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Conformal Generic Riemannian Maps from Almost Hermitian Manifolds

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Abstract. In the present paper, the notion of conformal generic Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds is defined. Examples for this type conformal maps are given. The concept of pluriharmonic map is used to get conditions defining totally geodesic foliations for certain distributions and being horizontally homothetic map on the base manifold.

1. Introduction

The notion of submersion was introduced by O'Neill [10] and Gray [6]. Then, this notion was widely studied [4] and new kind of Riemannian submersions like invariant submersion, anti-invariant submersion, slant submersion, generic submersion were introduced [1, 2, 11–13]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions [4–6, 10]. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{\dim M_1, \dim M_2\}$. Then the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = kerF_* \oplus (kerF_*)^{\perp}.$$

We always have $(rangeF_*)^{\perp}$ because of $rankF < min\{\dim M_1, \dim M_2\}$. Therefore tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^{\perp}.$$

A smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^{\perp} \longrightarrow (rangeF_*)$ is a linear isometry. Hence a Riemannian map satisfies the equation

$$g_1(X, Y) = g_2(F_*(X), F_*(Y)) \tag{1}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$ [5].

We say that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq min\{m, n\}$ and F_{*p} maps the horizontal space $(ker(F_{*p})^{\perp})$ conformally onto $range(F_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}(X), F_{*p}(Y)) = \lambda^2(p) g_M(X, Y) \tag{2}$$

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for $X, Y \in \Gamma((ker(F_{*p})^{\perp})$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$ [14, 15]. Here, λ is the dilation of F at a point $p \in M$ and it is a continuous function as $\lambda : M \rightarrow [0, \infty)$.

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on M such that $J^2 = -I$ where I denotes the identity transformation of TM and

$$g_M(X, Y) = g_M(JX, JY), \forall X, Y \in \Gamma(TM). \tag{3}$$

Let (M, g_M, J) be an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y = 0, \tag{4}$$

we say M is a Kaehlerian manifold [3, 21].

Riemannian maps would provide relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side and Maxwell's equation, Schrodinger's equation on the physical side [5]. Some application areas of conformal Riemannian maps are computer vision [7], geometric modelling [18] and medical imaging [19].

In this paper, conformal generic Riemannian maps from almost Hermitian manifolds to Riemannian manifolds were introduced, geometric properties of the base manifold and the total manifold by the existence of such maps were investigated and examples were given. Also, certain geodesicity conditions for conformal generic Riemannian maps were obtained. Moreover, several conditions for conformal generic Riemannian maps to be horizontally homothetic maps by using the adapted version of the notion of pluriharmonic maps were obtained.

2. Preliminaries

In this section, some definitions and useful results for conformal generic Riemannian maps are given. Let $(M, g_{_M})$ and $(N, g_{_N})$ be Riemannian manifolds and $F : M \longrightarrow N$ is a smooth map between them. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \tag{5}$$

for $X, Y \in \Gamma(TM)$. The second fundamental form ∇F_* is symmetric [8].

Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) to a Riemannian manifold (N^n, g_N). Then we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions as

$$\begin{aligned} \mathcal{A}_X Y &= h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, & (6)\\ \mathcal{T}_X Y &= h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY & (7) \end{aligned}$$

for vector fields $X, Y \in \Gamma(TM)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M [10]. For any $X \in \Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. The tensor field $\mathcal{T}$ is symmetric on the vertical distribution [10, 20]. On the other hand, from (6) and (7) we have

$$\begin{aligned} \overset{M}{\nabla}_U V &= \mathcal{T}_U V + \hat{\nabla}_U V, & (8)\\ \overset{M}{\nabla}_U X &= h\overset{M}{\nabla}_U X + \mathcal{T}_U X, & (9)\\ \overset{M}{\nabla}_X V &= \mathcal{A}_X V + v\overset{M}{\nabla}_X V, & (10)\\ \overset{M}{\nabla}_X Y &= h\overset{M}{\nabla}_X Y + \mathcal{A}_X Y & (11) \end{aligned}$$

for $X, Y \in \Gamma((ker\,F_*)^{\perp})$ and $U, V \in \Gamma(kerF_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [11, 12].

A vector field on M is called a projectable vector field if it is related to a vector field on N. Thus, we say a vector field is basic on M if it is both a horizontal and a projectable vector field. Hereafter, when we mention a horizontal vector field, we always consider a basic vector field [3].

