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NEW TYPES OF SETS IN ČECH CLOSURE SPACES

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Abstract. In this paper, we analysis and introduce the concepts of regular closed (open) sets and regular generalized closed (open) sets in Čech closure spaces. Also, we investigate the properties such as intersection, union, subspaces of regular generalized closed (open) sets of a Čech closure spaces. Moreover, by giving counter examples of one-sided theorems, it has been shown that the converse situation is not realized.

1. INTRODUCTION

In a topological space, many types of sets such as open set, closed set, generalized closed set, generalized open set, regular generalized closed set, regular open, regular closed are defined. Firstly, studies in this area started with the generalized closed set model that Levine [8] put forward by generalizing closed sets of any topological space. For example, it was shown that completeness, normality, compactness in a uniform space are inherited by generalized closed subsets. Balachandran et al. [9] introduced the concept of generalized continuous maps by using generalized closed sets. In the following years, Palaniappan and Chandrasekhara Rao [10] introduced regular generalized closed sets in topological spaces.

There are many methods researchers can use to define a topology. Čech closure space, one of these methods, are a set of axioms used to define a topology on a set other than any null set by E. Čech in [3]. After defining Čech closure spaces, it has managed to attract the attention of many researchers and then studied on these spaces, see e.g. [4, 5, 6, 7].

Thanks to this paper, regular generalized closed (open) sets and regular closed (open) sets, which are two new concepts for Čech closure spaces, are brought into literature. Moreover, the given sets were analyzed in detail and their properties such as subspaces, intersection, union were examined. In addition, the types of sets such as closed sets, generalized closed sets given previously for Čech closure spaces

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and the new set types given in this paper were compared and the relationships between each other were studied.

2. PRELIMINARIES

In this section, we recall some basic notions in Čech closure spaces.

Throughout this paper, let $\mathcal{U} \neq \emptyset$ be a set, $2^{\mathcal{U}}$ denotes the power set of $\mathcal{U}$ and X, Y be non-empty subsets of U.

Definition 2.1. [1] An operator $c : 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ defined satisfying the axioms:
$[c1]$ $c(\emptyset) = \emptyset$,
$[c2]$ $X \subseteq c(X)$ for all $X \subseteq \mathcal{U}$,
$[c3]$ $c(X \cup Y) = c(X) \cup c(Y)$ for all $X, Y \subseteq \mathcal{U}$
is called a Čech closure operator (briefly closure operator) and the pair $(\mathcal{U}, c)$ is called a Čech closure space (briefly closure space). Here, for $X \subset \mathcal{U}$, we call $c(X)$ the closure of X.

Definition 2.2. [1] Let c be a closure operator and $(\mathcal{U}, c)$ be a closure space. Then, for $\emptyset \neq X \subseteq \mathcal{U}$,
(i) A c on $\mathcal{U}$ is called idempotent if $c(X) = c(c(X))$.
(ii) X is closed set (briefly c-set) in $(\mathcal{U}, c)$ if $X = c(X)$.
(iii) X is open set (briefly o-set) in $(\mathcal{U}, c)$ if its complement is c-set.
(iv) The $\emptyset$ and $\mathcal{U}$ are both o-set and c-set.

Definition 2.3. [1] Let $(\mathcal{U}, c)$ be a closure space. A closure space $(\mathcal{V}, c_{\mathcal{V}})$ is called a subspace of $(\mathcal{U}, c)$ if $\mathcal{V} \subseteq \mathcal{U}$ and $c_{\mathcal{V}}(X) = c(X) \cap \mathcal{V}$, for all $X \subseteq \mathcal{V}$.

Definition 2.4. [1] Let $(\mathcal{V}, c_{\mathcal{V}})$ be a Čech closure subspace of $(\mathcal{U}, c)$. If K is a closed subset of $(\mathcal{V}, c_{\mathcal{V}})$, then K is a closed subset of $(\mathcal{U}, c)$.

Definition 2.5. [2] Let $(\mathcal{U}, c)$ be a closure space. Then,
(i) A $X \subseteq \mathcal{U}$ is called a generalized closed set (briefly gc-set), if $c(X) \subseteq K$ whenever K is an open subset of $(\mathcal{U}, c)$ with $X \subseteq Y$.
(ii) A $X \subseteq \mathcal{U}$ is called a generalized open set (briefly go-set), if its complement is gc-set.
(iii) If X and Y are generalized closed subsets of $(\mathcal{U}, c)$, then $X \cup Y$ is gc-set. Moreover, the $X \cap Y$ need not be a gc-set.

