0^1(\Omega), H^2(\Omega)$. Let $\mathcal{B} : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ be a symmetrical, uniform elliptical operator, defined as follows:

$$\mathcal{B}w(s) = b(s)w(s,t) - \sum_{r=1}^{n} \frac{\partial}{\partial s_r}\left(\mathcal{B}_{rl}(s)\frac{\partial}{\partial s_l}w(s)\right), s \in \overline{\Omega}, \tag{11}$$

where $D(\mathcal{B}) = H^2(\Omega) \cap H_0^1(\Omega)$, $b(s) \in C(\overline{\Omega}, [0,\infty))$, $\mathcal{B}_{rl} \in C^1(\overline{\Omega})$, $\mathcal{B}_{rl} = \mathcal{B}_{lr}$ for all $1 \le r, l \le n$, and there exist $\widetilde{B} > 0$ such that (see [49])

$$\widetilde{B}\sum_{i=1}^{n} e_i^2 \le \sum_{1\le r,l\le n} e_i \mathcal{B}_{rl}(x) e_k, \qquad \text{for all } x \in \overline{\Omega}, e = (e_1, e_2, \cdots, e_n) \subset \mathbb{R}^n.$$

We now consider the spectral problem

$$\begin{cases} \mathcal{B}\psi_j(x) = \lambda_j \psi_j(x) \\ \psi_j(x) = 0 \text{ on } \partial\Omega\cdot \end{cases} \tag{12}$$

where $\{\lambda_j\}_{j\ge1}$ (see e.g [55]) is an increasing sequence of positive real numbers that satisfies the condition that $\lim_{j\to\infty} \lambda_j = \infty$.

Furthermore, $\mathcal{B}^p w$ operator is defined as follows:

$$\begin{aligned} &\mathcal{B}^p w := \sum_{j=1}^{\infty} \langle \mathbf{w}, \psi_j\rangle \lambda_j^p \psi_j, \\ &w \in \mathbb{D}(\mathcal{B}^p) = \left\{ w \in \mathbb{L}^2(\Omega) : \sum_{j=1}^{\infty} |\langle w, \psi_j\rangle|^2 \lambda_j^{2p} < \infty \right\}, \end{aligned} \tag{13}$$

The Banach space $\mathbb{D}(\mathcal{B}^p)$ is equipped with the norm

$$\|u\|^2_{\mathbb{D}(\mathcal{B}^p)} := \sum_{j=1}^{\infty} \lambda_j^{2p} \big|\langle u, \psi_j\rangle\big|^2. \tag{14}$$

If $p = 1$, we have $\mathbb{D}(\mathcal{B}^1) = H^2(\Omega)$. For $p \ge 0$, the Hilbert space

$$\mathbb{H}^p(\Omega) = \left\{ u : u \in \mathbb{L}^2(\Omega) \text{ and} \sum_{j=1}^{\infty} |\langle u, \psi_j\rangle|^2 \lambda_j^{2p} < \infty \right\}, \tag{15}$$

it is equipped with the norm

$$\|u\|^2_{\mathbb{H}^p(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2p} \left|\langle u, \psi_j\rangle\right|^2.$$

If $p = 0$ then $\mathbb{H}^0(\Omega) = L^2(\Omega)$. Denote $\mathcal{H}^p(\Omega) = \mathbb{H}^p(\Omega) \times \mathbb{H}^p(\Omega)$. Let $w(u, v) \in \mathcal{H}^p(\Omega)$, we define the following norm:

$$\|w\|_{\mathcal{H}^p(\Omega)} = \sqrt{\|u\|^2_{\mathbb{H}^p(\Omega)} + \|v\|^2_{\mathbb{H}^p(\Omega)}} \tag{16}$$

For $\alpha \in (0, 1)$ and $\gamma > 0$, we use the notation $\mathbb{C}^\alpha_\gamma(0, T; \mathbb{H}^p(\Omega))$ to indicate the subspace of $\mathbb{C}(0, T; \mathbb{H}^p(\Omega))$ such that

$$\|u\|_{\mathbb{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))} := \sup_{0<t\le T} t^\alpha e^{-\gamma t}\|u(t)\|_{\mathbb{H}^p(\Omega)} < \infty, \qquad u \in \mathbb{C}(0, T; \mathbb{H}^p(\Omega)),$$

with the convention that $\mathbb{C}_\gamma(0,T;\mathbb{H}^p(\Omega)) := \mathbb{C}^0_\gamma(0,T;\mathbb{H}^p(\Omega))$ when $\alpha = 0$. The product space

$$\mathcal{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega)) = \mathbb{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega)) \times \mathbb{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))$$

has the norm

$$\|w\|_{\mathcal{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))} = \left(\|u\|^2_{\mathbb{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))} + \|v\|^2_{\mathbb{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))}\right)^{\frac{1}{2}} \tag{17}$$

Now, using the result [15], we can deduce the solution of the coupled nonlinear fractional pseudo-parabolic equation system (1) as below

$$\begin{cases} u_a(t,x) = \sum\limits_{j=1}^{\infty} E_{a,1}\left(\dfrac{-\lambda_j t^a}{1+k\lambda_j}\right) f_{0,j}\psi_j + \sum\limits_{j=1}^{\infty} \frac{1}{1+k\lambda_j}\left[\int\limits_0^t (t-s)^a E_{a,a}\left(\dfrac{-(t-s)^{a-1}\lambda_j}{1+k\lambda_j}\right)\mathcal{F}(u_j(s))ds\right]\psi_j. \\ v_a(t,x) = \sum\limits_{j=1}^{\infty} E_{a,1}\left(\dfrac{-\lambda_j t^a}{1+l\lambda_j}\right) g_{0,j}\psi_j + \sum\limits_{j=1}^{\infty} \frac{1}{1+l\lambda_j}\left[\int\limits_0^t (t-s)^a E_{a,a}\left(\dfrac{-(t-s)^{a-1}\lambda_j}{1+l\lambda_j}\right)\mathcal{G}(u_j(s))ds\right]\psi_j. \end{cases} \tag{18}$$

