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Değişken Gecikmeli Sinir Ağları için Geliştirilmiş Niteliksel Kriterler

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Anahtar Kelimeler
Gecikmeli Sinir Ağları,
Lyapunov–Krasovskiĭ Fonksiyoneli,
Zamanla Değişen Gecikme,
Asimptotik Karalılık,
Integrallenebilirlik,
Sınırlılık

Öz: "Phys. Lett. A 374 (2010), no. 7, 938–943" makalesinde Tian ve Xie [1, Teorem 1] değişken gecikmeli sinir ağlarının bir sistemini ele aldı. [1]'de, ele alınan gecikmeli sinir ağları sisteminin orijinin global asimptotik kararlı olması için, araştırmacılar tarafından yeni şartlar elde edildi. [1]' de bazı yeni ve az kısıtlayıcı gecikme bağımlı global asimptotik karalılık şartları elde etmek için, ispat tekniği yeni bir Lyapunov fonksiyonelinin tanımına bağlıdır. İzlenimlerimize göre, Tian ve Xie [1, Teorem 1]'nin global asimptotik karalılık kriterleri ilginç olmasına rağmen, bu kriterlerin kısıtlayıcı olduğu ve uygulamalar sırasında bu kriterlerin sağlatılması zor olabilir. Bu çalışmada, biz uygun ve yeni bir Lyapunov – Krasovskiĭ fonksiyoneli tanımlayarak, bu fonksiyonel yardımı ile Tian ve Xie [1, Teorem 1]'nin asimptotik karalılık kriterlerini daha az kısıtlayıcı koşullar altında elde etmekteyiz. Bununla birlikte çözümlerin integrallenebilirlik ve sınırlılık durumlarını incelemekteyiz. Bu çalışma ile değişken gecikmeli sinir ağlarının niteliksel teorisine yeni katkılar yapmayı amaçlıyoruz.

Improved Qualitative Criteria for Neural Networks with Variable Delays

Keywords
NNs,
LKF,
Time-varying delay,
Asymptotic stability,
Integrability,
Boundedness

Abstract: In the paper "Phys. Lett. A 374 (2010), no. 7, 938–943", Tian and Xie [1, Theorem 1] considered a system of neural networks (NNs) with variable delay. In [1], new global asymptotic stability criteria for the considered system of NNs are obtained. In [1], the technique of the proof depends upon the definition of a candidate Lyapunov functional to obtain some new and weaker delay-dependent global asymptotic stability criteria. To the best of the information, we would like to claim that in spite of the global asymptotic stability criteria in Tian and Xie [1, Theorem 1] are very interesting, however they are very strong and satisfactions of them during applications may be difficult. In this work, we define a novel suitable Lyapunov – Krasovskiĭ functional (LKF) and establish less conservative global asymptotic stability criteria than those given in Tian and Xie [1, Theorem 1] as well as we investigate the integrability and boundedness of solutions. By this way, we aim to do new contributions to the qualitative theory of NNs with time-varying delays.

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1. Introduction

It follows from relevant literature that in last years the stability of numerous systems of NNs with time-varying delay(s) has been extensively studied, because of their very effective roles in sciences, engineering, medicine and so on. For example, systems of NNs with time-varying delay(s) can appear during application prospective in pattern recognition, engineering systems, image processing , biological systems, fault diagnosis, economical systems, associative memories and so forth (see, the papers [2, 4-6, 9-13] and the references therein).

We now outline a few related works on the stability problems of various NNs with time-varying delays.

In [3], Hua et al. investigated the behavior of continues NNs with variable delay given by

$$\frac{dx}{dt} = -Cx(t) + Af(x(t)) + Bf(x(t-d(t))) + I$$

with

$$0 \le d(t) \le d, \ \frac{d}{dt}d(t) \le \mu.$$

In [3], a theorem on the asymptotic stability of this system is proved via an LKF and linear matrix inequalities (LMIs) by the authors. By the given examples, it is obtained upper bounds for the constant d.