On the other hand, let F be a conformal Riemannian map between Riemannian manifolds (M^m, g_M) and (N^n, g_N). Then, we have

$$\begin{aligned}(\nabla F_*)(X,Y)\mid_{rangeF_*} &= X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X)\\ &- g_M(X,Y)F_*(grad(\ln\lambda))\end{aligned} \tag{12}$$

where $X, Y \in \Gamma((kerF_*)^\perp)$. Hence from (12), we obtain $\overset{N}{\nabla}{}^F_X F_*(Y)$ as

$$\begin{aligned}\overset{N}{\nabla}{}^F_X F_*(Y) &= F_*(h\overset{M}{\nabla}_X Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X)\\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(X,Y)\end{aligned} \tag{13}$$

where $(\nabla F_*)^\perp(X,Y)$ is the component of $(\nabla F_*)(X,Y)$ on $(rangeF_*)^\perp$ for $X, Y \in \Gamma((kerF_*)^\perp)$ [16, 17].

Now, a map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X,Y) + (\nabla F_*)(JX,JY) = 0 \tag{14}$$

for $X, Y \in \Gamma(TM)$ [9].

3. Conformal Generic Riemannian Maps

Now, we define the notion of conformal generic Riemannian map and give its tangent space's decomposition.

Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the complex subspace of the vertical subspace $\mathcal{V}_p$ at $p \in M$ is

$$\mathcal{D}_p = (kerF_{*p} \cap J(kerF_{*p})).$$

Definition 3.1. *Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If the dimension of $\mathcal{D}_p$ is constant along M and it defines a differentiable distribution on M then we say that F is a conformal generic Riemannian map.*

Let F be a conformal generic Riemannian map. Then, we say F is purely real (respectively, complex) if $\mathcal{D}_p = \{0\}$ (respectively, $\mathcal{D}_p = kerF_{*p}$). Orthogonal complementary distribution $\mathcal{D}^\perp$ of a conformal generic Riemannian map F is called purely real distribution and it satisfies

$$kerF_* = \mathcal{D} \oplus \mathcal{D}^\perp \tag{15}$$

and

$$\mathcal{D} \cap \mathcal{D}^\perp = \{0\}. \tag{16}$$

Let F be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). For $U \in \Gamma(kerF_*)$, we write

$$JU = \phi U + \omega U \tag{17}$$

where $\phi U \in \Gamma(kerF_*)$ and $\omega U \in \Gamma((kerF_*)^\perp)$. We contemplate the complementary orthogonal distribution μ to $\omega\mathcal{D}^\perp$ in $(kerF_*)^\perp$. Therefore we have

$$\phi\mathcal{D}^\perp \subseteq \mathcal{D}^\perp, (kerF_*)^\perp = \omega\mathcal{D}^\perp \oplus \mu. \tag{18}$$

In addition, for $X \in \Gamma((kerF_*)^{\perp})$, we write

$$JX = BX + CX \tag{19}$$

where $BX \in \Gamma(\mathcal{D}^{\perp})$ and $CX \in \Gamma(\mu)$. Clearly, we get

$$B((kerF_*)^{\perp}) = \mathcal{D}^{\perp}. \tag{20}$$

From (15) for $U \in \Gamma(kerF_*)$, we can write

$$JU = \Phi_1 U + \Phi_2 U + \omega U \tag{21}$$

where Φ_1 and Φ_2 are the projections from $kerF_*$ to $\mathcal{D}$ and $\mathcal{D}^{\perp}$, respectively.

We say that a conformal generic Riemannian map is proper if $\mathcal{D}^{\perp}$ is neither complex nor purely real. Now, we give examples to conformal generic Riemannian maps.

Example 3.2. *Every conformal semi-invariant Riemannian map [17] F from an almost Hermitian manifold to a Riemannian manifold is a conformal generic Riemannian map with $\mathcal{D}^{\perp}$ is a totally real distribution.*

Example 3.3. *Let $F : (\mathbb{R}^8, g_{\mathbb{R}^8}, J) \longrightarrow (\mathbb{R}^5, g_{\mathbb{R}^8})$ be a map defined by*

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \longrightarrow (\frac{x_1 - x_2 + x_6}{\sqrt{3}}, \frac{x_1 + x_2}{\sqrt{2}}, 0, x_4, x_3)$$

for any point $x \in \mathbb{R}^8$. We obtain the horizontal distribution and the vertical distributions

$$\mathcal{H} = (kerF_*)^{\perp} = \{H_1 = \frac{1}{\sqrt{3}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}), H_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), H_3 = \frac{\partial}{\partial x_4}, H_4 = \frac{\partial}{\partial x_3}\}$$

and

$$\mathcal{V} = (kerF_*) = \{V_1 = \frac{\partial}{\partial x_5}, V_2 = \frac{\partial}{\partial x_7}, V_3 = \frac{\partial}{\partial x_8}, V_4 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{2}{\sqrt{3}}\frac{\partial}{\partial x_6}\},$$

respectively. Thus, using (2) we have

$$g_{\mathbb{R}^5}(F_*(H_i), F_*(H_i)) = \lambda^2 g_{\mathbb{R}^8}(H_i, H_i), i = 1, 2, 3, 4$$

and

$$g_{\mathbb{R}^5}(F_*(H_i), F_*(H_j)) = \lambda^2 g_{\mathbb{R}^8}(H_i, H_j) = 0, i \neq j.$$