We can rewritten as follows:

$$\begin{cases} u_a(t,x) = \mathcal{R}_{a,k}(t)f_0 + \int\limits_0^t \mathcal{S}_{a,k}(t-s)\mathcal{F}(u_a(s))ds, \\ v_a(t,x) = \mathcal{R}_{a,l}(t)g_0 + \int\limits_0^t \mathcal{S}_{a,l}(t-s)\mathcal{G}(u_a(s))ds, \end{cases} \tag{19}$$

where

$$\mathcal{R}_{a,k}(t)\Psi = \sum_{j=1}^{\infty} E_{a,1}\left(\frac{-\lambda_j t^a}{1+k\lambda_j}\right)\langle\Psi,\psi_j\rangle\psi_j \tag{20}$$

$$\mathcal{R}_{a,l}(t)\Psi = \sum_{j=1}^{\infty} E_{a,1}\left(\frac{-\lambda_j t^a}{1+l\lambda_j}\right)\langle\Psi,\psi_j\rangle\psi_j \tag{21}$$

$$\mathcal{S}_{a,k}(t-s)\Psi = \sum_{j=1}^{\infty} \frac{1}{1+k\lambda_j}\left[(t-s)^{a-1}E_{a,a}\left(\frac{-(t-s)^a\lambda_j}{1+k\lambda_j}\right)\langle\Psi,\psi_j\rangle\right]\psi_j, \tag{22}$$

$$\mathcal{S}_{a,l}(t-s)\Psi = \sum_{j=1}^{\infty} \frac{1}{1+l\lambda_j}\left[(t-s)^{a-1}E_{a,a}\left(\frac{-(t-s)^a\lambda_j}{1+l\lambda_j}\right)\langle\Psi,\psi_j\rangle\right]\psi_j. \tag{23}$$

Replace a with a', we have

$$\begin{cases} u_{a'}(t,x) = \mathcal{R}_{a',k}(t)f_0 + \int\limits_0^t \mathcal{S}_{a',k}(t-s)\mathcal{F}(u_{a'}(s))ds, \\ v_{a'}(t,x) = \mathcal{R}_{a',l}(t)g_0 + \int\limits_0^t \mathcal{S}_{a',l}(t-s)\mathcal{G}(u_{a'}(s))ds. \end{cases} \tag{24}$$

Next, we give some lemmas to evaluate the solution's dependence on the parameter. And, we can refer to the proof in [15].

Lemma 2.9. *For $0 < a_1 < a < a_2 < 1$, $\theta \in (0,1)$, $p \geq 0$ and $\Psi \in \mathbb{H}^p(\Omega)$. The following inequalities hold:*

$$\|\mathcal{R}_{a,k}(t)\Psi\|_{\mathbb{H}^p(\Omega)} \leq \overline{D}_1(a_1,a_2,k,\theta)t^{-a_2\theta}\|\Psi\|_{\mathbb{H}^p(\Omega)}, \tag{25}$$

$$\|\mathcal{R}_{a,l}(t)\Psi\|_{\mathbb{H}^p(\Omega)} \leq \overline{D}_1(a_1,a_2,l,\theta)t^{-a_2\theta}\|\Psi\|_{\mathbb{H}^p(\Omega)}, \tag{26}$$

$$\|\mathcal{S}_{a,k}(t-s)\Psi\|_{\mathbb{H}^p(\Omega)} \leq \overline{D}_2(a_1,a_2,k,\theta)(t-s)^{(a_1-1-a_1\theta)}\|\Psi\|_{\mathbb{H}^p(\Omega)}. \tag{27}$$

$$\|\mathcal{S}_{a,l}(t-s)\Psi\|_{\mathbb{H}^p(\Omega)} \leq \overline{D}_2(a_1,a_2,l,\theta)(t-s)^{(a_1-1-a_1\theta)}\|\Psi\|_{\mathbb{H}^p(\Omega)}.$$

Lemma 2.10. *For $a \in (0,1)$, $p \geq 0$ with $0 < \theta < 1$ and $\Psi \in \mathbb{H}^p(\Omega)$, we get the following inequalities:*

