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# EXACT SOLUTIONS FOR SOME FRACTIONAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS VIA KUDRYASHOV METHOD

**ABSTRACT**

In this study the Kudryashov method have been implemented for the exact solutions of the fractional RLW Burgers equation and the fractional Clannish Random Walker's Parabolic (CRWP) Equation. Some new solutions of the fractional RLW Burgers equation and the fractional CRWP equation have been obtained by using this method.

**Keywords:** The Fractional Nonlinear Partial Differential Equation, The Fractional RLW Burgers Equation, The Fractional CRW Equation, Kudryashov Method, Nonlinear Partial Differential Equation

# BAZI KESİRLİ LİNEER OLMAYAN KISMİ DİFERENSİYEL DENKLEMLER İÇİN KUDRYASHOV METODU İLE TAM ÇÖZÜM

**ÖZET**

Bu çalışmada kesirli RLW Burgers ve CRWP denklemlerinin tam çözümleri için Kudryashov metodunu uygulanmıştır. Bu metodu kullanılarak, kesirli RLW Burgers ve kesirli CRWP denklemleri için bazı yeni çözümler elde edilmiştir.

**Anahtar Kelimeler:** Kesirli Lineer Olmayan Kısmi Diferensiyel Denklemler, Kesirli RLW Burgers Denklemi, Kesirli CRWP Denklemi, Kudryashov Metod, Lineer Olmayan Kısmi Differansiyel Denklem

## 1. INTRODUCTION (GİRİŞ)

In the recent years, remarkable progress has been made in the construction of the approximate solutions for fractional nonlinear partial differential equations (fnPDE)[1, 2 and 3]. In particular fractional differential equations could be helpful to understand the behavior of the physical problems. Therewithal reaching to the exact solutions of fractional differential equations is very important. In this stage, it is not possible to solve the fnPDE before converting these equations into integer-order differential equations, doing this conversion we need to have a variable transformation by using a kind of fractional derivative and some useful formulas such as a modified Riemann-Liouville derivative which are proposed by Jumarie [4, 5 and 6].

Many explicit exact methods and analytic methods have been introduced in literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 and 26]. In our present work, we implement Kudryashov method for constructing exact solutions of fnPDE. We can therefore easily convert fnPDE into nPDE or ODE using suitable transformation, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty.

## 2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this article, the first section presents the scope of the study as an introduction. The second section contains some basic definitions of the modified Riemann-Liouville derivative, analyze of the Kudryashov method. In the third section, we will obtain solutions of the fractional RLW Burgers and the fractional CRWP equations by using Kudryashov method. In the last section, we implement the conclusion.

## 3. PRELIMINARIES AND NOTATIONS (TEMEL KAVRAMLAR VE GÖSTERIMLER)

In this part of the paper, it would be helpful to give some definitions and properties of the fractional calculus theory. For an introduction to the classical fractional calculus we direct the reader to [1, 2 and 3]. Here, we briefly review the modified Riemann-Liouville derivative from the recent fractional calculus proposed by Jumarie [4, 5 and 6]. Let $f$: [0, 1]→$R$ be a continuous function and $\alpha \in (0,1)$. The Jumarie modified fractional derivative of order $\alpha$ and $f$ may be defined by expression of [9] as follows:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} \left[ f(\xi) - f(0) \right] d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left[ f(\xi) - f(0) \right] d\xi, & 0 < \alpha < 1, \\ \left( f^{(n)}(x) \right)^{(\alpha-n)}, & n \le \alpha \le n+1, n \ge 1. \end{cases} \tag{3.1}$$

In addition to this expression, we may give a summary of the fractional modified Riemann-Liouville derivative properties which are used further in this paper. Some of the useful formulas are given as

$$D_x^{\alpha} k = 0 \quad (k \text{ is a constant}) \tag{3.2}$$

$$D_x^{\alpha} x^{\mu} = \begin{cases} 0, & (\mu \leq \alpha - 1), \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, & (\mu > \alpha - 1). \end{cases} \quad (3.3)$$

