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Almost Contraction Mappings and (S,T)-Stability of Jungck Iteration in Cone Metric Spaces over Banach Algebras

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Abstract. In this paper, we first introduce almost contraction mappings for a pair of two mappings in cone metric spaces over Banach algebras (CMSBA). Then, we observe that the class of such mappings in this setting contains those of many well known mappings. Finally, based on the fixed point theorem of the mappings belonging to this class, we obtain (S,T)-stability results of Jungck iterations for some mappings in CMSBA.

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1. Introduction

In 1922, Banach showed that T has a unique fixed point if the conditions below hold: Given a complete metric space (X,d) and a self mapping T satisfying

$$d(T\nu, T\vartheta) \leq \lambda d(\nu,\vartheta) \text{ for all } \nu,\vartheta \in X,$$

where $\lambda \in (0,1)$. In the literature, there are many studies aiming to improve this result known as Banach's contraction principle. In 1973, one of them has been obtained by Hardy and Rogers [4], considering the following class of mappings:

$$d(T\nu, T\vartheta) \leq l_1 d(\nu,\vartheta) + l_2 d(\nu, T\nu) + l_3 d(\vartheta, T\vartheta) + l_4 d(\nu, T\vartheta) + l_5 d(\vartheta, T\nu), \tag{1.1}$$

where l_i $(i:1,...,5)$ are non-negative constants satisfying $\sum_{i=1}^{5} l_i < 1$ and $\nu,\vartheta \in X$. Moreover, in 1976, Jungck (see [10]) gave some results about common fixed points for given commuting mappings. In 2004, another kind of mapping known as almost contraction was introduced by Berinde in [2] as follows: There exist $h \in (0,1)$ and $L \geq 0$ such that

$$d(T\nu, T\vartheta) \leq h d(\nu,\vartheta) + L d(\vartheta, T\nu) \tag{1.2}$$

for all $\nu,\vartheta \in X$, which contains well known contractive mappings such as Banach, Kannan, Chatterjea, Zamfirescu. On the other hand, Harder and Hicks in [5] initiated a great interest in the study of stability for procedures of fixed point iterations in 1988. In 1995, Osilike extended the results of [5] by considering the following class of mappings:

$$d(T\nu, T\vartheta) \leq a d(\nu,\vartheta) + L d(\nu, T\nu) \tag{1.3}$$

for all $\nu,\vartheta \in X$, where $a \in [0,1)$ and $L \geq 0$ [16]. Then, in 2005, Singh et al., in [18] introduced the concept of stability of Jungck-type iteration procedures for a pair of mappings.

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In 2013, Liu and Xu in [14] extended Banach's contraction principle by replacing the usual Lipschitz constant with a vector in the setting of CMSBA with normal solid cones. To emphasize the merit of such an extension, they also presented a mapping which is contraction in this new setting, but not in the usual sense. Furthermore, Xu and Radenovic in [19] showed that the normality condition of cone can be removed to obtain the main results in [14]. Moreover, Huang et al., in [8] presented the notion of T-stability of Picard iteration in CMSBA. In 2018, Ozavsar in [15] extended the notion of almost contraction to the setting of CMSBA, and observed that Banach, Kannan and Chatterjea type mappings are contained by the class of almost contraction mappings in such spaces. Also, the fixed point results for almost contraction mappings have been studied recently in cone b-metric spaces over Banach algebras [3].

Now, in the sequel, we present some basic definitions and properties, which will be necessary to obtain our main results.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra so that e and θ are unit and zero elements of this algebra, respectively. In order to establish a partial ordered relation on $\mathcal{A}$, we need to the concept of cone: A nonempty closed subset $\mathcal{P}$ of $\mathcal{A}$ is said to be a cone if

- **P1:** $\{\theta, e\} \subseteq \mathcal{P}$,
- **P2:** $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subseteq \mathcal{P}$,
- **P3:** $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$,
- **P4:** $\alpha\mathcal{P} + \beta\mathcal{P} \subseteq \mathcal{P}$ for all nonnegative real numbers α and β.