$$\begin{aligned}
\Big\|\Big[\mathcal{R}_{a',k}(t) - \mathcal{R}_{a,k}(t)\Big]\Psi\Big\|_{\mathbb{H}^p(\Omega)} &\leq \overline{D}_1(a_1,a_2,\epsilon,\theta,k,T)t^{-a_2(1-\theta)-\epsilon}\|\Psi\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^\epsilon + (a'-a)\Big],\\
\Big\|\Big[\mathcal{R}_{a',l}(t) - \mathcal{R}_{a,l}(t)\Big]\Psi\Big\|_{\mathbb{H}^p(\Omega)} &\leq \overline{D}_1(a_1,a_2,\epsilon,\theta,l,T)t^{-a_2(1-\theta)-\epsilon}\|\Psi\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^\epsilon + (a'-a)\Big],\\
\Big\|\Big[\mathcal{S}_{a',k}(t-s) - \mathcal{S}_{a,k}(t-s)\Big]\Psi\Big\|_{\mathbb{H}^p(\Omega)} &\leq \overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)(t-s)^{a_1\theta-\epsilon-1}\|\Psi\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^\epsilon + (a'-a)\Big],\\
\Big\|\Big[\mathcal{S}_{a',l}(t-s) - \mathcal{S}_{a,l}(t-s)\Big]\Psi\Big\|_{\mathbb{H}^p(\Omega)} &\leq \overline{D}_2(a_1,a_2,\epsilon,\theta,l,T)(t-s)^{a_1\theta-\epsilon-1}\|\Psi\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^\epsilon + (a'-a)\Big].
\end{aligned}$$

where $\overline{D}_1, \overline{D}_2$ are independent of ω and also defined in the proof.

3. The fractional-order stability of a nonlinear FPPES.

In this section, we investigate the existence of a unique mild solution of the problem (1) and how its continuity depends on the parameters with input (the fractional-order a and the initial state f_0, g_0). The function $\mathcal{F}$ and $\mathcal{G}$ are assumed to meet the following criteria:
(**F**).

$$\begin{cases}
\|\mathcal{F}(u,v)(\cdot,t)\|_{\mathbb{H}^p(\Omega)} < C_{\mathcal{F}}\Big(1 + \|u(\cdot,t)\|_{\mathbb{H}^p(\Omega)} + \|v(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big),\\
\|\mathcal{F}(u_1,v_1)(\cdot,t) - \mathcal{F}(u_2,v_2)(\cdot,t)\|_{\mathbb{H}^p(\Omega)} < K_{\mathcal{F}}\Big(\|u_1(\cdot,t) - u_2(\cdot,t)\|_{\mathbb{H}^p(\Omega)} + \|v_1(\cdot,t) - v_2(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big),\\
\mathcal{F}(0) = 0,
\end{cases} \tag{28}$$

where $(u_1,v_1),(u_2,v_2) \in \mathcal{H}^p(\Omega)$.

(**G**).

$$\begin{cases}
\|\mathcal{G}(u,v)(\cdot,t)\|_{\mathbb{H}^p(\Omega)} < C_{\mathcal{G}}\Big(1 + \|u(\cdot,t)\|_{\mathbb{H}^p(\Omega)} + \|v(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big),\\
\|\mathcal{G}(u_1,v_1)(\cdot,t) - \mathcal{G}(u_2,v_2)(\cdot,t)\|_{\mathbb{H}^p(\Omega)} < K_{\mathcal{G}}\Big(\|u_1(\cdot,t) - u_2(\cdot,t)\|_{\mathbb{H}^p(\Omega)} + \|v_1(\cdot,t) - v_2(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big),\\
\mathcal{G}(0) = 0,
\end{cases} \tag{29}$$

here $(u_1,v_1),(u_2,v_2) \in \mathcal{H}^p(\Omega)$.

Definition 3.1. $w(u(t,x),v(t,x)) \in \mathcal{C}^\alpha_\gamma(0,T;\mathbb{H}^p(\Omega))$ is called a mild solution to the problem (1) if it satisfies system (19).

Theorem 3.2. *Assume that* $0 < a_1 < a < a' < a_2 < 1$, $0 < \theta < 1$, $0 < \alpha < a_1(1-\theta) < 1$ *and* $f_0 \in \mathbb{H}^p(\Omega)$, $g_0 \in \mathbb{H}^p(\Omega)$. *Then Problem (1) has unique solution* $w(u,v) \in \mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))$, *and the following estimate holds*

$$\|w_a\|_{\mathcal{C}^{\theta a_2}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \leq \frac{\mathfrak{D}^{k,l}_{1,\theta}(a_1,a_2)}{1-\mathfrak{D}^{k,l}_{2,\theta}(a_1,a_2)}. \tag{30}$$

here $\mathfrak{D}^{k,l}_{1,\theta}(a_1,a_2)$ *and* $\mathfrak{D}^{k,l}_{2,\theta}(a_1,a_2)$ *is defined in* (45). *Furthermore, consider* w_a *and* $w_{a'}$ *are solutions of Problem* (1) *for fractional orders* a *and* a' *respectively. If exist* θ *and* ϵ *such that* $0 < \epsilon < a_1\theta$ *and* $1 - \frac{1}{a_2} < \theta < \frac{a_1}{a_2+a_1}$ *then we get*

$$\|w_{a'} - w_a\|_{\mathcal{C}^{a_2(1-\theta)+\epsilon}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \leq \frac{\Big(\overline{U}_1(a_1,a_2,\epsilon,\theta,k,T) + \overline{V}_1(a_1,a_2,\epsilon,\theta,l,T)\Big)\Big[(a'-a)^{\epsilon} + (a'-a)\Big]}{1-\Big(\overline{V}_1(a_1,a_2,\epsilon,\theta,k,T) + \overline{V}_2(a_1,a_2,\epsilon,\theta,l,T)\Big)}. \tag{31}$$

where $\overline{U}_1(a_1,a_2,\epsilon,\theta,k,T)$, $\overline{U}_2(a_1,a_2,\epsilon,\theta,k,T)$,$\overline{V}_1(a_1,a_2,\epsilon,\theta,l,T)$ *and* $\overline{V}_2(a_1,a_2,\epsilon,\theta,l,T)$ *are positive constants.*