Afterwards, Sun et al. [6] considered the following NNs with variable delay:

$$\frac{dx}{dt} = -Cx(t) + Ag(x(t)) + A_1 g(x(t-\tau(t))) + u$$

with

$$0 \le \tau(t) \le h, \mu_2 \le \frac{d}{dt}\tau(t) \le \mu_2 < 1.$$

In [6], the stability of the above NNs with time-varying delay is investigated via constructing a new LKF, which includes a triple-integral term. Here, using the free-weighting matrices method, some delay-dependent stability criteria are derived in terms of LMIs. Indeed, in [6], a theorem, which includes sufficient conditions on the asymptotic stability of the above NNs, is proved. The rate-range of the delay, i.e., upper bounds of h for different μ is also obtained.

Kwon and Park [4] considered an uncertain NNs with discrete time-varying delays. Here, the problem of stability analysis for the considered NNs with time-varying delays is investigated. By constructing a new LKF, a new delay-dependent stability criterion for the considered NNs is established in terms of LMIs. Numerical examples are included to show the effectiveness of proposed criterion.

Kwon et al. [5] considered the following NNs with time-varying delay, $h(t)$:

$$\frac{dy}{dt} = -Ay(t) + W_0 g(y(t)) + W_1 g(y(t-h(t))) + b$$

with

$$0 \le h(t) \le h_U, -\infty < \frac{d}{dt}h(t) \le h_D.$$

In Kwon et al. [5], certain sufficient conditions to satisfy the asymptotic stability of the above NNs are established in terms of LMIs via definition of an LKF and some novel techniques.

In 2009, 2010, 2011 and 2015, Tian and Xu [7], Tian and Xie [1], Tian and Zhong [8], Tian and Liu [9] considered the following system of NNs with time-varying delay, $\tau(t)$:

$$\frac{dx}{dt} = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + \mu, \tag{1}$$

where $t \in \mathbb{R}^+, \mathbb{R}^+ = [0,\infty)$, $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $g(x) = [g_1(x_1(t)), g_2(x_2(t)), \dots, g_n(x_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation, $g(x(t-\tau(t))) = [g_1(x_1(t-\tau(t))), g_2(x_2(t-\tau(t))), \dots, g_n(x_n(t-\tau(t)))]^T \in \mathbb{R}^n$ and $\mu = [\mu_1, \mu_2, \dots, \mu_n]^T \in \mathbb{R}^n$ is a constant input vector. Next, $A, B \in \mathbb{R}^{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. $C = diag(C_{11}, C_{22}, \dots, C_{nn})$ with $C_{ii} > 0, i = 1,2,\dots,n,$ and $\tau(t)$ is time-varying delay function and satisfies the following conditions:

$$\tau(t) \in C^1(\mathbb{R}^+, (0,\infty)),\ \mathbb{R}^+ = [0,\infty),\ 0 \le \tau(t) \le h,\ 0 \le \tau'(t) \le h_0 < 1,$$

where h and h_0 are constants. In addition, it is assumed that each neuron activation function in system (1), i.e. $g_i(x(.))$, satisfies the following condition:

$$\gamma_i \le \frac{g_i(x) - g_i(y)}{x - y} \le \sigma_i \text{ for all } x,\ y \in \mathbb{R},\ i = 1,2,\dots,n, \tag{2}$$

where γ_i, σ_i are positive constants.

We assume that $x^* = [x_1^*, x_2^*, \dots, x_n^*]^T$ is the equilibrium point of system(1). Using the transformation $z(.) = x(.) - x^*$, then the system of NNs (1) can be converted to the following system of delay differential equations (DDEs):

$$\frac{dz}{dt} = -Cz(t) + Af(z(t)) + Bf(z(t - \tau(t))), \tag{3}$$

where

$z(t) = [z_1(t), z_2(t), \dots, z_n(t)]^T \in \mathbb{R}^n$, $f(z) = [f_1(z_1(t)), f_2(z_2(t)), \dots, f_n(z_n(t))]^T \in \mathbb{R}^n$ and $f_i(z_i(t))$
$= g_i(z_i(t) + z_i^*) - g_i(z_i^*),\ i = 1,2,\dots,n.$

In view of the inequality (2), we can derive that

$$\gamma_i \le \frac{f_i(z_i(t))}{z_i(t)} \le \sigma_i,\ f_i(0) = 0, i = 1,2,\dots,n\ . \tag{4}$$