On this basis, we define a partial ordering $\preceq$ as follows:

$$\nu \preceq \vartheta \text{ if and only if } \vartheta - \nu \in \mathcal{P}.$$

Moreover, $\nu \ll \vartheta$ stands for $\vartheta - \nu \in int\mathcal{P}$, where $int\mathcal{P}$ denotes the interior of $\mathcal{P}$. A cone $\mathcal{P}$ is called a solid cone if $int\mathcal{P} \neq \emptyset$. Throughout the paper, we suppose that $\mathcal{A}$ stands for a Banach algebra with a unit e, $\mathcal{P}$ is a a solid cone in $\mathcal{A}$, and $\preceq$ is a partial ordering with respect to $\mathcal{P}$.

Definition 2.1 ([6, 14])**.** Let X be a nonempty set. A cone metric d over $\mathcal{A}$ is defined as a mapping given by $d : X \times X \to A$ satisfying

- **d1:** $\theta \preceq d(\nu, \vartheta)$ and $d(\nu, \vartheta) = \theta$ if and only if $\nu = \vartheta$;
- **d2:** $d(\nu, \vartheta) = d(\vartheta, \nu)$;
- **d3:** $d(\nu, \vartheta) \preceq d(\nu, w) + d(w, \vartheta)$

for all $\nu, \vartheta, w \in X$.

The pair (X, d) is called the cone metric space on $\mathcal{A}$, and we assume this pair as such from now on.

Definition 2.2 ([6, 14])**.** Let $\{\nu_n\}$ be sequence in X. Then,

(1) $\{\nu_n\}$ converges to $\nu \in X$ if for each $c \in \mathcal{A}$ with $\theta \ll c$ there exists $n_c \in \mathbb{N}$ such that $d(\nu_n, \nu) \ll c$ for all $n \geq n_c$. This is denoted by $\lim_{n\to\infty} \nu_n = \nu$ or $\nu_n \to \nu, n \to \infty$.
(2) $\{\nu_n\}$ is a Cauchy sequence if for each $c \in \mathcal{A}$ with $\theta \ll c$ there exists $n_c \in \mathbb{N}$ such that $d(\nu_n, \nu_m) \ll c$ for all $n, m \geq n_c$.
(3) (X, d) is called complete if every Cauchy sequence $\{\nu_n\}$ in X is convergent.

Definition 2.3 ([1, 11])**.** Let $S, T : X \to X$ be mappings on X.

(1) If $w = Sz = Tz$ for some $z \in X$, then z is said to be a coincidence point of S and T, and w is said a point of coincidence of S and T.
(2) The pair $\{S, T\}$ is said to be weakly compatible if S and T commute at all of coincidence points, that is, $STz = TSz$ for all $z \in \{z \in X : Sz = Tz\}$.

Proposition 2.4 ([1])**.** *Suppose that S and T are weakly compatible self-mappings of a set X such that these mappings have a unique point of coincidence $w = Sz = Tz$. Then, w is the unique common fixed point for S and T.*

Definition 2.5 ([19])**.** A sequence $\{\nu_n\}$ in $\mathcal{A}$ is a c-sequence if for each $\theta \ll c$, there exists $n_c \in \mathbb{N}$ such that $\nu_n \ll c$ for all $n > n_c$.

Note that by the above definition, if $\{\nu_n\}$ convergens to $\nu \in X$, $\{d(\nu_n, \nu)\}$ is a c-sequence.

Lemma 2.6 ([19]). *If a sequence $\{\nu_n\}$ in $\mathcal{A}$ is a c-sequence, then $\{\alpha\nu_n\}$ is a c-sequence, where $\alpha \in \mathcal{P}$.*

Lemma 2.7 ([13]). *If two sequences $\{\nu_n\}$ and $\{\vartheta_n\}$ in $\mathcal{A}$ are c-sequences, then $\{\alpha\nu_n + \beta\vartheta_n\}$ is a c-sequence, where $\alpha, \beta > 0$.*

Lemma 2.8 ([12]). *The conditions listed below hold:*

a: *If $\nu \preceq \vartheta$ and $\vartheta \ll c$, then $\nu \ll c$.*
b: *If $\theta \preceq \nu \ll c$ for each $\theta \ll c$, then $\nu = \theta$.*

In the following, some definitions and properties, which are necessary to give our results, are introduced. The spectral radius $\rho(\nu)$ of $\nu \in \mathcal{A}$ satisfies

$$\rho(\nu) = \lim_{n\to\infty} \|\nu^n\|^{\frac{1}{n}} = \inf \|\nu^n\|^{\frac{1}{n}} .$$

If $\rho(\nu) < |\lambda|$, then $\lambda e - \nu$ invertible in $\mathcal{A}$, and also $(\lambda e - \nu)^{-1} = \sum_{i=0}^{\infty} \frac{\nu^i}{\lambda^{i+1}}$, where λ is a complex constant [17].