Proof. We divide the proof into three parts:

Part 1. We use the Banach fixed-point theorem to prove the existence and uniqueness of the solution of equation (18). For $w_a \in \mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))$ and operate Υ is defined by

$$\Upsilon w_a := \Big(\Upsilon_k u_a(\cdot,t), \Upsilon_l v_a(\cdot,t)\Big) \tag{32}$$

where

$$\begin{cases} \Upsilon_k u_a(t,x) = \mathcal{R}_{a,k}(t) f_0 + \int\limits_0^t \mathcal{S}_{a,k}(t-s)\mathcal{F}(u_a(s))ds, \\ \Upsilon_l v_a(t,x) = \mathcal{R}_{a,l}(t) g_0 + \int\limits_0^t \mathcal{S}_{a,l}(t-s)\mathcal{G}(u_a(s))ds, \end{cases} \tag{33}$$

We will show that for all $w_a \in \mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))_{@} \in \mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))$ the equation $\Upsilon w_a = w_a$ has a unique solution. Indeed, let $w_{a,1}(u_{a,1}(t,x), v_{a,1}(t,x)), w_{a,2}(u_{a,2}(t,x), v_{a,2}(t,x)) \in \mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))$, we get

$$\Upsilon_k\Big(u_{a,1}(t,\cdot) - u_{a,2}(t,\cdot)\Big) = \int\limits_0^t \mathcal{S}_{a,k}(t-s)\Big[\mathcal{F}(u_{a,1},v_{a,2})(s,\cdot) - \mathcal{F}(u_{a,2},v_{a,2})(s,\cdot)\Big]ds, \tag{34}$$

$$\Upsilon_l\Big(v_{a,1}(t,\cdot) - v_{a,2}(t,\cdot)\Big) = \int\limits_0^t \mathcal{S}_{a,l}(t-s)\Big[\mathcal{G}(u_{a,1},v_{a,1})(s,\cdot) - \mathcal{G}(u_{a,2},v_{a,2})(s,\cdot)\Big]ds. \tag{35}$$

Using Lemma (2.9) and condition (28), (34) will be

$$\begin{aligned}
&\left\| t^{\alpha} e^{-\gamma t}\Big[\Upsilon_k\Big(u_{a,1}(t,\cdot)-u_{a,2}(t,\cdot)\Big)\Big]\right\|_{\mathbb{H}^p(\Omega)} \\
&= \left\| \int_0^t \mathcal{S}_{a,k}(t-s) t^{\alpha} e^{-\gamma t}\Big[\mathcal{F}(u_{a,1},v_{a,1})(s,\cdot)-\mathcal{F}(u_{a,2},v_{a,2})(s,\cdot)\Big]ds \right\|_{\mathbb{H}^p(\Omega)} \\
&\le \int_0^t \overline{D}_2(a_1,a_2,k,\theta)(t-s)^{(a_1-1-a_1\theta)} t^{\alpha} e^{-\gamma t} K_{\mathcal{F}}\Big(\|u_{a,1}(s,\cdot)-u_{a,2}(s,\cdot)\|_{\mathbb{H}^p(\Omega)}+\|v_{a,1}(s,\cdot)-v_{a,2}(s,\cdot)\|_{\mathbb{H}^p(\Omega)}\Big)ds \\
&\le \overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}} \\
&\quad \times t^{\alpha}\int_0^t s^{-\alpha}(t-s)^{(a_1-1-a_1\theta)}e^{-\gamma(t-s)}\Big[s^{\alpha}e^{-\gamma s}\Big(\|u_{a,1}(s,\cdot)-u_{a,2}(s,\cdot)\|_{\mathbb{H}^p(\Omega)}+\|v_{a,1}(s,\cdot)-v_{a,2}(s,\cdot)\|_{\mathbb{H}^p(\Omega)}\Big)\Big]ds \\
&\le \overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}}\Big(\|u_{a,1}-u_{a,2}\|_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}+\|v_{a,1}-v_{a,2}\|_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}\Big) \\
&\qquad\qquad\qquad\qquad \times t^{a_1-a_1\theta}\int_0^1 s^{-\alpha}(1-s)^{a_1-1-a_1\theta}e^{-\gamma t(1-s)}ds.
\end{aligned}$$

By setting $\mu_1 := -\alpha; \mu_2 := a_1 - 1 - a_1\theta$ and

$$\mathscr{C}^{\mu_2+1}_{\mu_1,\mu_2}(\gamma) := \sup_{t\in[0,T]} t^{a_1-a_1\theta}\int_0^1 s^{-\alpha}(1-s)^{a_1-a_1\theta-1}e^{-\gamma t(1-s)}ds,$$

appling inequality $(m+n)^2 \le 2(m^2+n^2)$, we can get

$$\left\|\Upsilon_k\Big(u_{a,1}(t,\cdot)-u_{a,2}(t,\cdot)\Big)\right\|_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \le \sqrt{2}\overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}}\|w_{a,1}-w_{a,2}\|_{\mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}\mathscr{C}^{\mu_2+1}_{\mu_1,\mu_2}(\gamma). \tag{36}$$

Similarly, we also get

$$\left\|\Upsilon_l\Big(u_{a,1}(t,\cdot)-u_{a,2}(t,\cdot)\Big)\right\|_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \le \sqrt{2}\overline{D}_2(a_1,a_2,l,\theta)K_{\mathcal{G}}\|w_{a,1}-w_{a,2}\|_{\mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}\mathscr{C}^{\mu_2+1}_{\mu_1,\mu_2}(\gamma). \tag{37}$$