It is worth to mention that, in the present literature, there are numerous and interesting papers on the various stability problems related the system of NNs (3) with time-varying delays and its modified different forms. In fact, here, our aim is not to focus on the details of those works. However, we would like to attract the attentions of researchers to a different point such as the positive effect of proper LKFs during the investigation of stability problems of NNs with time-varying delays. Next, we should mention that, in the above mentioned papers, one of the important aim of the researchers is to obtain weaker stability criteria on the upper bound of the time variable delays of the considered NNs in terms of LMIs, using LKF method, free-weighting matrices method, convex optimization approach, reciprocally convex approach and so on. In all of the above mentioned papers, the technique of the proofs is based on construction of various different LKFs. For illustrative aims, in particular cases, some numerical examples are also provided to demonstrate the obtained criteria and to show applications of them therein. Here, for the sake of the brevity, we would not like to give more details on the subject.

In this paper, we also consider the system of NNs (1) with time-varying delay and its equivalent system of DDEs (3). Indeed, we take into account the paper of Tian and Xie [1] as a reference paper. The above mentioned papers can also be taken as the references papers, however, for the sake of the suitability, we would not to do that ones. Here, we define a very different LKF from those are available in the above mentioned papers and literature. We aim to show the positive effect of this LKF to get weaker and more suitable conditions. Here, we would not obtain the upper bound of the variable time delay in a general form. This fact can be specified according to proper stability problems. In this paper, we would not like to discuss this case.

Now, from above discussion, it can be seen that the stability problem related to the equilibrium point x^* of the system of NNs (1) with time-varying delay can be transformed into the stability of the origin of the system of DDEs (3) with time-varying delay. Hence, instead of the investigation of stability of the system of NNs (1) with time-varying delay, it can be investigated the stability problem of the origin of the system of DDEs (3). In this paper, we also do the same, i.e., under the above assumptions and transform, instead of the stability problem of the system of NNs (1), we investigate the stability of the system of DDEs (3) with time variable delay.

2. Material and Method

In this paper, the technique of the proofs is based on the Lyapunov-Krasovskiĭ functional method. When we use this method, it is needed to construct or to define a suitable LKF.

3. Results

Consider the following system of DDEs:

$$\frac{dz}{dt} = F(z_t), z_t(\theta) = z(t+\theta), -r \le \theta \le 0, \tag{5}$$

where $z \in \mathbb{R}^n$. We assume that $C = C([-r,0],\mathbb{R}^n)$ is the space of continuous functions from $[-r,0]$ into $\mathbb{R}^n$ and $F: C \to \mathbb{R}^n$ is continuous and $F(0)=0.$ Let C_H be the open H-ball in $C;$ $C_H := \{\phi \in C([-r,0],\ \mathbb{R}^n): \|\phi\| < H\}.$ Let S be the set of $\phi \in C$ such that $\|\phi\| \ge H,$ denote by $S^\bullet$ the set of all functions $\phi \in C$ such that $|\phi(0)| \ge H,$ where H is large enough.

Let $z(t) = z(t,t_0,\phi)$ be a solution of (5) on $[t_0-\tau, t_0],$ $t_0 \ge 0,$ such that $z(t) = \phi(t)$ on $[t_0-\tau,t_0],$ where $\phi:[t_0-\tau,t_0] \to \mathbb{R}^n$ is a continuous initial function (see, Burton [10] and Yoshizawa [11]).

Definition 1. (Burton [10].) The zero solution of DDE (5) is asymptotically stable if it is stable if for each $t_1 \ge t_0 \ge 0,$ there is an $\eta > 0$ such that $[\phi \in C(t_1),\ \|\phi\| < \eta]$ imply that $z(t,t_1,\phi) \to 0$ as $t \to \infty.$ If this is true for every $\eta > 0,$ then $x = 0$ is asymptotically stable in the large or globally asymptotically stable.

Lemma 1. (Sinha [12, Lemma 1].) Suppose $F(0)=0.$ Let V be a continuous LKF defined on C_H with $V(0)=0$ and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty,$ $u(s) \to \infty$ as $s \to \infty$ with $u(0)=0.$ If for all ϕ in $C,$ $u(\phi(0)) \le V(\phi),$ $V(\phi) \ge 0,$ $\frac{d}{dt}V(\varphi) \le 0,$ then the zero solution of the system of DDEs (4) is stable.