Lemma 2.9 ([17]). *Let $\nu, \vartheta \in \mathcal{A}$ with $\nu\vartheta = \vartheta\nu$. Then, the following inequalities hold:*

a: $\rho(\nu\vartheta) \leq \rho(\nu)\rho(\vartheta)$,
b: $\rho(\nu + \vartheta) \leq \rho(\nu) + \rho(\vartheta)$.

Lemma 2.10 ([7]). *If $\rho(\nu) < |\lambda|$ and λ is a complex constant, then*

$$\rho((\lambda e - \nu)^{-1}) \leq \frac{1}{|\lambda| - \rho(\nu)}.$$

Lemma 2.11 ([7]). *If $\rho(\nu) < 1$, then $\{\nu^n\}$ is a c-sequence.*

Lemma 2.12 ([8]). *Let $\{\nu_n\}$ and $\{c_n\}$ be two sequences in $\mathcal{A}$ satisfying the following inequality:*

$$\nu_{n+1} \preceq h\nu_n + c_n,$$

where $h \in \mathcal{P}$ with $\rho(h) < 1$. If $\{c_n\}$ is a c-sequence, then $\{\nu_n\}$ is a c-sequence.

3. Main Results

We first consider the versions of (1.1), (1.2) and (1.3) for a pair of self-mappings in cone metric spaces over $\mathcal{A}$, respectively, as follows:

(HR): There exist $l_i \in \mathcal{P}$ $(i : 1, ..., 5)$ with $\rho(l_1) + \rho(l_2 + l_3 + l_4 + l_5) < 1$ such that

$$d(T\nu, T\vartheta) \preceq l_1 d(S\nu, S\vartheta) + l_2 d(S\nu, T\nu) + l_3 d(S\vartheta, T\vartheta) + l_4 d(S\nu, T\vartheta) + l_5 d(S\vartheta, T\nu),$$

(B): There exist $h \in \mathcal{P}$ with $\rho(h) < 1$ and $\theta \preceq L$ such that

$$d(T\nu, T\vartheta) \preceq hd(S\nu, S\vartheta) + Ld(S\vartheta, T\nu),$$

(O): There exist $a \in \mathcal{P}$ with $\rho(a) < 1$ and $\theta \preceq L$ such that

$$d(T\nu, T\vartheta) \preceq ad(S\nu, S\vartheta) + Ld(S\nu, T\nu),$$

for all $\nu, \vartheta \in X$.

Proposition 3.1. *If l_1 commutes with $l_2 + l_3 + l_4 + l_5$, then $(HR) \Rightarrow (B)$.*

Proof. By the condition (HR) and triangle rule, we get

$$\begin{aligned} d(T\nu, T\vartheta) &\preceq l_1 d(S\nu, S\vartheta) + l_2 d(S\nu, T\nu) + l_3 d(S\vartheta, T\vartheta) + l_4 d(S\nu, T\vartheta) + l_5 d(S\vartheta, T\nu) \\ &\preceq l_1 d(S\nu, S\vartheta) + l_2 d(S\nu, S\vartheta) + l_2 d(S\vartheta, T\nu) + l_3 d(S\vartheta, T\nu) + l_3 d(T\nu, T\vartheta) \\ &\quad + l_4 d(S\nu, S\vartheta) + l_4 d(S\vartheta, T\nu) + l_4 d(T\nu, T\vartheta) + l_5 d(S\vartheta, T\nu) \\ &= (l_1 + l_2 + l_4) d(S\nu, S\vartheta) + (l_2 + l_3 + l_4 + l_5) d(S\vartheta, T\nu) + (l_3 + l_4) d(T\nu, T\vartheta), \end{aligned}$$