From (36)-(37), we can obtain estimate as follow

$$\begin{aligned}
&\left\|\Upsilon\Big(w_{a,1}(t,\cdot)-w_{a,2}(t,\cdot)\Big)\right\|_{\mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \\
&\qquad = \sqrt{\left\|\Upsilon_k\Big(u_{a,1}(\cdot,t)-u_{a,2}(\cdot,t)\Big)\right\|^2_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}+\left\|\Upsilon_l\Big(v_{a,1}(\cdot,t)-v_{a,2}(\cdot,t)\Big)\right\|^2_{\mathbb{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}} \\
&\qquad \le \sqrt{2\overline{D}_2^2(a_1,a_2,k,\theta)K^2_{\mathcal{F}}+2\overline{D}_2^2(a_1,a_2,l,\theta)K^2_{\mathcal{G}}}\,\mathscr{C}^{\mu_2+1}_{\mu_1,\mu_2}(\gamma)\|w_{a,1}-w_{a,2}\|_{\mathcal{C}^{\alpha}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}.
\end{aligned} \tag{38}$$

Using the assumption $0<\alpha<a_1(1-\theta)$, we see that

$$\begin{cases} -\alpha > -1, \\ -\alpha + a(1-\theta) - 1 > -1. \end{cases}$$

Hence, by applying Lemma (2.8), we deduce that

$$\mathscr{C}^{\mu_2+1}_{\mu_1,\mu_2}(\gamma) \xrightarrow{\gamma\to\infty} 0. \tag{39}$$

Therefore, we can choose γ_0 such that

$$\sqrt{2\overline{D}_2^2(a_1,a_2,k,\theta)K_{\mathcal{F}}^2+2\overline{D}_2^2(a_1,a_2,l,\theta)K_{\mathcal{G}}^2}\mathscr{C}_{\mu_1,\mu_2}^{\mu_2+1}(\gamma_0)<1.$$

Accordingly, we discover that Υ is a contractive mapping in $\mathbb{C}_\gamma^\alpha(0,T;\mathbb{H}^p(\Omega))$. Following the Banach fixed point theorem, we confirm that (1) has a uniques solution in $\mathbb{C}_\gamma^\alpha(0,T;\mathbb{H}^p(\Omega))$.

Part 2. Estimates $\|w_a(t,\cdot)\|_{\mathcal{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}$. From (19) we get

$$\left\|u_a(t,\cdot)\right\|_{\mathbb{H}^p(\Omega)}\le\left\|\mathcal{R}_{a,k}(t)f_0\right\|_{\mathbb{H}^p(\Omega)}+\left\|\int_0^t\mathcal{S}_{a,k}(t-s)\mathcal{F}(u_a,v_a)(\cdot,s)ds\right\|.$$

Using Lemma (2.9), and (28). we can deduce

$$\begin{aligned}
&\|t^{a_2\theta}e^{-\gamma t}u_a(t,\cdot)\|_{\mathbb{H}^p(\Omega)}\\
&\le t^{a_2\theta}e^{-\gamma t}\overline{D}_1(a_1,a_2,k,\theta)t^{-a_2\theta}\|f_0\|_{\mathbb{H}^p(\Omega)}+t^{a_2\theta}e^{-\gamma t}\int_0^t\overline{D}_2(a_1,a_2,k,\theta)(t-s)^{(a_1-1-a_1\theta)}\|\mathcal{F}(u_a,v_a)(\cdot,s)\|_{\mathbb{H}^p(\Omega)}ds\\
&\le\overline{D}_1(a_1,a_2,k,\theta)e^{-\gamma t}\|f_0\|_{\mathbb{H}^p(\Omega)}\\
&+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}t^{a_2\theta}\int_0^t s^{-a_2\theta}(t-s)^{(a_1-1-a_1\theta)}e^{-\gamma(t-s)}s^{a_2\theta}e^{-\gamma s}\Big(1+\|u_a(\cdot,s)\|_{\mathbb{H}^p(\Omega)}+\|v_a(\cdot,s)\|_{\mathbb{H}^p(\Omega)}\Big)ds,
\end{aligned}\tag{40}$$

It is clear that $e^{-bt}<1$ for all $t>0$. Using the same evaluation technique as (36) with asumption $0<\theta<\frac{a_1}{a_2+a_1}<1$, we can deduce

$$\begin{aligned}
&\|t^{a_2\theta}e^{-\gamma t}u_a(t,\cdot)\|_{\mathbb{H}^p(\Omega)}\\
&\le\overline{D}_1(a_1,a_2,k,\theta)\|f_0\|_{\mathbb{H}^p(\Omega)}+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}t^{a_2\theta}\int_0^t(t-s)^{a_1-1-a_1\theta}ds\\
&+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}t^{a_1-a_1\theta}\int_0^1 s^{-a_2\theta}(1-s)^{(a_1-1-a_1\theta)}e^{-\gamma t(1-s)}\Big[s^{a_2\theta}e^{-\gamma s}\Big(\|u_a(\cdot,s)\|_{\mathbb{H}^p(\Omega)}+\|v_a(\cdot,s)\|_{\mathbb{H}^p(\Omega)}\Big)\Big]ds\\
&\le\overline{D}_1(a_1,a_2,k,\theta)\|f_0\|_{\mathbb{H}^p(\Omega)}+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\frac{T^{a_1+(a_2-a_1)\theta}}{a_1-a_1\theta}\\
&\qquad+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\mathscr{C}_{\mu_1',\mu_2}^{\mu_2+1}(\gamma)\Big(\|u_a\|_{\mathbb{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}+\|v_a\|_{\mathbb{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}\Big)\\
&\le\overline{D}_1(a_1,a_2,k,\theta)\|f_0\|_{\mathbb{H}^p(\Omega)}+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\frac{T^{a_1+(a_2-a_1)\theta}}{a_1-a_1\theta}\\
&\qquad\qquad+\sqrt{2}\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\mathscr{C}_{\mu_1',\mu_2}^{\mu_2+1}(\gamma)\|w_a\|_{\mathcal{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}.
\end{aligned}\tag{41}$$