Remark 2.6. [2] Every c-set is gc-set. The converse need not be a c-set.

Definition 2.7. [1] An interior operator on $\mathcal{U}$ is a map $int : 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ which satisfies
(i) $int(\mathcal{U}) = \mathcal{U}$,
(ii) $int(X) \subseteq X$ for all $X \subseteq \mathcal{U}$,
(iii) $int(X \cap Y) = int(X) \cap int(Y)$ for all $X, Y \subseteq \mathcal{U}$.

In other words, the set $int(X)$ with respect to the closure operator c is defined as $int(X) = \mathcal{U}$–$c(\mathcal{U} - X)$.

3. REGULER GENERALIZED CLOSED SETS

Definition 3.1. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular closed set (briefly rc-set) [regular open set (briefly ro-set)], if $X = c(int(X))$ $[X = int(c(X))]$.

Remark 3.2. Every ro-set is o-set. The converse need not be a ro-set as can be seen from the following example.

Example 3.3. Let $\mathcal{U} = \{m, n\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$ and $c(\{m\}) = c(\{n\}) = c(\mathcal{U}) = \mathcal{U}$. Then $\{m\}$ and $\{n\}$ are o-sets but they are not ro-sets.

Definition 3.4. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular generalized closed set (briefly rgc-set) iff $c(X) \subseteq K$ whenever $X \subseteq K$, where K is ro-set.

Proposition 1. Let $(\mathcal{U}, c)$ be a closure space. If X and Y are regular generalized closed subsets of $(\mathcal{U}, c)$, then $X \cup Y$ is rgc-set.

Proof. Let K be a regular open subset of $(\mathcal{U}, c)$ such that $X \cup Y \subseteq K$. Then $X \subseteq K$ and $Y \subseteq K$. Since X and Y are rgc-set, $c(X) \subseteq K$ and $c(Y) \subseteq K$. Therefore $c(X) \cup c(Y) \subseteq K$ and hence $c(X \cup Y) \subseteq K$. Consequently $X \cup Y$ is rgc-set. □

Remark 3.5. The intersection of two rgc-sets is generally not a rgc set.

Example 3.6. Let $\mathcal{U} = \{m, n, r\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$, $c(\{m\}) = \{m, n\}$, $c(\{n\}) = c(\{r\}) = c(\{n, r\}) = \{n, r\}$, $c(\{m, n\}) = c(\{m, r\}) = c(\mathcal{U}) = \mathcal{U}$. Then $\{m, n\}$ and $\{m, r\}$ are rgc-set but $\{m, n\} \cap \{m, r\} = \{m\}$ is not rgc-set.

Proposition 2. Let $(\mathcal{U}, c)$ be a closure space. If X is a rgc-set and K is a c-set in $(\mathcal{U}, c)$, then $X \cap K$ is rgc-set.

Proof. Let G be an regular open subset of $(\mathcal{U}, c)$ such that $X \cap K \subseteq G$. Then $X \subseteq G \cup (\mathcal{U} - K)$ and so $c(X) \subseteq G \cup (\mathcal{U} - K)$. Therefore $c(X) \cap K \subseteq G$. Since K is a c-set, $c(X \cap K) \subseteq G$. Hence, $X \cap K$ is a rgc-set. □

Proposition 3. Let $(\mathcal{V}, c_{\mathcal{V}})$ be a closed subspace of $(\mathcal{U}, c)$. If K is a regular generalized closed subset of $(\mathcal{V}, c_{\mathcal{V}})$, then K is a regular generalized closed subset of $(\mathcal{U}, c)$.

Proof. Let G be an regular open subset of $(\mathcal{U}, c)$ such that $K \subseteq G$. Then $K \subseteq G \cap \mathcal{V}$. Since K is a rgc-set and $G \cap \mathcal{V}$ is a ro-set in $(\mathcal{V}, c_{\mathcal{V}})$, $c(K) \cap \mathcal{V} = c_{\mathcal{V}}(K) \subseteq G$. But $\mathcal{V}$ is a closed subset of $(\mathcal{U}, c)$ and $c(K) \subseteq G$. Hence, K is a regular generalized closed subset of $(\mathcal{U}, c)$. □

Theorem 3.7. *Let $(\mathcal{U}, c)$ be a closure space and c be idempotent. If X is a regular generalized closed subset of $(\mathcal{U}, c)$ and $X \subseteq Y \subseteq c(X)$, then $c(Y) - Y$ contains no nonempty rc-set.*