It follows that F is a conformal Riemannian map at any point $x \in \mathbb{R}^8$ with $0 < rankF_* = 4 \leq min\{\dim(\mathbb{R}^8), \dim(\mathbb{R}^5)\}$ and $\lambda = 1$. On the other hand, by using the standard complex structure $J = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$ on $\mathbb{R}^8$, one can see that

$$\begin{aligned} JV_1 &= \frac{3}{2+\sqrt{3}}H_1 - \frac{3}{3+2\sqrt{3}}V_4, \\ JV_4 &= aH_1 + \sqrt{2}H_2 + \frac{2}{\sqrt{3}}V_1 - \frac{a}{\sqrt{3}}V_4, a \in \mathbb{R}, \\ JV_2 &= V_3, \quad JH_3 = -H_4. \end{aligned}$$

Hence, F is a conformal generic Riemannian map with $\mathcal{D} = span\{V_2, V_3\}$, $\mathcal{D}^{\perp} = span\{V_1, V_4\}$ and $\mu = span\{H_3, H_4\}$.

Now, we examine some geometric properties on the total manifold and the base manifold of a proper conformal generic Riemannian map.

Lemma 3.4. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}$ is integrable if and only if the following condition is satisfied*

$$(\nabla F_*)(U, JV) = (\nabla F_*)(JU, V) \tag{22}$$

for $U, V \in \Gamma(\mathcal{D})$.

Proof. Since M is a Kaehlerian manifold, from (4), (8), (19) and (21) we have

$$\mathcal{T}_U JV + v\overset{M}{\nabla}_U JV = B\mathcal{T}_U V + C\mathcal{T}_U V + \Phi_1 v\overset{M}{\nabla}_U V + \Phi_2 v\overset{M}{\nabla}_U V + \omega v\overset{M}{\nabla}_U V \tag{23}$$

and changing the role of U and V in (23) we have

$$\mathcal{T}_V JU + v\overset{M}{\nabla}_V JU = B\mathcal{T}_V U + C\mathcal{T}_V U + \Phi_1 v\overset{M}{\nabla}_V U + \Phi_2 v\overset{M}{\nabla}_V U + \omega v\overset{M}{\nabla}_V U. \tag{24}$$

Since $\mathcal{T}$ is symmetric on $kerF_*$, taking horizontal parts of (23) and (24) we get

$$\mathcal{T}_U JV - \mathcal{T}_V JU = \omega\{v\overset{M}{\nabla}_U V - v\overset{M}{\nabla}_V U\}. \tag{25}$$

From equation (5) we obtain

$$-(\nabla F_*)(U, JV) + (\nabla F_*)(JU, V) = F_*(\omega v[U, V]). \tag{26}$$

The proof is clear from (26). □

Lemma 3.5. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}^\perp$ is integrable if and only if the following condition is satisfied*

$$v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \mathcal{T}_{V_2}\omega V_1 - \mathcal{T}_{V_1}\omega V_2 \in \Gamma(\mathcal{D}^\perp) \tag{27}$$

for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$.

Proof. The real distribution $\mathcal{D}^\perp$ is integrable if and only if $g_M([V_1, V_2], U) = 0$ and $g_M([V_1, V_2], X) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D})$ and $X \in \Gamma(kerF_*)^\perp$.Since $kerF_*$ is always integrable we have $g_M([V_1, V_2], X) = 0$. Hence, we only examine $g_M([V_1, V_2], U) = 0$. For $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$ we have

$$\begin{aligned} \overset{M}{\nabla}_{V_1} V_2 &= -B\mathcal{T}_{V_1}\Phi_2 V_2 - C\mathcal{T}_{V_1}\Phi_2 V_2 + \Phi_1 v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 + \Phi_2 v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 \\ &+ \omega v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - \Phi_1\mathcal{T}_{V_1}\omega V_2 - \Phi_2\mathcal{T}_{V_1}\omega V_2 - \omega\mathcal{T}_{V_1}\omega V_2 \\ &- Bh\overset{M}{\nabla}_{V_1}\omega V_2 - Ch\overset{M}{\nabla}_{V_1}\omega V_2. \end{aligned} \tag{28}$$