Let $\mathbb{R} \subset C_H$ be a set of all functions $\phi \in C_H$, where $V'(\phi)=0.$ If $\{0\}$ is the largest invariant set in $\mathbb{R}$, then the zero solution of the system of DDEs (5) is asymptotically stable.

Theorem 1. (Smith [13, Theorem 5.17].**)** If V is a LKF on the set $\Omega \subset C$ and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in \Omega$ for $t \ge 0,$ then $\omega(\phi)$ is contained in the largest invariant subset of $E \equiv \{\psi \in \bar{\Omega}: \frac{d}{dt}V(\psi) = 0\}.$

Remark 1. If the largest invariant set contained in E is $\{0\},$ then the zero solution of (3) is globally asymptotically stable (see Graef and Tunç [14, Theorem 2.1]).

Let $z \in \mathbb{R}^n$ and norm $\|.\|$ be defined by

$$\|z\| = \sum_{i=1}^{n} |z_i|.$$

Next, let $M \in \mathbb{R}^{n\times n}$. Then, the norm of this matrix is defined by

$$\|M\| = \max_{1\le j\le n}\left(\sum_{i=1}^{n}|m_{ij}|\right).$$

We should note that, in some places, it will be written z instead of $z(t)$, without mention.

For any $\phi \in C$, let

$$\|\phi\|_C = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-\tau,0]}$$

and

$$C_H = \{\phi : \phi \in C \text{ and } \|\phi\|_C \le H < \infty\}.$$

We should note that the neural system (3) with time variable delay is a particular case of the system of DDEs (5).

Let us give the main result of Tian and Xie [1, Theorem 1].

Theorem 2. (Tian and Xie [1, Theorem 1]). For given scalars $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n)$, $u \ge 0$, $h \ge 0$, the origin of the NNs (3) with time variable delay is globally asymptotically stable if there exist symmetric positive matrices P, Q_1, Q_2, $\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$, $\begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}$, positive diagonal matrices T_1, T_2, $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, $D = diag(d_1, d_2, ..., d_n)$ and any matrices P_1, P_2, N_i, M_i, L_i, S_i, $(i = 1, 2, ..., 7)$, with appropriate dimensions, such that the following LMIs hold:

$$E_1 = \begin{bmatrix} \bar{E} & -\frac{h}{2}N \\ * & -\frac{h^2}{4}Q_1 \end{bmatrix} < 0,$$

$$E_2 = \begin{bmatrix} \bar{E} & -\frac{h}{2}L \\ * & -\frac{h^2}{4}Q_1 \end{bmatrix} < 0,$$

$$\Phi_1 = \begin{bmatrix} \bar{\Phi} & -\frac{h}{2}M \\ * & -\frac{h^2}{4}Q_2 \end{bmatrix} < 0,$$

$$\Phi_2 = \begin{bmatrix} \bar{\Phi} & -\frac{h}{2}S \\ * & -\frac{h^2}{4}Q_2 \end{bmatrix} < 0.$$

Remark 2. For the details of some the other terms included in this theorem, see, Tian and Xie [1, Theorem 1]. In fact, this theorem includes the terms formulated as $E_{11} - E_{17}$, $E_{22} - E_{27}$, $E_{33} - E_{37}$, $E_{44} - E_{47}$, $E_{55} - E_{57}$,

$E_{66}-E_{67}$, E_{77}, $\Phi_{11}-\Phi_{17}$, $\Phi_{22}-\Phi_{27}$, $\Phi_{33}-\Phi_{37}$, $\Phi_{44}-\Phi_{47}$, $\Phi_{55}-\Phi_{57}$, $\Phi_{66}-\Phi_{67}$ and Φ_{77}. These terms can lead very stronger stability conditions.

3.1. New Qualitative Analyses

In this section, we obtain the global asymptotic stability result of Tian and Xie [1, Theorem 1] under less conservative conditions and also establish two new qualitative results for the system of DDEs (3).

The first new result of this paper is on the global asymptotic stability of the origin and given by the following theorem, Theorem 2.