here $\mu_1':=-a_2\theta$. Therefor, we obtain

$$\begin{aligned}
\|u_a(t,\cdot)\|_{\mathbb{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}&\le\overline{D}_1(a_1,a_2,k,\theta)\|f_0\|_{\mathbb{H}^p(\Omega)}+\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\frac{T^{a_1+(a_2-a_1)\theta}}{a_1-a_1\theta}\\
&\qquad+\sqrt{2}\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\mathscr{C}_{\mu_1',\mu_2}^{\mu_2+1}(\gamma)\|w_a\|_{\mathcal{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}.
\end{aligned}\tag{42}$$

By the same evaluation technique above, we also have

$$\|v_a(t,\cdot)\|_{\mathbb{C}^{a_2\theta}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \le \overline{D}_1(a_1,a_2,l,\theta)\|g_0\|_{\mathbb{H}^p(\Omega)} + \overline{D}_2(a_1,a_2,l,\theta)C_{\mathcal{G}}\frac{T^{a_1+(a_2-a_1)\theta}}{a_1-a_1\theta} + \sqrt{2}\overline{D}_2(a_1,a_2,l,\theta)C_{\mathcal{G}}\mathscr{C}^{\mu_2+1}_{\mu_1',\mu_2}(\gamma)\|w_a\|_{\mathcal{C}^{a_2\theta}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}. \tag{43}$$

From (42)-(43), we get

$$\|w_a(t,\cdot)\|_{\mathcal{C}^{a_2\theta}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \le \mathfrak{D}^{k,l}_{1,\theta}(a_1,a_2) + \mathfrak{D}^{k,l}_{2,\theta}(a_1,a_2)\|w_a\|_{\mathcal{C}^{a_2\theta}_{\gamma}(0,T;\mathbb{H}^p(\Omega))}. \tag{44}$$

where the following symbols are used

$$\begin{aligned}\mathfrak{D}^{k,l}_{1,\theta}(a_1,a_2) &:= \overline{D}_1(a_1,a_2,k,\theta)\|f_0\|_{\mathbb{H}^p(\Omega)} + \overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\frac{T^{a+(a_2-a)\theta}}{a-a\theta} \\ &\quad + \overline{D}_1(a_1,a_2,l,\theta)\|g_0\|_{\mathbb{H}^p(\Omega)} + \overline{D}_2(a_1,a_2,l,\theta)C_{\mathcal{G}}\frac{T^{a+(a_2-a)\theta}}{a-a\theta}, \\ \mathfrak{D}^{k,l}_{2,\theta}(a_1,a_2) &:= \sqrt{2}\overline{D}_2(a_1,a_2,k,\theta)C_{\mathcal{F}}\mathscr{C}^{\mu_2+1}_{\mu_1',\mu_2}(\gamma) + \sqrt{2}\overline{D}_2(a_1,a_2,l,\theta)C_{\mathcal{G}}\mathscr{C}^{\mu_2+1}_{\mu_1',\mu_2}(\gamma).\end{aligned} \tag{45}$$

Therefore, we obtain

$$\|w_a\|_{\mathcal{C}^{a_2\theta}_{\gamma}(0,T;\mathbb{H}^p(\Omega))} \le \frac{\mathfrak{D}^{k,l}_{1,\theta}(a_1,a_2)}{1-\mathfrak{D}^{k,l}_{2,\theta}(a_1,a_2)} \tag{46}$$

Part 3. Similar as part 2, from (19)-(24), we can show that

$$\begin{aligned}&\|u_{a'}(t,\cdot)-u_a(t,\cdot)\|_{\mathbb{H}^p(\Omega)} \\ &\le \Big\|\big[\mathcal{R}_{a',k}-\mathcal{R}_{a,k}(t)\big](t)f_0\Big\|_{\mathbb{H}^p(\Omega)} + \Big\|\int_0^t\big[\mathcal{S}_{a',k}(t-s)-\mathcal{S}_{a,k}(t-s)\big]\mathcal{F}(u_{a'},v_{a'})(\cdot,s)ds\Big\|^2_{\mathbb{H}^p(\Omega)} \\ &\quad + \Big\|\int_0^t\mathcal{S}_{a,k}(t-s)\Big[\mathcal{F}(u_{a'},v_{a'})(\cdot,s)-\mathcal{F}(u_a,v_a)(\cdot,s)\Big]ds\Big\|^2_{\mathbb{H}^p(\Omega)}.\end{aligned}$$