which implies that

$$(e - l_3 - l_4) d(T\nu, T\vartheta) \preceq (l_1 + l_2 + k_4) d(S\nu, S\vartheta) + (l_2 + l_3 + l_4 + l_5) d(S\vartheta, T\nu). \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} d(Tv, T\vartheta) &= d(T\vartheta, Tv) \preceq l_1 d(S\vartheta, Sv) + l_2 d(S\vartheta, T\vartheta) + l_3 d(Sv, Tv) + l_4 d(S\vartheta, Tv) + l_5 d(Sv, T\vartheta) \\ &\preceq l_1 d(S\vartheta, Sv) + l_2 d(S\vartheta, Tv) + l_2 d(Tv, T\vartheta) + l_3 d(Sv, S\vartheta) + l_3 d(S\vartheta, Tv) + l_4 d(S\vartheta, Tv) \\ &\quad + l_5 d(Sv, S\vartheta) + l_5 d(S\vartheta, Tv) + l_5 d(Tv, T\vartheta) \\ &= (l_1 + l_3 + l_5) d(Sv, S\vartheta) + (l_2 + l_3 + l_4 + l_5) d(S\vartheta, Tv) + (l_2 + l_5) d(Tv, T\vartheta), \end{aligned}$$

which means that

$$(e - l_2 - l_5) d(Tv, T\vartheta) \preceq (l_1 + l_3 + l_5) d(Sv, S\vartheta) + (l_2 + l_3 + l_4 + l_5) d(S\vartheta, Tv). \tag{3.2}$$

Adding up (3.1) and (3.2), we obtain

$$(2e - l) d(Tv, T\vartheta) \preceq (2l_1 + l) d(Sv, S\vartheta) + 2l d(S\vartheta, Tv), \tag{3.3}$$

where $l = l_2 + l_3 + l_4 + l_5$. Since $\rho(l) \leq \rho(l_1) + \rho(l) < 1 < 2$, $(2e - l)$ is invertible. By multiplying in both sides of (3.3) by $(2e - l)^{-1}$, one can write

$$d(Tv, T\vartheta) \preceq (2e - l)^{-1}(2l_1 + l) d(Sv, S\vartheta) + (2e - l)^{-1} 2l d(S\vartheta, Tv).$$

Moreover, since l_1 commutes with l, we can obtain that

$$\begin{aligned} (2e - l)^{-1}(2l_1 + l) &= \Big(\sum_{i=0}^{\infty} \frac{l^i}{2^{i+1}}\Big)(2l_1 + l) \\ &= 2l_1\Big(\sum_{i=0}^{\infty} \frac{l^i}{2^{i+1}}\Big) + l\Big(\sum_{i=0}^{\infty} \frac{l^i}{2^{i+1}}\Big) \\ &= (2l_1 + l)\Big(\sum_{i=0}^{\infty} \frac{l^i}{2^{i+1}}\Big) = (2l_1 + l)(2e - l)^{-1}, \end{aligned}$$

that is, $(2e - l)^{-1}$ commutes with $(2l_1 + l)$. Letting $h = (2e - l)^{-1}(2l_1 + l)$, by Lemma 2.9 and Lemma 2.10, we get

$$\begin{aligned} \rho(h) &= \rho((2e - l)^{-1}(2l_1 + l)) \leq \rho((2e - l)^{-1})\rho(2l_1 + l) \\ &\leq \frac{1}{2 - \rho(l)}[2\rho(l_1) + \rho(l)] < 1. \end{aligned}$$

Considering $h = (2e - l)^{-1}(2l_1 + k)$ with $\rho(h) < 1$ and $L = (2e - l)^{-1} 2l$, we can easily see that the condition (B) holds. □

Proposition 3.2. *If l_1 commutes with $l_2 + l_3 + l_4 + l_5$, then $(HR) \Rightarrow (O)$.*

Proof. Since the proof is very similar to Proposition 3.1, it is left to the reader. □

Now, we introduce the following theorem which will be necessary in our results.

Theorem 3.3 ([3]). *Let S and T be two self-mappings of X such that TX is a subset of SX, which is a complete subspace of X. Assume that there exist $h \in \mathcal{P}$ with $\rho(h) < 1$ and $\theta \preceq L$ such that*

$$d(Tv, T\vartheta) \preceq h d(Sv, S\vartheta) + L d(S\vartheta, Tv)$$

for all $v, \vartheta \in X$. Then, S and T have a point of coincidence in X.