Theorem 2. We assume the following conditions, $(C1)$, $(C2)$, hold:

$(C1)$ There is a positive constant f_0 such that

$$f(0)=0,\ \|f(u)\|\le f_0\|u\| \text{ for all } u\in\mathbb{R}^n.$$

$(C2)$ There are constants C_0, h_0, $0<h_0<1$, and Δ_0 such that

$$C=diag(C_{11},C_{22},...,C_{nn}) \text{ with } C_{ii}>0,$$

$$C_0=\sum_{i=1}^{n}C_{ii}$$

and

$$C_0(1-h_0)-f_0(1-h_0)\|A\|-f_0\|B\|\ge\Delta_0.$$

Then the origin of the system of DDEs (3) is globally asymptotically stable.

Proof. For the poof of this theorem, we define a LKF $W=W(z(t))$ by

$$W(z(t)):=\|z(t)\|+\gamma\int_{t-\tau(t)}^{t}\|f(z(s)\|ds$$

$$=|z_1(t)|+...+|z_n(t)|+\gamma\int_{t-\tau(t)}^{t}|f_1(z_1(s))|ds+...+\gamma\int_{t-\tau(t)}^{t}|f_n(z_n(s))|ds, \tag{6}$$

where $\gamma>0$, $\gamma\in\mathbb{R}$, and γ will be chosen later in the proof.

As for the next step, it follows that

$$W(0)=0,|z_1(t)|+...+|z_n(t)|\le W(z(t)).$$

Calculating the time derivative of the LKF W in (6) along the trajectories of the system of DDEs (3) and using the condition $0\le\tau'(t)\le h_0<1$, we obtain

$$\frac{d}{dt}W(z(t))=z'(t)\operatorname{sgn}z(t+0)+\gamma\|f(z(t))\|-\gamma\|f(z(t-\tau(t)))\|(1-\tau'(t))$$

$$\leq \sum_{i=1}^{n} z_i'(t)\operatorname{sgn} z_i(t+0) + \gamma \| f(z(t))\| - \gamma(1-h_0)\| f(z(t-\tau(t)))\|. \qquad (7)$$

According to the first term in (7), using condition $(C2)$, we derive

$$\sum_{i=1}^{n} \operatorname{sgn} z_i(t+0) z_i'(t) \leq -\sum_{i=1}^{n} C_{ii} |z_i(t)|$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| \; |f_j(z(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| \; |f_j(z(t-\tau(t)))|$$

$$= -\sum_{i=1}^{n} C_{ii} |z_i(t)| + \|A\| \; \|f(z(t))\| + \|B\| \; \|f(z(t-\tau(t)))\|$$

$$= -C_0 \|z(t)\| + \|A\| \; \|f(z(t))\| + \|B\| \; \|f(z(t-\tau(t)))\|$$

with $C_0 = \sum_{i=1}^{n} C_{ii}$.

From (7) and the above discussion, it follows that

$$\frac{d}{dt} W(z(t)) \leq -C_0 \|z(t)\| + \|A\| \; \|f(z(t))\| + \|B\| \; \|f(z(t-\tau(t)))\|$$

$$+\gamma \| f(z(t))\| - \gamma(1-h_0)\| f(z(t-\tau(t)))\|.$$

Let $\gamma = \frac{\|B\|}{1-h_0}$. Then, according to the condition $(C1)$, we have

$$\frac{d}{dt} W(z(t)) \leq -C_0 \|z(t)\| + \|A\| \; \|f(z(t))\| + \frac{\|B\|}{1-h_0} \| f(z(t))\|$$

$$\leq -C_0 \|z(t)\| + f_0 \|A\| \; \|z(t)\| + \frac{f_0 \|B\|}{1-h_0} \|z(t)\|$$

$$= -\left(C_0 - f_0 \|A\| - \frac{f_0 \|B\|}{1-h_0} \right) \|z(t)\|$$

$$= -\frac{1}{1-h_0} \left[C_0(1-h_0) - f_0(1-h_0)\|A\| - f_0 \|B\| \right] \|z(t)\|$$

$$\leq -\frac{\Delta_0}{1-h_0} \|z(t)\|$$

$$= -M_0 \|z(t)\|,$$

where $M_0 = \frac{\Delta_0}{1-h_0}$. Hence, we have

$$\frac{d}{dt}W(z(t)) \le -C_0 \|z(t)\| \le 0. \qquad (8)$$

Next , we define a set E such that $E \equiv \{\psi \in \bar{\Omega} : \frac{d}{dt}W(\psi) = 0\}$. Hence, it is derived from (8) that

$$\frac{d}{dt}W(z) = 0 \text{ if and only if } z = 0.$$

From this point of view, we can obtain that the largest invariant subset of the set E is the set $\{0\}$. From the discussion of the proof, we conclude the origin of the system of DDEs (3) is globally asymptotically stable (**s**ee, [1, 3, 14]).