Proof. Since $Y \subseteq c(X)$ and c is idempotent, then $c(Y) \subseteq c(c(X)) = c(X)$. That is $c(Y) \subseteq c(X)$. Since $X \subseteq Y$, we obtain $\mathcal{U} - Y \subseteq \mathcal{U} - X$. Form $c(Y) \subseteq c(X)$ and $\mathcal{U} - Y \subseteq \mathcal{U} - X$, $(u(Y) \cap (\mathcal{U} - Y)) \subseteq (c(X) \cap (\mathcal{U} - X))$ which implies $(c(Y) - Y) \subseteq (c(X) - X)$. Now X is a rgc-set. Hence $c(X) - X$ has no nonempty regular generalized closed subset, neither does $c(Y) - Y$. □

Theorem 3.8. *Let $(\mathcal{U}, c)$ be a closure space and $X \subseteq \mathcal{U}$. If X is a rgc-set, then $c(X) - X$ contains no nonempty rc-set.*

Proof. Suppose that X is a rgc-set. Let Y be a regular generalized closed subset of $c(X) - X$. Then $Y \subseteq c(X) \cap (\mathcal{U} - X)$ and so $X \subseteq \mathcal{U} - Y$. But X is a rgc-set. Therefore $c(X) \subseteq \mathcal{U} - Y$. Consequently $Y \subseteq \mathcal{U} - c(X)$. Since $Y \subseteq c(X)$, $Y \subseteq c(X) \cap (\mathcal{U} - c(X)) = \emptyset$. Thus $B = \emptyset$. Therefore $c(X) - X$ contains no nonempty rc-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.9. Let $\mathcal{U} = \{x, y, z\}$ and define a closure operator c on $\mathcal{U}$ by $c(\emptyset) = \emptyset$, $c(\{x\}) = \{x, y\}$, $c(\{y\}) = c(\{z\}) = c(\{y, z\}) = \{y, z\}$, $c(\{x, y\}) = c(\{x, z\}) = c(\mathcal{U}) = \mathcal{U}$. Then $c(\{x\}) - \{x\} = \{y\}$ does not contain nonempty rc-set. But $\{x\}$ is not rgc-set.

Corollary 1. Let $(\mathcal{U}, c)$ be a closure space and X be a rgc-subset of $(\mathcal{U}, c)$. Then X is a rc-set if and only if $c(int(X)) - X$ is a rc-set.

Proof. Let X be regular generalized closed subset of $(\mathcal{U}, c)$. If X is a rc-set, then $c(int(X)) - X = \emptyset$. But $\emptyset$ is always a rc-set. Therefore $c(int(X)) - X$ is a rc-set.

Conversely, suppose that $c(int(X)) - X$ is a rc-set. But X is a rgc-set. Also $c(X) - X$ contains the rc-set $c(int(X)) - X$. By Theorem 3.8, we have $c(int(X)) - X = \emptyset$. Hence $c(int(X)) = X$. Therefore X is a rc-set. □

Theorem 3.10. *Let $(\mathcal{U}, c)$ be a closure space and $X \subseteq \mathcal{U}$. If X is a gc-set, then X is a rgc-set.*

Proof. Suppose that $X \subseteq K$, where K is a ro-set. Now K is a ro-set, implies that K is a open. Since X is a gc-set, then $c(X) \subseteq K$. Therefore X is a rgc-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.11. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and define a closure operator c on $\mathcal{U}$ by

$$c(\emptyset) = \emptyset, \quad c(\{1\}) = c(\{1, 2\}) = \{1, 2\}, \quad c(\{2\}) = \{2\},$$
$$c(\{3\}) = c(\{2, 3\}) = \{2, 3\}, \quad c(\{4\}) = \{4\}, \quad c(\{2, 4\}) = \{2, 4\},$$
$$c(\{1, 3\}) = c(\{1, 2, 3\}) = \{1, 2, 3\}, \quad c(\{1, 4\}) = c(\{1, 2, 4\}) = \{1, 2, 4\},$$
$$c(\{3, 4\}) = c(\{2, 3, 4\}) = \{2, 3, 4\}, \quad c(\{1, 3, 4\}) = c(\mathcal{U}) = \mathcal{U}.$$

Then $\{1, 3\}$ is a rgc-set but it is not gc-set.

Definition 3.12. Let $(\mathcal{U}, c)$ be a closure space. A $X \subseteq \mathcal{U}$ is called a regular generalized open set (briefly a rgo-set) if and only if its complement is a rgc-set.