Proof. The proof can be easily seen by setting up $s = 1$ in Theorem 3.4 given in the paper [3]. □

If $S = I_X$ is the identity map on X in Theorem 3.3, we have the following Corollary.

Corollary 3.4 ([15]). *Let (X, d) be a complete cone metric space over $\mathcal{A}$. Assume that there exist $h \in \mathcal{P}$ with $\rho(h) < 1$ and $\theta \preceq L$ such that*

$$d(Tv, T\vartheta) \preceq h d(v, \vartheta) + L d(\vartheta, Tv)$$

for all $v, \vartheta \in X$. Then, T has a fixed point in X.

Theorem 3.5. *Let S, T be two self-mappings of X and w be a point of coincidence of these mappings, that is $w = Sz = Tz$. Assume that there exist $a \in \mathcal{P}$ with $\rho(a) < 1$ and $\theta \preceq L$ such that*

$$d(Tv, T\vartheta) \preceq ad(Sv, S\vartheta) + Ld(Sv, Tv) \tag{3.4}$$

for all $v, \vartheta \in X$. Then, the point of coincidence w is unique. Furthermore, if $\{S, T\}$ is weakly compatible pair, then w is a unique common fixed point of S and T.

Proof. Suppose that w^* is a different point of coincidence for these mappings, that is $w^* = Sz^* = Tz^*$. Then, we get by (3.4)

$$d(Sz, Sz^*) = d(Tz, Tz^*) \preceq ad(Sz, Sz^*),$$

which implies that

$$d(w, w^*) \preceq ad(w, w^*) \preceq \cdots \preceq a^n d(w, w^*).$$

Using Lemma 2.6 and Lemma 2.11, $a^n d(w, w^*)$ is a c-sequence and by Lemma 2.8, it is clear that $d(w, w^*) = \theta$, that is $w = w^*$. Finally, if $\{S, T\}$ is weakly compatible pair, w is a unique common fixed point of these two mappings by Proposition 2.4. □

Corollary 3.6 ([7]). *Let S and T be two self-mappings of X such that TX is a subset of SX, which is a complete subspace of X. Assume that there exist $l_i \in \mathcal{P}$ $(i : 1, ..., 5)$ with $\rho(l_1) + \rho(l_2 + l_3 + l_4 + l_5) < 1$ such that*

$$d(Tv, T\vartheta) \preceq l_1 d(Sv, S\vartheta) + l_2 d(Sv, Tv) + l_3 d(S\vartheta, T\vartheta) + l_4 d(Sv, T\vartheta) + l_5 d(S\vartheta, Tv)$$

for all $v, \vartheta \in X$, and l_1 commutes with $l_2 + l_3 + l_4 + l_5$. Then, there is a unique point of coincidence in X for given S and T mappings. Furthermore, if $\{S, T\}$ is weakly compatible pair, then S and T have a unique common fixed point.

Proof. By Proposition 3.1 and Theorem 3.3, T and S have a point of coincidence. Also, by Proposition 3.2 and Theorem 3.5, the proof is completed. □

4. (S, T)-Stability in Cone Metric Spaces over Banach Algebras

Let (X, d) cone metric space over $\mathcal{A}$, and $T, S : X \to X$ satisfy $TX \subset SX$. Assume that an iteration procedure is given as follows:

$$Sv_{n+1} = f(T, v_n), \quad n = 0, 1, 2, ..., \tag{4.1}$$

holds. As an example, in the case of Jungck iteration we have $Sv_{n+1} = Tv_n$, which reduces to the Picard iteration when S is the idendity mapping.

Definition 4.1. Let $\{S\vartheta_n\}$ be any sequence in X and $\{Sv_n\}$ converges to a point of coincidence w of S and T, that is $w = Sz = Tz$. Let define $\varepsilon_n = d(S\vartheta_{n+1}, f(T, \vartheta_n))$ for $n = 0, 1, 2, ...$ The iteration procedure (4.1) is said to be (S, T)-stable if

$$\varepsilon_n \text{ is a } c\text{-sequence } \Rightarrow \lim_{n\to\infty} S\vartheta_n = w.$$

If the conditions of Definition 4.1 hold for $Sv_{n+1} = Tv_n$, then we will say that Jungck's iteration is (S, T)-stable.