The next new result of this paper is related to the integrability of the solutions and given by Theorem 3 below.

Theorem 3. The norm of solutions of the system of DDEs (3) are integrable in the sense of Lebesgue on $\mathbb{R}^+$ if the conditions $(C1)$ and $(C2)$ hold.

Proof. The main tool to prove Theorem 3 is the LKF $W = W(z(t))$ defined by (6). According to the conditions $(C1)$ and $(C2)$, we have the inequality (8). Since the LKF $W(z(t))$ is decreasing, integrating the inequality (8), we get

$$C_0 \int_{t_0}^{t} \|z(s)\| ds \le W(\phi(t_0)) - W(z(t)) \le W(\phi(t_0)) = W_0, \text{ (a positive constant).}$$

According to this result, we can conclude that

$$\int_{t_0}^{\infty} \|z(s)\| ds < \infty,$$

which proves Theorem 3.

The last new result of this paper is on the boundedness of the solutions and given by the following theorem, Theorem 4.

Theorem 4. The solutions of the system of DDEs (3) are bounded if the conditions $(C1)$ and $(C2)$ hold.

Proof. As for the proof of Theorem 4, again we consider the LKF $W = W(z(t))$ defined by (6). Indeed, from the conditions $(C1)$, $(C2)$ and (8), it follows that

$$\frac{d}{dt}W(z(t)) \le 0. \qquad (9)$$

By the integration of the inequality (9), we have

$$W(z(t)) \le W(\phi(t_0)) = W_0.$$
(10)

To this end, the relations (6) and (10) readily imply the inequality:

$$\|z(t)\| \le \|z(t)\| + \gamma \int_{t-\tau(t)}^{t} \|f(z(s)\| ds \le W_0,$$

i.e., we find

$$\|z(t)\| \le W_0 \text{ for all } t \ge t_0 \ge 0.$$

Thus, we have arrived at the solutions of the nonlinear system of DDEs (3) are bounded. Hence, the proof of Theorem 4 is complete.

4. Discussion and Conclusion

We now explain the contributions of Theorems 2-4 to the qualitative properties of the system of DDEs (3), the topic and the former literature on the subject.

1^0) We consider the same system of DDEs of Tian and Xie [1, Theorem 1], the system of DDEs (3). Tian and Xie [1, Theorem 1] investigated global asymptotic stability of the origin of the system of DDEs (3). Here, we also study the global asymptotic stability of the origin of the system of DDEs (3) as well as integrabilty and boundedness of solutions of the system of DDEs (3). We obtain the global asymptotic stability result of Tian and Xie [1, Theorem 1] under very weaker and suitable conditions via a suitable LKF, which is different from that in [1]. Next, to the best of our information, the integrabilty and boundedness of the system of DDEs (3) were not investigated in the literature yet. This is the first paper that investigates the integrabilty and boundedness of the system of DDEs (3). By this way, we added two new results to those of Tian and Xie [1, Theorem 1]. These are contributions to the work of the Tian and Xie [1, Theorem 1], the qualitative theory of for neural networks with time-varying delays and the relevant literature.

2^0) The proof of the result of Tian and Xie [1, Theorem 1] was completed by the LKF defined as the following:

$$V(z) = V_1(z) + V_2(z) + V_3(z) + V_4(z), \text{ where } z = z(t),$$

with

$$V_1(z) = z^T Pz + 2\sum_{i=1}^{n}\left\{ \int_0^{z_i(t)} \lambda_i (f_i(s) - \gamma_i s)ds + \int_0^{z_i(t)} d_i(\sigma_i s - f_i(s))ds \right\},$$

$$V_2(z) = \int_{t-\frac{h}{2}}^{t} \begin{bmatrix} z(s) \\ z(s-\frac{h}{2}) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} z(s) \\ z(s-\frac{h}{2}) \end{bmatrix} ds,$$

$$V_3(z) = \int_{t-\tau(t)}^{t} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds$$

and

$$V_4(z) = \frac{h}{2}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Q_1\dot{z}(s)dsd\theta + \frac{h}{2}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t} \dot{z}^T(s)Q_2\dot{z}(s)dsd\theta. \tag{11}$$