Interchanging the role of V_1 and V_2 in (28) we have

$$\begin{aligned} \overset{M}{\nabla}_{V_2} V_1 &= -B\mathcal{T}_{V_2}\Phi_2 V_1 - C\mathcal{T}_{V_2}\Phi_2 V_1 + \Phi_1 v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \Phi_2 v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 \\ &+ \omega v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 - \Phi_1\mathcal{T}_{V_2}\omega V_1 - \Phi_2\mathcal{T}_{V_2}\omega V_1 - \omega\mathcal{T}_{V_2}\omega V_1 \\ &- Bh\overset{M}{\nabla}_{V_2}\omega V_1 - Ch\overset{M}{\nabla}_{V_2}\omega V_1. \end{aligned} \tag{29}$$

Now, using (28) and (29) we get

$$g_M([V_1, V_2], U) = g_M(\Phi_1\{v\overset{M}{\nabla}_{V_1}\Phi_2 V_2 - v\overset{M}{\nabla}_{V_2}\Phi_2 V_1 + \mathcal{T}_{V_2}\omega V_1 - \mathcal{T}_{V_1}\omega V_2\}, U). \tag{30}$$

The proof is complete from (30). □

Lemma 3.6. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the horizontal distribution $(kerF_*)^\perp$ is integrable if and only if the following condition is satisfied*

$$\begin{aligned} &\frac{1}{\lambda^2} g_N((\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) + F_*(h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX), F_*(\omega U)) \\ &= g_M(v\overset{M}{\nabla}_Y BX - v\overset{M}{\nabla}_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \phi U) \end{aligned} \tag{31}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. The horizontal distribution $(kerF_*)^\perp$ is integrable if and only if $g_M([X,Y],U) = 0$ for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$. From (4) we have

$$J\nabla_X Y = \mathcal{A}_X BY + v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY + h\overset{M}{\nabla}_X CY. \tag{32}$$

After changing the roles of X and Y, we get

$$\begin{aligned} J[X,Y] &= \mathcal{A}_X BY - \mathcal{A}_Y BX + v\overset{M}{\nabla}_X BY - v\overset{M}{\nabla}_Y BX \\ &+ \mathcal{A}_X CY - \mathcal{A}_Y CX + h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX. \end{aligned} \tag{33}$$

Now, from (17) we get for $U \in \Gamma(kerF_*)$

$$\begin{aligned} 0 = -g_M([X,Y],U) &= -g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX + h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX, \omega U) \\ &- g_M(v\overset{M}{\nabla}_X BY - v\overset{M}{\nabla}_Y BX + \mathcal{A}_X CY - \mathcal{A}_Y CX, \phi U). \end{aligned} \tag{34}$$

Hence, from (2) and (5) we obtain

$$\begin{aligned} &\frac{1}{\lambda^2} g_N((\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) + F_*(h\overset{M}{\nabla}_X CY - h\overset{M}{\nabla}_Y CX), F_*(\omega U)) \\ &= g_M(v\overset{M}{\nabla}_Y BX - v\overset{M}{\nabla}_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \phi U). \end{aligned} \tag{35}$$

The proof is complete from (35). □

Now, we remark some useful notions.

Definition 3.7. *Let $F : M \longrightarrow N$ be a conformal Riemannian map. Then, if*

$$\mathcal{H}(grad(\ln\lambda)) = 0, \tag{36}$$

we say F is a horizontally homothetic map [3].

Definition 3.8. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $kerF_*$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(U_1,U_2) + (\nabla F_*)(JU_1,JU_2) = 0 \tag{37}$$

for $U_1, U_2 \in \Gamma(kerF_*)$ [16, 17].

Theorem 3.9. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i- $C\{\mathcal{T}_{U_1}\phi_{U_2} + h\overset{M}{\nabla}_{U_1}\omega U_2\} = \mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1,$

ii- F is a $kerF_*$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\omega U_1, \omega U_2) = 0$

for any $U_1, U_2 \in \Gamma(kerF_*)$.

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. From (5), (13), (14) and (37), we get

$$\begin{array}{rcl} 0 & = & F_*(\mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1) + F_*(C\mathcal{T}_{U_1}\phi_{U_2} + Ch\overset{M}{\nabla}_{U_1}\omega U_2) \\ & + & (\nabla F_*)^{\perp}(\omega U_1, \omega U_2) + \omega U_1(\ln\lambda)F_*(\omega U_2) \\ & + & \omega U_2(\ln\lambda)F_*(\omega U_1) - g_M(\omega U_1, \omega U_2)F_*(grad(\ln\lambda)) \end{array} \tag{38}$$

for any $U_1, U_2 \in \Gamma(kerF_*)$. Suppose that (i) and (ii) are satisfied in (38). Then, we have $C\{\mathcal{T}_{U_1}\phi_{U_2} + h\overset{M}{\nabla}_{U_1}\omega U_2\} = \mathcal{T}_{\phi U_1}\phi U_2 + \mathcal{A}_{\omega U_1}\phi U_2 + \mathcal{A}_{\omega U_2}\phi U_1$ and F is a $kerF_*$-pluriharmonic map for any $U_1, U_2 \in \Gamma(kerF_*)$, respectively. Thus, we have

$$\begin{array}{rcl} 0 & = & (\nabla F_*)^{\perp}(\omega U_1, \omega U_2) + \omega U_1(\ln\lambda)F_*(\omega U_2) \\ & + & \omega U_2(\ln\lambda)F_*(\omega U_1) - g_M(\omega U_1, \omega U_2)F_*(grad(\ln\lambda)). \end{array} \tag{39}$$