Similar to integer-order differentiation, the Jumarie' modified fractional differentiation is a linear operation:

$$D_x^{\alpha}\left(\beta u(x)+\gamma v(x)\right) = \beta D_x^{\alpha} u(x) + \gamma D_x^{\alpha} v(x) \quad (3.4)$$

where $\beta$ and $\gamma$ are constants, and satisfies the so called Leibniz rule for the Jumarie' modified fractional derivative is equal to the standard one:

$$D_x^{\alpha}\left(u(x)v(x)\right) = v(x)\, D_x^{\alpha}\, u(x) + u(x)\, D_x^{\alpha} v(x)\; v(x) = \sum_{k=0}^{\infty}\binom{n}{k} u^{(k)}(x) D_x^{\alpha-k} v(x) \quad (3.5)$$

if $v(x)$ is continuous in $[0,x]$ and $u(x)$ has continuous derivative in $[0,x]$. The last properties is

$$D_x^{\alpha}\left[f\left(u(x)\right)\right] = f_u'(u) = D_x^{\alpha} u(x) = D_x^{\alpha} f(u)(u_x)^{\alpha}, \quad (3.6)$$

which are direct consequences of the equality $d^{\alpha}x(t) = \Gamma(1+\alpha)dx(t)$.

In the last, let us consider the time fractional differential equation with independent variables $x = (x_1, x_2, x_3 ..., x_m, t)$ and a dependent variable $u$,

$$G(u, D_x^{\alpha} u, u_{x_1}, u_{x_2}, u_{x_3}, D_x^{2\alpha} u, u_{x_1x_1}, u_{x_2x_2}, u_{x_3x_3}, ...) = 0 . \quad (3.7)$$

Let us give a variable transformation for Eq. (1.7) as

$$u(x_1, x_2, x_3 ..., x_m, t) = u(\xi), \; \xi = x_1 + l_1 x_2 + \ldots + l_{m-1} x_m + \frac{ct^{\alpha}}{\Gamma(\alpha+1)}, \quad (3.8)$$

where $l_i$ and $c$ are constants to be determined later; after the transformation the fractional differential equation (3.7) is reduced to an ordinary differential equation

$$H(u(\xi), u'(\xi), u''(\xi), \ldots), \quad (3.9)$$

where $' = \dfrac{d}{d(\xi)}$ .

For more information on the mathematical properties of fractional derivatives the mentioned references can be consulted.

## 4. ANALYSIS OF THE METHOD (METODUN ANALİZİ)

Let us simply describe the Kudryashov method. Consider a given time fractional partial differential equation in two variables and a dependent variable $u$,

$$W(u, D_t^{\alpha} u, u_x, u_{xx}, u_{xxx}, ...) = 0 . \quad (4.1)$$

Let us give a variable transformation for Eq. (3.1) as

$$u(x,t) = u(\xi), \; \xi = x + \frac{ct^{\alpha}}{\Gamma(\alpha+1)}, \quad (4.2)$$

where $c$ is constant to be determined later; after the transformation the fractional differential equation (4.1) is reduced to an ordinary differential equation (3.9). Now we show how one could obtain the exact solution of the equation (4.1) helping the approach by Kudryashov.

The first step: Determination of the dominant terms.

To find dominant terms we substitute

$$u = \xi^{p} \tag{4.3}$$

into all terms of equation (3.9). Then we compare degrees of all terms in equation (4.9) and choose two or more with the smallest degree. The minimum value of $p$ define the pole of solution of equation (3.9) and we denote it as $N$. We have to point out that method can be applied when $N$ is integer and one can transform the equation studied and repeat the procedure.

The second step: The solution structure.

We look for exact solution of equation (3.9) in the form

$$u = a_0 + a_1 F(\xi) + a_2 F^2(\xi) + ... + a_N F^N(\xi) \tag{4.4}$$

where $a_i$ are unknown constants, $F(\xi)$ is the function as:

$$F(\xi) = \frac{1}{1+e^{\xi}}. \tag{4.5}$$

This function satisfies to the first order ordinary differential equation

$$F_{\xi} = F^2 - F \ . \tag{4.6}$$

Equation (4.6) is necessary to calculate the derivatives of function $u(\xi)$.

The Third step: Derivatives calculation.