The time derivative of this LKF $V(z)$ along the trajectory of system of DDEs (3) was calculated in [8] as follows:

$$\dot{V}(z) = \dot{V}_1(z) + \dot{V}_2(z) + \dot{V}_3(z) + \dot{V}_4(z),$$

where

$$\dot{V}_1(z) = 2z^T P\dot{z} + 2[f^T(z) - z^T\Gamma]\Lambda\dot{z} + 2[z^T(\Sigma - f^T(z)]D\dot{z},$$

$$\dot{V}_2(z) = \begin{bmatrix} z(t) \\ z(t-\frac{h}{2}) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} z(t) \\ z(t-\frac{h}{2}) \end{bmatrix} - \begin{bmatrix} z(t-\frac{h}{2}) \\ z(t-h) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} z(t-\frac{h}{2}) \\ z(t-h) \end{bmatrix},$$

$$\dot{V}_3(z) \le \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix} \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}$$

$$-(1-u)\begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix} \begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix},$$

$$\dot{V}_4(z) = \left(\frac{h}{2}\right)^2 \dot{z}^T(t)(Q_1+Q_2)\dot{z}(t) - \frac{h}{2}\int_{t-\frac{h}{2}}^{t} \dot{z}^T(s)Q_1\dot{z}(s)ds - \frac{h}{2}\int_{t-h}^{t-\frac{h}{2}} \dot{z}^T(s)Q_2\dot{z}(s)ds.$$

In [1], via some calculations, it was obtained the following estimates, respectively:

$$\dot{V}(z(t)) \le \varsigma^T(t)E\varsigma(t),$$

where

$$\varsigma(t) = [z^T(t) \quad z^T(t-\tau(t)) \quad z^T(t-\frac{h}{2}) \quad z^T(t-h) \quad \dot{z}^T(t) \quad f^T(z(t)) \quad f^T(z(t-\tau(t))) \quad V_1^T \quad V_2^T],$$

$$E = \begin{bmatrix} \bar{E} & -(\frac{h}{2}-\tau(t))N & -\tau(t)L \\ * & -\frac{h}{2}(\frac{h}{2}-\tau(t))Q_1 & 0 \\ * & * & -\frac{h}{2}\tau(t)Q_1 \end{bmatrix}$$

and

$$\dot{V}(z(t)) \le \xi^T(t)\Phi\xi(t),$$

where

$$\xi(t) = [z^T(t) \quad z^T(t-\tau(t)) \quad z^T(t-\frac{h}{2}) \quad z^T(t-h) \quad \dot{z}^T(t) \quad f^T(z(t)) \quad f^T(z(t-\tau(t))) \quad U_1^T \quad U_2^T],$$

$$\Phi = \begin{bmatrix} \bar{\Phi} & -(h-\tau(t))M & -(\tau(t)-\frac{h}{2})S \\ * & -\frac{h}{2}(h-\tau(t))Q_2 & 0 \\ * & * & -\frac{h}{2}(\tau(t)-\frac{h}{2})Q_2 \end{bmatrix}.$$

Since $E < 0$ and $\Phi < 0$, it is proved that the origin of the system (3) is asymptotically stable.

In this paper, instead of the LKF (11), we define completely a different LKF:

$$W(z(t)) := \|z(t)\| + \gamma \int_{t-\tau(t)}^{t} \| f(z(s)\| ds. \tag{12}$$

The time derivative of this LKF along the trajectories of the system of DDEs (3) is given by

$$\frac{d}{dt}W(z(t)) = z'(t)\operatorname{sgn} z(t+0) + \gamma \| f(z(t))\| - \gamma \| f(z(t-\tau(t)))\|(1-\tau'(t))$$