It is clear from (39) that $(\nabla F_*)^{\perp}(\omega U_1, \omega U_2) = 0$. Now, we obtain from (2), (18) and (39)

$$0 = \lambda^2 \omega U_2(\ln\lambda) g_M(\omega U_1, \omega U_1) \tag{40}$$

for $\omega U_1 \in \Gamma(\omega(\mathcal{D}^{\perp}))$. So, we get $\omega U_2(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^{\perp})$. Similarly, we obtain from (39)

$$0 = -\lambda^2 CX(\ln\lambda) g_M(\omega U_1, \omega U_2) \tag{41}$$

with $\omega U_1 = \omega U_2$ for $CX \in \Gamma(\mu)$. So, we get $CX(\ln\lambda) = 0$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (40) and (41). The proof is complete. □

Definition 3.10. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $(kerF_*)^{\perp}$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(Z_1, Z_2) + (\nabla F_*)(JZ_1, JZ_2) = 0 \tag{42}$$

for $Z_1, Z_2 \in \Gamma((kerF_*)^{\perp})$ [16, 17].

Theorem 3.11. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any three conditions below imply the fourth condition:*

i- $\overset{N}{\nabla}{}^F_{Z_1}F_*(Z_2) = F_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1 + \mathcal{A}_{CZ_1}BZ_2)$,

ii- F is a $(kerF_*)^{\perp}$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^{\perp}(CZ_1, CZ_2) = 0$,

iv- The distribution $(kerF_*)^{\perp}$ defines a totally geodesic foliation in M

for any $Z_1, Z_2 \in \Gamma((kerF_*)^{\perp})$.

Proof. We only show the proof of (iii) and (iv). The proof of (i) and (ii) are clear. From (5), (13), (14) and (42), we get

$$\begin{array}{rcl} F_*(\overset{M}{\nabla}_{Z_1}Z_2) & = & \overset{N}{\nabla}{}^F_{Z_1}F_*(Z_2) + (\nabla F_*)^{\perp}(CZ_1, CZ_2) \\ & - & F_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1 + \mathcal{A}_{CZ_1}BZ_2) \\ & + & CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) \\ & - & g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) \end{array} \tag{43}$$

for any $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. Suppose that (i), (ii) and (iii) are satisfied in (43). Then, we have

$$\begin{aligned}
&\overset{N}{\nabla}{}^F_{Z_1} F_*(Z_2) = F_*(\mathcal{T}_{BZ_1} BZ_2 + \mathcal{A}_{CZ_2} BZ_1 + \mathcal{A}_{CZ_1} BZ_2),\\
&(\nabla F_*)(Z_1, Z_2) + (\nabla F_*)(JZ_1, JZ_2) = 0,\\
&CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) - g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) = 0,\\
&(\nabla F_*)^\perp(CZ_1, CZ_2) = 0,
\end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{Z_1} Z_2) = 0$ for $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. Therefore, the distribution $(kerF_*)^\perp$ defines a totally geodesic foliation in M. Suppose that (i), (ii) and (iv) are satisfied in (43). Then, it is clear from (43) that $(\nabla F_*)^\perp(CZ_1, CZ_2) = 0$ and we obtain

$$0 = CZ_1(\ln\lambda)F_*(CZ_2) + CZ_2(\ln\lambda)F_*(CZ_1) - g_M(CZ_1, CZ_2)F_*(grad(\ln\lambda)) \tag{44}$$

for any $Z_1, Z_2 \in \Gamma((kerF_*)^\perp)$. From (2) and (44), we get

$$0 = \lambda^2 CZ_2(\ln\lambda) g_M(CZ_1, CZ_1) \tag{45}$$

for $CZ_1 \in \Gamma(\mu)$. So, we get $CZ_2(\ln\lambda) = 0$. It means λ is a constant on μ. Similarly, we obtain from (18) and (44)

$$0 = -\lambda^2 \omega U_1(\ln\lambda) g_M(CZ_1, CZ_2) \tag{46}$$

with $CZ_1 = CZ_2$ for $\omega U_1 \in \Gamma(\omega(\mathcal{D}^\perp))$. So, we get $\omega U_1(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Thus, F is a horizontally homothetic map from (45) and (46). The proof is complete. □

Definition 3.12. *Let F be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $\mathcal{D}^\perp$-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0 \tag{47}$$

for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$ [16, 17].