We should calculate the derivatives of function $u$. One can do it using the computer algebra systems Mapple or Mathematica. As an example we consider the general case when $N$ is arbitrary. Differentiating the expression (4.4) with respect to $\xi$ and taking into account (4.6) we have

$$\begin{aligned} u_{\xi} &= \sum_{i=1}^{N} a_i i (F-1) F^{i}, \\ u_{\xi\xi} &= \sum_{i=1}^{N} a_i i (i+1) F^2 - (2i+1) F + i) F^{i}. \end{aligned} \tag{4.7}$$

The high order derivatives of function $u(\xi)$ can be found in [26].

The fourth step: Defining the values of unknown parameters.

We substitute expressions (4.7) in equation (3.9). Later we take $u(\xi)$ from (4.4) into account. Thus equation (4.1) takes the form

$$p[F(\xi)] = 0, \tag{4.8}$$

where $p[F(\xi)]$ is a polynomial of function $F(\xi)$. Then we collect all terms with the same powers of function $F(\xi)$ and equate these expressions equal to zero. As a result we find system of algebraic equations. Solving this system we get the values of unknown parameters.

## 5. APPLICATION OF THE METHOD (METODUN UYGULAMASI)

**Example 1. (Örnek 1)**

Let's consider the fractional RLW Burgers equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial u}{\partial x}+12u\frac{\partial u}{\partial x}-\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)=0, t>0, 0<\alpha, \tag{5.1}$$

with the initial condition $u(x,0)=f(x)$. For the fractional RLW Burgers equation with the transform $u(x,t)=u(\xi),\ \xi=x-\frac{ct^\alpha}{\Gamma(1+\alpha)}$, we have the ordinary differential equation as following

$$-cu'+u'+12uu'-u''+cu'''=0. \tag{5.2}$$

The pole of the equation is given by $N=2$. Therefore, we may choose

$$u=a_0+a_1F+a_2F^2. \tag{5.3}$$

Substituting (5.3) into equation (5.2) yields a set of algebraic equations for $a_0,a_1,a_2$ and $c$. We obtain the algebraic equations as follows:

$$\begin{aligned}
&-a_1+6a_0a_1=0,\\
&a_1-6a_0a_1+6a_1^{\,2}-2a_2+12a_0a_2+6a_1c-6a_2c=0,\\
&-6a_1^{\,2}+2a_2-12a_0a_2+18a_1a_2-12a_1c+36a_2c=0,\\
&-18a_1a_2+12a_2^{\,2}+6a_1c-54a_2c=0,\\
&-12a_2^{\,2}+24a_2c=0.
\end{aligned} \tag{5.4}$$

From the solutions of the system (5.4), we have the following case.

$$a_0=\frac{1}{6},\ a_1=-2c,\ a_2=2c,\ c\neq 0. \tag{5.5}$$

By means of Mathematica, substituting (5.5) into (5.3), we have obtained the following exact solution of equation (5.1). This solution is as following:

$$u(x,t)=\frac{1}{6}-\frac{c}{2}sech\left(\frac{1}{2}\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)^2.$$

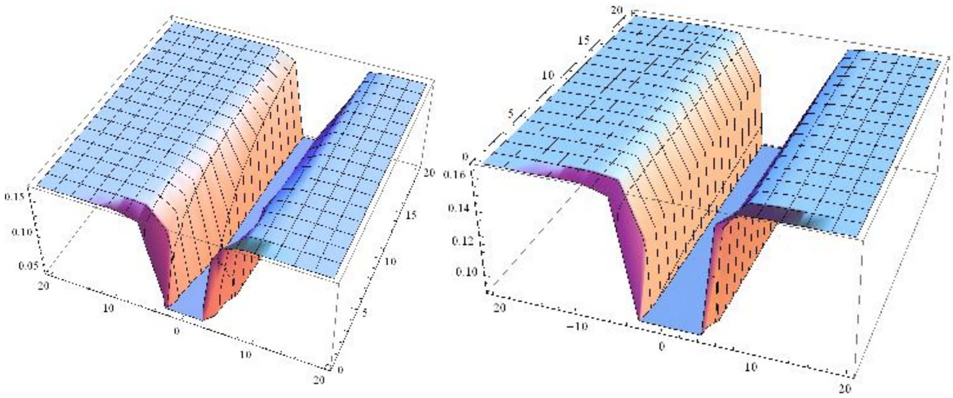

Figure 1. Graphs of the solution of (5.1) for *u*(*x*, *t*) corresponding to the values $c=1, \alpha=0.5$ and $\alpha=0.1$