Theorem 3.13. *Let $(\mathcal{U}, c)$ be a closure space. A set $X \subseteq \mathcal{U}$ is a rgo-set if and only if $H \subseteq int(X)$ whenever H is a rc-set and $H \subseteq X$.*

Proof. Let $H \subseteq int(X)$ whenever H is a rc-set, $H \subseteq X$ and $K = \mathcal{U} - X$. Suppose that $K \subseteq G$ where G is a ro-set.

Now $T \subseteq G$ implies $H = \mathcal{U} - G \subseteq X$ and H is a rc-set which implies $H \subseteq int(X)$. Also $H \subseteq int(X)$ implies $\mathcal{U} - int(X) \subseteq \mathcal{U} - H = G$. This inturn implies $\mathcal{U} - int(\mathcal{U} - K) \subseteq G$. Or equivalently $c(K) \subseteq G$. Thus K is a rgc-set. Hence we obtain X is a a rgo-set.

Conversely, suppose that X is a rgo-set, $H \subseteq X$ and H is a rc-set. Moreover $\mathcal{U}-H$ is a ro-set. Then $\mathcal{U}-X \subseteq \mathcal{U}-H$. Since $\mathcal{U}-X$ is a rgc-set, $\mathcal{U}-c(X) \subseteq \mathcal{U}-H$. Therefore $H \subseteq \mathcal{U}-(\mathcal{U}-c(X)) = int(X)$. □

Theorem 3.14. *Let $(\mathcal{U}, c)$ be a closure space. If X is a rgo-subset of $(\mathcal{U}, c)$, then $G = \mathcal{U}$ whenever G is a ro-set and $int(X) \cup (\mathcal{U}-X) \subseteq G$.*

Proof. Suppose that X is a rgo-set in $(\mathcal{U}, c)$. Let G be a ro-set and $int(X) \cup (\mathcal{U}-X) \subseteq G$. This implies $\mathcal{U}-G \subseteq (\mathcal{U}-int(X)) \cap (\mathcal{U}-(\mathcal{U}-X))$. That is $\mathcal{U}-G \subseteq (\mathcal{U}-int(X)) \cap X$ or equivalently $\mathcal{U}-G \subseteq (\mathcal{U}-int(X))-(\mathcal{U}-X) = (\mathcal{U}-(\mathcal{U}-c(\mathcal{U}-X)))-(\mathcal{U}-X) = c(\mathcal{U}-X)-(\mathcal{U}-X) = c(\mathcal{U}-X)-(\mathcal{U}-X)$. Now $\mathcal{U}-G$ is a rc-set and $\mathcal{U}-X$ is rgc-set. By Theorem 3.8, it follows taht $\mathcal{U}-G = \emptyset$. Hence we obtain $G = \mathcal{U}$. □

Theorem 3.15. *Let $(\mathcal{U}, c)$ be a closure space. If X is a rgc-subset of $(\mathcal{U}, c)$, then $c(X)-X$ is a rgo-set.*

Proof. Suppose that X is a rgc-set and $H \subseteq c(X)-X$, where H is a rc-set. By Theorem 3.8, $H = \emptyset$ and so $H \subseteq int(c(X)-X)$. By Theorem 3.13, $c(X)-X$ is a rgo-set. □

The converse of this theorem is not true in general as can be seen from the following example.

Example 3.16. Let $\mathcal{U} = \{m, n, r, s\}$ and define a closure operator c on $\mathcal{U}$ by

$$c(\emptyset) = \emptyset, \quad c(\{m\}) = c(\{m,n\}) = \{m,n\}, \quad c(\{n\}) = \{n\},$$
$$c(\{r\}) = c(\{n,r\}) = \{n,r\}, \quad c(\{s\}) = \{s\}, \quad c(\{n,s\}) = \{n,s\},$$
$$c(\{m,r\}) = c(\{m,n,r\}) = \{m,n,r\}, \quad c(\{m,s\}) = c(\{m,n,s\}) = \{m,n,s\},$$
$$c(\{r,s\}) = c(\{n,r,s\}) = \{n,r,s\}, \quad c(\{m,r,s\}) = c(\mathcal{U}) = \mathcal{U}.$$

Then $c(\{r\})-\{r\} = \{n\}$ is a rgo-set. But $\{r\}$ is not a rgc-set in $(\mathcal{U}, c)$.

4. Conclusion

In the present paper, we have introduced regular closed (open) sets and regular generalized closed (open) sets in Čech closure spaces. In addition, some basic properties of new concepts for Čech closure spaces were examined. We have investigated behavior relative to union, intersection, subspaces of regular closed (open) sets and regular generalized closed (open) sets. We hope that the findings in this paper will help researcher enhance and promote the further study on Čech closure spaces to carry out a general framework.

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