Theorem 3.13. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any three conditions below imply the fourth condition:*

i- $\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2 = 0$,

ii- F is a $\mathcal{D}^\perp$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0$,

iv- The distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation in M

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$.

Proof. We only show the proof of (iii) and (iv). The proof of (i) and (ii) are clear. From (5), (13), (14) and (47), we get

$$\begin{aligned}
F_*(\overset{M}{\nabla}_{V_1} V_2) &= -F_*(\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2)\\
&+ \omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1)\\
&- g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(\omega V_1, \omega V_2)
\end{aligned} \tag{48}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. Suppose that (i), (ii) and (iii) are satisfied in (48). Then, we have

$$\begin{aligned}
&\mathcal{T}_{\phi V_1}\phi V_2 + \mathcal{A}_{\omega V_2}\phi V_1 + \mathcal{A}_{\omega V_1}\phi V_2 = 0,\\
&(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0,\\
&\omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1) - g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) = 0,\\
&(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0,
\end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{V_1}V_2) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. Therefore, the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation in M. Suppose that (i), (ii) and (iv) are satisfied in (48). Then, it is clear from (48) that $(\nabla F_*)^\perp(\omega V_1, \omega V_2) = 0$ and we obtain

$$0 = \omega V_1(\ln\lambda)F_*(\omega V_2) + \omega V_2(\ln\lambda)F_*(\omega V_1) - g_M(\omega V_1, \omega V_2)F_*(grad(\ln\lambda)) \tag{49}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}^\perp)$. From (2) and (49), we get

$$0 = \lambda^2\omega V_2(\ln\lambda)g_M(\omega V_1, \omega V_1) \tag{50}$$

for $\omega V_1 \in \Gamma(\omega(\mathcal{D}^\perp))$. So, we get $\omega V_2(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Similarly, we obtain from (18) and (49)

$$0 = -\lambda^2 CX(\ln\lambda)g_M(\omega V_1, \omega V_2) \tag{51}$$

with $\omega V_1 = \omega V_2$ for $CX \in \Gamma(\mu)$. So, we get $CX(\ln\lambda) = 0$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (50) and (51). The proof is complete. □

Definition 3.14. *Let F be a map from a complex manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then F is called a* $\mathcal{D}$*-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0 \tag{52}$$

for $V_1, V_2 \in \Gamma(\mathcal{D})$ *[16, 17].*

Theorem 3.15. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then any two conditions below imply the third condition:*

- *i-* $C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2 = 0,$
- *ii- F is a* $\mathcal{D}$*-pluriharmonic map,*
- *iii- The distribution* $\mathcal{D}$ *defines a totally geodesic foliation in M*

for any $V_1, V_2 \in \Gamma(\mathcal{D})$*.*

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. From (5), (14), (17), (18), and (52), we get

$$F_*(\overset{M}{\nabla}_{V_1}V_2) = F_*(C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2) \tag{53}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D})$. Suppose that (i) and (ii) are satisfied in (53). Then, we have

$$\begin{aligned} &C\mathcal{T}_{\phi V_1}\phi^2 V_2 + \omega v\overset{M}{\nabla}_{\phi V_1}\phi^2 V_2 = 0, \\ &(\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) = 0, \end{aligned}$$

respectively. Thus, we have $F_*(\overset{M}{\nabla}_{V_1}V_2) = 0$ for $V_1, V_2 \in \Gamma(\mathcal{D})$. Therefore, the distribution $\mathcal{D}$ defines a totally geodesic foliation in M. □

Definition 3.16. *Let F be a map from a complex manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then F is called a* $\{(kerF_*)^\perp - kerF_*\}$*-pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(X, V) + (\nabla F_*)(JX, JV) = 0 \tag{54}$$

for $X \in \Gamma((kerF_*)^\perp)$ *and* $V \in \Gamma(kerF_*)$ *[17].*

Theorem 3.17. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition:*

i- $C\{\mathcal{A}_X\phi V + h\overset{M}{\nabla}_X\omega V\} + \omega\{\mathcal{A}_X\omega V + v\overset{M}{\nabla}_X\phi V\} = -\{\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V\}$,

ii- F is a $\{(kerF_*)^\perp - kerF_*\}$-pluriharmonic map,

iii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(CX, \omega V) = 0$

for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$.