Remark 4.2. The concept of stability given in Definition 4.1 is nothing else that one in obtained in [8] when we take $S = I_X$.

Theorem 4.3. *Let S, T be two self-mappings of X and w be a point of coincidence of these mappings, that is $w = Sz = Tz$. Assume that there exist $a \in \mathcal{P}$ with $\rho(a) < 1$ and $\theta \preceq L$ such that*

$$d(Tv, T\vartheta) \preceq ad(Sv, S\vartheta) + Ld(Sv, Tv)$$

for all $v, \vartheta \in X$. Then Jungck's iteration is (S, T)-stable.

Proof. Let $\{S\vartheta_n\} \subseteq X$, $\varepsilon_n = d(S\vartheta_{n+1}, T\vartheta_n)$ and ε_n is a c-sequence. We shall show that $S\vartheta_n \to w$. Since

$$\begin{aligned} d(S\vartheta_{n+1}, w) &\preceq d(S\vartheta_{n+1}, T\vartheta_n) + d(T\vartheta_n, w) = d(Tz, T\vartheta_n) + \varepsilon_n \\ &\preceq ad(Sz, S\vartheta_n) + Ld(Sz, Tz) + \varepsilon_n = ad(w, S\vartheta_n) + \varepsilon_n, \end{aligned}$$

by taking $v_n = d(S\vartheta_n, w)$ and $c_n = \varepsilon_n$ in Lemma 2.12, we get that $d(S\vartheta_n, w)$ is a c-sequence and then $S\vartheta_n \to w$. □

Corollary 4.4. *Let S and T be two self-mappings of X such that TX is a subset of SX, which is a complete subspace of X. Assume that there exist $l_i \in \mathcal{P}$ $(i : 1, ..., 5)$ with $\rho(l_1) + \rho(l_2 + l_3 + l_4 + l_5) < 1$ such that*

$$d(T\nu, T\vartheta) \preceq l_1 d(S\nu, S\vartheta) + l_2 d(S\nu, T\nu) + l_3 d(S\vartheta, T\vartheta) + l_4 d(S\nu, T\vartheta) + l_5 d(S\vartheta, T\nu)$$

for all $\nu, \vartheta \in X$, and l_1 commutes with $l_2 + l_3 + l_4 + l_5$. Then, Jungck's iteration is (S,T)-stable.

Proof. By Corallary 3.6, it is seen that there is a point of coincidence for these mappings S and T. By Proposition 3.2 and Theorem 4.3, the proof is completed. □

Taking $S = I_X$ in Corallary 4.4, the following result is clear.

Corollary 4.5 ([8])**.** *Let (X, d) be a complete cone metric space over $\mathcal{A}$. Assume that there exist $l_i \in \mathcal{P}$ $(i : 1, ..., 5)$ with $\rho(l_1) + \rho(l_2 + l_3 + l_4 + l_5) < 1$ such that*

$$d(T\nu, T\vartheta) \preceq l_1 d(\nu, \vartheta) + l_2 d(\nu, T\nu) + l_3 d(\vartheta, T\vartheta) + l_4 d(\nu, T\vartheta) + l_5 d(\vartheta, T\nu)$$

for all $\nu, \vartheta \in X$, and l_1 commutes with $l_2 + l_3 + l_4 + l_5$. Then, Picard's iteration is T-stable.

5. Conclusion

In this paper, we first give a relation between Hardy-Rogers type mappings (HR) and almost contraction type mappings (B) (see Proposition 3.1). Then, we observe that the fixed point result of mappings in the class (B) improves the result of those in (HR) as a consequence of this relation. Finally, we extend the notion of (S,T)-stability to the setting of CMSBA. Notice that $\varepsilon_n \to \theta$ $(n \to \infty)$, which is called θ-sequence, implies that $\{\varepsilon_n\}$ is a c-sequence in solid cone, but its converse is not true (see [9]). This fact says that there may be a pair of mappings $T, S : X \to X$ such that some related iterations are not (S,T)-stable in usual sense but (S,T)-stable in the setting obtained in this extension. For this reason, Definition 4.1 improves the notion of (S,T)-stability in [18].

Authors Contibution Statement

All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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