Applying Lemma (2.9-2.10), we obtain

$$\begin{aligned}&t^{a_2(1-\theta)+\epsilon}e^{-\gamma t}\|\mathbf{u}_{a'}(t,\cdot)-\mathbf{u}_a(t,\cdot)\|^2_{\mathbb{H}^p(\Omega)} \\ &\le \underbrace{t^{a_2(1-\theta)+\epsilon}e^{-\gamma t}\overline{D}_1(a_1,a_2,\epsilon,\theta,k,T)t^{-a_2(1-\theta)-\epsilon}\|f_0\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^{\epsilon}+(a'-a)\Big]}_{\mathcal{U}_1} \\ &\quad + \underbrace{t^{a_2(1-\theta)+\epsilon}e^{-\gamma t}\Big[(a'-a)^{\epsilon}+(a'-a)\Big]\int_0^t\overline{D}_2(a_1,a_2,\epsilon,\rho,k,T)(t-s)^{a_1\theta-\epsilon-1}\|\mathcal{F}(u_{a'},v_{a'})(\cdot,s)\|_{\mathbb{H}^p(\Omega)}ds}_{\mathcal{U}_2} \\ &\quad + \underbrace{t^{a_2(1-\theta)+\epsilon}e^{-\gamma t}\int_0^t\overline{D}_2(a_1,a_2,k,\theta)(t-s)^{(a-1-a\theta)}\|\mathcal{F}(u_{a'},v_{a'})(\cdot,s)-\mathcal{F}(u_a,v_a)(\cdot,s)\|_{\mathbb{H}^p(\Omega)}ds}_{\mathcal{U}_3}.\end{aligned} \tag{47}$$

Estimate $\mathcal{U}_1$: using inequality $e^{-bt}<1$ for all $t>0$, we obtain

$$\mathcal{U}_1 \le \overline{D}_1(a_1,a_2,\epsilon,\theta,k,T)\|f_0\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^{\epsilon}+(a'-a)\Big]. \tag{48}$$

Estimate $\mathcal{U}_2$: Using the same estimation (42) with assumption (28), we get

$$\begin{aligned}
\mathcal{U}_2 &\leq \overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)\Big[(a'-a)^\epsilon+(a'-a)\Big]C_{\mathcal{F}} \\
&\quad\times t^{2a_2(1-\theta)+2\epsilon}\int\limits_0^t s^{-a_2\theta}(t-s)^{(a_1\theta-\epsilon-1)}e^{-\gamma(t-s)}\Big[s^{a_2\theta}e^{-\gamma s}\Big(1+\|u_{a'}(\cdot,t)\|_{\mathbb{H}^p(\Omega)}+\|v_{a'}(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big)\Big]ds. \\
&\leq \overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)C_{\mathcal{F}}\Big[(a'-a)^\epsilon+(a'-a)\Big]t^{a_2(1-\theta)+\epsilon}\int\limits_0^t (t-s)^{a_1\theta-\epsilon-1}ds. \\
&\quad+\overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)C_{\mathcal{F}}\Big[(a'-a)^\epsilon+(a'-a)\Big] \\
&\qquad\qquad\times\sqrt{2}\|w_{a'}\|_{\mathcal{C}_\gamma^{a_2\theta}(0,T;\mathbb{H}^p(\Omega))}t^{a_2(1-\theta)+\epsilon}\int\limits_0^t s^{-a_2\theta}(t-s)^{a_1\theta-\epsilon-1}ds.
\end{aligned}$$

Assume that $0<\epsilon<a_1\theta$, $\theta<\frac{a_1}{a_2+a_1}<1$ then

$$\begin{cases} -a_2\theta>-1, \\ a_1\theta-\epsilon-1>-1. \end{cases}$$

Following the Beta's function property

$$\int_0^t s^{m-1}(t-s)^{n-1}ds=t^{m+n-1}B(m,n)$$

where

$$B(m,n):=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m,n)}, m>0, n>0,$$

from (46) we can deduce

$$\begin{aligned}
\mathcal{U}_2 &\leq \overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)\Big[(a'-a)^\epsilon+(a'-a)\Big]C_{\mathcal{F}}\frac{T^{a_2(1-\theta)+a_1\theta}}{a_1\theta-\epsilon} \\
&\quad+\overline{D}_2(a_1,a_2,\epsilon,\theta,k,T)\Big[(a'-a)^\epsilon+(a'-a)\Big]\frac{C_{\mathcal{F}}\sqrt{2}\mathfrak{D}_{1,\theta}^{k,l}(a_1,a_2)}{1-\mathfrak{D}_{2,\theta}^{k,l}(a_1,a_2)}T^{a_2-2a_2\theta+a_1\theta}B(1-a_2\theta,a_1\theta-\epsilon).
\end{aligned} \tag{49}$$

Estimate $\mathcal{U}_3$: Using globally Lipschitz property of $\mathcal{F}$ (29), $0<\epsilon<a_1\theta$ and $1-\frac{1}{a_2}<\theta<\frac{a_1}{a_2+a_1}$, we also have

$$\begin{aligned}
\mathcal{U}_3 &\leq \overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}}t^{a_2(1-\theta)+\epsilon}\int\limits_0^t (t-s)^{a_1-1-a_1\theta}e^{-\gamma t}\Big(\|u_{a'}(\cdot,t)-u_a(\cdot,t)\|_{\mathbb{H}^p(\Omega)}+\|v_{a'}(\cdot,t)-v_a(\cdot,t)\|_{\mathbb{H}^p(\Omega)}\Big)ds, \\
&\leq \overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}}\Big(\|u_{a'}-u_a\|_{\mathbb{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}+\|v_{a'}-v_a\|_{\mathbb{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}\Big) \\
&\qquad\qquad\times t^{a_2(1-\theta)+\epsilon}\int\limits_0^t s^{-a_2(1-\theta)-\epsilon}(t-s)^{a_1-1-a_1\theta}e^{-\gamma(t-s)}ds, \\
&\leq \sqrt{2}\overline{D}_2(a_1,a_2,k,\theta)K_{\mathcal{F}}T^{a_1(1-\theta)}B(1-a_2(1-\theta)-\epsilon,a_1-a_1\theta)\|w_{a'}-w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}.
\end{aligned} \tag{50}$$