Proof. We only show the proof of (iii). The proof of (i) and (ii) are clear. Since second fundamental form of a map (∇F_*) is symmetric from (5), (12), (13), (14), (18) and (54), we get

$$\begin{aligned} 0 &= F_*(C\mathcal{A}_X\phi V + \omega v\overset{M}{\nabla}_X\phi V + \omega\mathcal{A}_X\omega V + Ch\overset{M}{\nabla}_X\omega V) \\ &- F_*(\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V) + (\nabla F_*)^\perp(CX, \omega V) \\ &+ CX(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(CX) \end{aligned} \tag{55}$$

for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$. Suppose that (i) and (ii) are satisfied in (55). Then, we have

$$\begin{aligned} &C\{\mathcal{A}_X\phi V + h\overset{M}{\nabla}_X\omega V\} + \omega\{\mathcal{A}_X\omega V + v\overset{M}{\nabla}_X\phi V\} = -\{\mathcal{T}_{BX}\phi V + \mathcal{A}_{\omega V}BX + \mathcal{A}_{CX}\phi V\}, \\ &(\nabla F_*)(X, V) + (\nabla F_*)(JX, JV) = 0, \end{aligned}$$

respectively. Then, it is clear from (55) that $(\nabla F_*)^\perp(CX, \omega V) = 0$. Thus, we have

$$0 = CX(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(CX) \tag{56}$$

for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$. From (2) and (56), we get

$$0 = \lambda^2\omega V(\ln\lambda)g_M(CX, CX) \tag{57}$$

for $CX \in \Gamma(\mu)$. So, we get $\omega V(\ln\lambda) = 0$. It means λ is a constant on $\omega(\mathcal{D}^\perp)$. Similarly, we obtain from (18) and (56)

$$0 = \lambda^2 CX(\ln\lambda)g_M(\omega V, \omega V) \tag{58}$$

for $\omega V \in \Gamma(\omega(\mathcal{D}^\perp))$. It means λ is a constant on μ. Thus, F is a horizontally homothetic map from (57) and (58). The proof is complete. □

Now, we investigate totally geodesicness of distributions in M.

Theorem 3.18. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, $kerF_*$ defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_N((\nabla F_*)(U, \phi V), F_*(\omega Z))$
$= \lambda^2\{g_M(\hat{\nabla}_U V, \phi^2 Z) - g_M(h\overset{M}{\nabla}_U\omega V, \omega Z)\}$,

ii- $g_N((\nabla F_*)(U, V), F_*(\omega BX)) + g_N((\nabla F_*)(U, \phi V), F_*(CX))$
$= \lambda^2\{g_M(\hat{\nabla}_U V, \phi BX) + g_M(h\overset{M}{\nabla}_U\omega V, CX)\}$

are satisfied for any $U, V \in \Gamma(kerF_*)$, $X \in \Gamma(\mu)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Firstly, we show (i). Since M is a Kaehlerian manifold from (17), we have

$$g_M(\overset{M}{\nabla}_U V, Z) = g_M(\overset{M}{\nabla}_U \phi V + \omega V, \phi Z + \omega Z)$$

for any $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Then, from (2), (8) and (9) we have

$$= g_M(\overset{M}{\nabla}_U JV, \phi Z) + g_M(\mathcal{T}_U \phi V, \omega Z) + g_M(h\overset{M}{\nabla}_U \omega Z, \omega Z).$$

Since $(\nabla F_*)(U, \phi V) = -F_*(\mathcal{T}_U \phi V)$, we obtain

$$= g_M(\overset{M}{\nabla}_U JV, \phi Z) + g_M(h\overset{M}{\nabla}_U \omega V, \omega Z) - \frac{1}{\lambda^2} g_N((\nabla F_*)(U, \phi V), F_*(\omega Z)) \tag{59}$$

for any $U, V \in \Gamma(kerF_*)$. On the other hand, we have from (8)

$$\begin{aligned} g_M(\overset{M}{\nabla}_U JV, \phi Z) &= -g_M(\overset{M}{\nabla}_U V, J\phi Z) \\ &= -g_M(\mathcal{T}_U V, \omega\phi Z) - g_M(\hat{\nabla}_U V, \phi^2 Z) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_M(\hat{\nabla}_U V, \phi^2 Z). \end{aligned} \tag{60}$$

Now, using (60) in (59) we get

$$\begin{aligned} 0 &= \frac{1}{\lambda^2}\{g_N((\nabla F_*)(U, V), F_*(\omega\phi Z)) - g_N((\nabla F_*)(U, \phi V), F_*(\omega Z))\} \\ &+ g_M(h\overset{M}{\nabla}_U \omega V, \omega Z) - g_M(\hat{\nabla}_U V, \phi^2 Z). \end{aligned} \tag{61}$$