Next, depending on the conditions $(C1)$ and $(C2)$, some clear and simple calculations implies that

$$\frac{d}{dt}W(z(t)) = -\frac{1}{1-h_0}\left[C_0(1-h_0) - f_0(1-h_0)\|A\| - f_0\|B\|\right]\|z(t)\|.$$

When we compare the LKFs (11) and (12), it can be seen that in spite of our LKF (12) has two terms, however the LKF (11) of Tian and Xie [1, Theorem 1] has numerous terms and includes five matrices. Indeed, naturally, these data leads very stronger asymptotic stability conditions as seen in Theorem 1. Namely, the given LKFs have to be positive definite and their derivatives along the trajectories of the system of DDEs (3) have to be negative definite. According to the qualitative theory of the functional differential equations, suitable LKFs can lead very less conservative and suitable conditions. This fact can be seen when we compare the conditions of Tian and Xie [1, Theorem 1], which has been stated above, and those of our theorem, Theorem 2. Hence, it follows that the conditions of Theorem 2 are very convenient, less conservative and more optimal for applications. Here, we would not like to give more details for the sake of the brevity. These are the novelty, originality and contributions of this paper. These are also desirable facts for a proper scientific work on the qualitative theory of solutions of various kind of differential equations.

In this paper, a system of NNs with time-varying delays is considered. New sufficient conditions are obtained on the asymptotic stability of the origin and as well as on the integrability and boundedness of solutions of the considered system of NNs. Here, the technique of the proof is based on the LKF method. Hence, depending upon the definition of a new and suitable LKF, the result of Tian and Xie [1, Theorem 1] can be obtained under very less restrictive and more suitable conditions. In addition, to the best of knowledge, here, the results of the integrability and boundedness of solutions are new and they did not discussed for the considered system of NNs with time-varying delays in the literature by this time. As a result, the results of this paper have new contributions to theory of the system of NNs and the relevant literature.

References

[1] Tian, J., Xie, X. 2010. New asymptotic stability criteria for neural networks with time-varying delay. Phys. Lett. A., 374(2010), 938–943.

[2] Gao, Z.M., He, Y., Wu, M. 2019. Improved stability criteria for the neural networks with time-varying delay via new augmented Lyapunov- Krasovskiĭ functional. Appl. Math. Comput., 349(2019), 258–269.

[3] Hua, C., Long, C., Guan, X. 2006. New results on stability analysis of neural networks with time-varying delays. Phys. Lett. A, 352(2006), 335–340.
[4] Kwon, O. M., Park, J. H. 2009. Improved delay-dependent stability criterion for neural networks with time-varying delays. Phys. Lett. A, 373(2009), 529–535.
[5] Kwon, O. M., Park, M. J., Park, J. H., Lee, S. M., Cha, E.J. 2013. Improved approaches to stability criteria for neural networks with time-varying delays. J. Franklin Inst., 350(2013), 2710–2735.
[6] Sun, J., Liu, G. P., Chen, J., Rees, D. 2009. Improved stability criteria for neural networks with time-varying delay. Phys. Lett. A, 373(2019), 342–348.
[7] Tian, J., Xu, D. 2009 New asymptotic stability criteria for neural networks with time-varying delays. Chaos Solitons Fractals, 41(2009), 1916–1922.
[8] Tian, J., Zhong, S. 2011. Improved delay-dependent stability criterion for neural networks with time-varying delay. Appl. Math. Comput., 217(2011), 10278–10288.
[9] Tian, J. Liu, Y. 2015. Improved delay-dependent stability analysis for neural networks with interval time-varying delays. Math. Probl. Eng., (2015), 10.
[10] Burton, T. A. 2005. Stability and periodic solutions of ordinary and functional differential equations, Corrected version of the 1985 original, Dover Publications, Inc., Mineola, NY.
[11] Yoshizawa, T. 1966. Stability theory by Liapunov's second method, Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo.
[12] Sinha, A. S. C. 1973. On stability of solutions of some third and fourth order delay-differential equation. Information and Control, 23(1973), 165–172.
[13] Smith, H. 2011. An introduction to delay differential equations with applications to the life sciences, Texts in Applied Mathematics, 57, Springer, New York.
[14] Graef, J. R., Tunç, C. 2015. Global asymptotic stability and boundedness of certain multi-delay functional differential equations of third order. Math. Methods Appl. Sci., 38(2015), 3747–3752.