(Şekil 1. $\alpha=0.5$ ve $\alpha=0.1$ değerlerine karşılık gelen *u*(*x*,*t*) için (5.1) denkleminin çözümünün grafiği)

**Example 2. (Örnek 2)**

Let's consider the fractional CRWP equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial u}{\partial x}+2u\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}=0, t>0, 0<\alpha, \tag{5.6}$$

with the initial condition $u(x,0)=f(x)$. For the fractional CRWP equation with the transform $u(x,t)=u(\xi),\ \xi=x-\frac{ct^\alpha}{\Gamma(1+\alpha)}$, we have the ordinary differential equation as following

$$-cu'-u'+2uu'-u''=0. \tag{5.7}$$

The pole of the equation is given $N=1$. Therefore, we may choose

$$u=a_0+a_1F\ . \tag{5.8}$$

Substituting (5.8) into (5.7) and using Mathematica yields a system of equations. Setting the coefficients of $F^i$ in the obtained system of equations to zero, we can deduce the following set of algebraic polynomials with the respect unknowns $a_0, a_1, a_2, ...$ namely:

$$\begin{aligned}&-2a_0a_1+a_1c=0\\&2a_1+2a_0a_1-2a_1^{\ 2}-a_1c=0\\&-2a_1+2a_1^{\ 2}=0\end{aligned} \tag{5.9}$$

From the solutions of the system (5.9), we find that

$$a_0=\frac{c}{2}, a_1=1, c\neq 0 \tag{5.10}$$

By means of Mathematica, substituting (5.10) into (5.8), we have obtained the following exact travelling wave solutions of equation (5.6). This solution is:

$$u(x,t)=\frac{c}{2}+\frac{1}{1+cosh\left[x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right]-sinh\left[x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right]}.$$

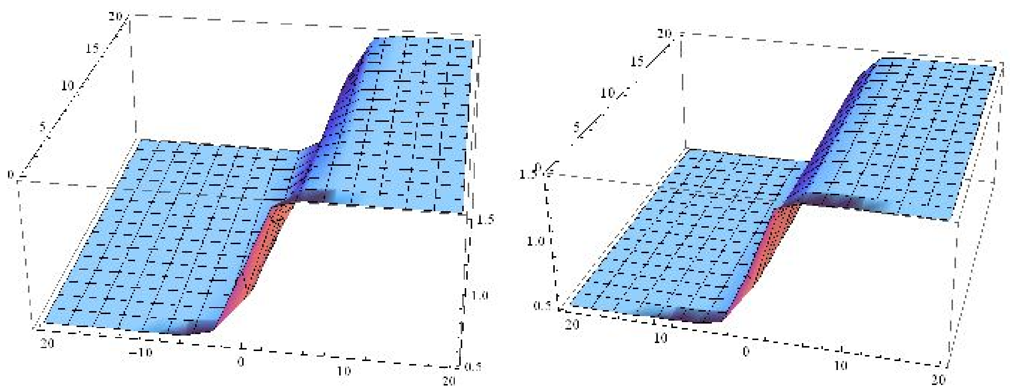

Figure 2. Graphs of the solution of (5.6) for *u(x, t)* corresponding to the values $c=1, \alpha=0.5$ and $\alpha=0.1$

(Şekil 2. $\alpha=0.5$ ve $\alpha=0.1$ değerlerine karşılık gelen *u(x,t)* için (5.6) denkleminin çözümünün grafiği)

## 6. CONCLUSION (SONUÇ)

In this paper, we apply the Kudryashov method [26] to show that this method is efficient and practically well suited to use in finding exact solutions for the fractional RLW equation and the fractional CRWP equation. The most important advantage of this method is that it can be reached more easily to the solutions than the other analytical methods. These solutions will be useful for further studies in applied sciences.

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