Therefore, we obtain (i). Now, we show (ii). Thus, from (8), (9), (17) and (19) we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, X) &= g_M(\overset{M}{\nabla}_U V, JBX) + g_M(\overset{M}{\nabla}_U \phi V + \omega V, CX) \\ &= g_M(\mathcal{T}_U V, \omega BX) + g_M(\hat{\nabla}_U V, \phi BX) \\ &+ g_M(\mathcal{T}_U \phi V, CX) + g_M(h\overset{M}{\nabla}_U \omega V, CX) \\ &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(U, V), F_*(\omega BX)) + g_M(\hat{\nabla}_U V, \phi BX) \\ &- \frac{1}{\lambda^2} g_N((\nabla F_*)(U, \phi V), F_*(CX)) + g_M(h\overset{M}{\nabla}_U \omega V, CX) \end{aligned} \tag{62}$$

for any $U, V \in \Gamma(kerF_*)$ and $X \in \Gamma(\mu)$. Hence, we obtain (ii) from (62). The proof is complete. □

Theorem 3.19. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, $(kerF_*)^{\perp}$ defines a totally geodesic foliation in M if and only if*

$$g_N((\nabla F_*)(X, BY), F_*(\omega U)) = \lambda^2\{g_M(h\overset{M}{\nabla}_X CY, \omega U) + g_M(v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY, \phi U)\}$$

is satisfied for any $X, Y \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma(kerF_*)$.

Proof. From (17) and (19), we have

$$g_M(\overset{M}{\nabla}_X Y, U) = g_M(\overset{M}{\nabla}_X BY + CY, \phi U + \omega U)$$

for any $X, Y \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma(kerF_*)$. Since $(\nabla F_*)(X, BY) = -F_*(\mathcal{A}_X BY)$ we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(X, BY), F_*(\omega U)) + g_M(h\overset{M}{\nabla}_X CY, \omega U) \\ &+ g_M(v\overset{M}{\nabla}_X BY + \mathcal{A}_X CY, \phi U). \end{aligned} \tag{63}$$

We obtain the proof from (63). □

Theorem 3.20. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, the distribution* $\mathcal{D}$ *defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(U_1, \phi U_2), F_*(\omega V)) = \lambda^2 g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V)$,

ii- $g_N((\nabla F_*)(U_1, \phi U_2), F_*(CX)) = \lambda^2 g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX)$

are satisfied for any $U_1, U_2 \in \Gamma(\mathcal{D})$*,* $X \in \Gamma((kerF_*)^{\perp})$ *and* $V \in \Gamma(\mathcal{D}^{\perp})$*.*

Proof. From (16) and (17) we know $\omega U_2 = 0$. Then, we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, V) &= g_M(\overset{M}{\nabla}_{U_1}\phi U_2, \phi V + \omega V) \\ &= g_M(\mathcal{T}_{U_1}\phi U_2, \omega V) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V) \end{aligned}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{D}^{\perp})$. Since $(\nabla F_*)(U_1, \phi U_2) = -F_*(\mathcal{T}_{U_1}\phi U_2)$, we have

$$g_M(\overset{M}{\nabla}_{U_1}U_2, V) = -\frac{1}{\lambda^2}g_N((\nabla F_*)(U_1, \phi U_2), F_*(\omega V)) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, \phi V). \tag{64}$$

From (64) we have (i). Similarly, we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, X) &= g_M(\overset{M}{\nabla}_{U_1}\phi U_2, BX + CX) \\ &= g_M(\mathcal{T}_{U_1}\phi U_2, CX) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX) \\ &= -\frac{1}{\lambda^2}g_N((\nabla F_*)(U_1, \phi U_2), F_*(CX)) + g_M(v\overset{M}{\nabla}_{U_1}\phi U_2, BX) \end{aligned} \tag{65}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D})$ and $X \in \Gamma((kerF_*)^{\perp})$. From (65) we have (ii). The proof is complete. □

In a similar way, we get the following theorem.

Theorem 3.21. *Let F be a proper conformal generic Riemannian map from a Kaehlerian manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, the distribution* $\mathcal{D}^{\perp}$ *defines a totally geodesic foliation in M if and only if*

i- $g_N((\nabla F_*)(V_1, \phi U), F_*(\omega V_2)) = \lambda^2 g_M(v\overset{M}{\nabla}_{V_1}\phi U, \phi V_2)$,

ii- $g_N((\nabla F_*)(V_1, BX), F_*(\omega V_2)) = \lambda^2 \{g_M(h\overset{M}{\nabla}_{V_1}CX, \omega V_2) + g_M(v\overset{M}{\nabla}_{V_1}BX + \mathcal{T}_{V_1}CX, \phi V_2)\}$

are satisfied for any $V_1, V_2 \in \Gamma(\mathcal{D}^{\perp})$*,* $X \in \Gamma((kerF_*)^{\perp})$ *and* $U \in \Gamma(\mathcal{D})$*.*

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