From (47)-(50), we obtain

$$
\begin{aligned}
&\|u_{a'} - u_a\|_{\mathbb{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))} \\
&\le \overline{D}_1(a_1, a_2, \epsilon, \theta, k, T)\|f_0\|_{\mathbb{H}^p(\Omega)}\Big[(a'-a)^\epsilon + (a'-a)\Big] \\
&+ \overline{D}_2(a_1, a_2, \epsilon, \theta, k, T)C_{\mathcal{F}}\frac{T^{a_2(1-\theta)+a_1\theta}}{a_1\theta - \epsilon}\Big[(a'-a)^\epsilon + (a'-a)\Big] \\
&+ \overline{D}_2(a_1, a_2, \epsilon, \theta, k, T)\frac{C_{\mathcal{F}}\sqrt{2}\mathfrak{D}_{1,\theta}^{k,l}(a_1, a_2)}{1 - \mathfrak{D}_{2,\theta}^{k,l}(a_1, a_2)}T^{a_2 - 2a_2\theta + a_1\theta}B(1 - a_2\theta, a_1\theta - \epsilon)\Big[(a'-a)^\epsilon + (a'-a)\Big] \\
&+ \sqrt{2}\overline{D}_2(a_1, a_2, k, \theta)K_{\mathcal{F}}T^{a_1(1-\theta)}B(1 - a_2(1-\theta) - \epsilon, a_1 - a_1\theta)\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}. \qquad (51)
\end{aligned}
$$

To facilitate the calculation, we set

$$
\begin{aligned}
\overline{U}_1(a_1, a_2, \epsilon, \theta, k, T) &:= \overline{D}_1(a_1, a_2, \epsilon, \theta, k, T)\|f_0\|_{\mathbb{H}^p(\Omega)} + \overline{D}_2(a_1, a_2, \epsilon, \theta, k, T)C_{\mathcal{F}}\frac{T^{a_2(1-\theta)+a_1\theta}}{a_1\theta - \epsilon} \\
&\quad + \overline{D}_2(a_1, a_2, \epsilon, \theta, k, T)\frac{C_{\mathcal{F}}\sqrt{2}\mathfrak{D}_{1,\theta}^{k,l}(a_1, a_2)}{1 - \mathfrak{D}_{2,\theta}^{k,l}(a_1, a_2)}T^{a_2 - 2a_2\theta + a_1\theta}B(1 - a_2\theta, a_1\theta - \epsilon), \\
\overline{U}_2(a_1, a_2, \epsilon, \theta, k, T) &:= \sqrt{2}\overline{D}_2(a_1, a_2, k, \theta)K_{\mathcal{F}}T^{a_1(1-\theta)}B(1 - a_2(1-\theta) - \epsilon, a_1 - a_1\theta). \qquad (52)
\end{aligned}
$$

We obtain the following estimate

$$
\begin{aligned}
&\|u_{a'} - u_a\|_{\mathbb{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))} \\
&\le \overline{U}_1(a_1, a_2, \epsilon, \theta, k, T)\Big[(a'-a)^\epsilon + (a'-a)\Big] + \overline{U}_2(a_1, a_2, \epsilon, \theta, k, T)\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}. \qquad (53)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|v_{a'} - v_a\|_{\mathbb{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))} \\
&\le \overline{V}_1(a_1, a_2, \epsilon, \theta, l, T)\Big[(a'-a)^\epsilon + (a'-a)\Big] + \overline{V}_2(a_1, a_2, \epsilon, \theta, l, T)\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}. \qquad (54)
\end{aligned}
$$

From (53)-(54), we can deduce

$$
\begin{aligned}
\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))} &\le \Big(\overline{U}_1(a_1, a_2, \epsilon, \theta, k, T) + \overline{V}_1(a_1, a_2, \epsilon, \theta, l, T)\Big)\Big[(a'-a)^\epsilon + (a'-a)\Big] \\
&+ \Big(\overline{V}_1(a_1, a_2, \epsilon, \theta, k, T) + \overline{V}_2(a_1, a_2, \epsilon, \theta, l, T)\Big)\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))}.
\end{aligned}
$$

Finally, we get the following evaluation:

$$
\|w_{a'} - w_a\|_{\mathcal{C}_\gamma^{a_2(1-\theta)+\epsilon}(0,T;\mathbb{H}^p(\Omega))} \le \frac{\Big(\overline{U}_1(a_1, a_2, \epsilon, \theta, k, T) + \overline{V}_1(a_1, a_2, \epsilon, \theta, l, T)\Big)\Big[(a'-a)^\epsilon + (a'-a)\Big]}{1 - \Big(\overline{V}_1(a_1, a_2, \epsilon, \theta, k, T) + \overline{V}_2(a_1, a_2, \epsilon, \theta, l, T)\Big)}.
$$

The theorem (3.2) has been proved. □

3.1. Data availability statement

No data were used to support this study.

3.2. Conflicts of Interest

The authors declare that they have no competing interests.

3.3. Authors Contributions

Authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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