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ON CAUCHY-RIEMANN EQUATIONS AND HOLOMORPHIC (ANALYTICAL) FUNCTIONS

ABSRACT

Let P and Q be real functions with two real variables. In this work, first we obtain for P and Q "Cauchy- Riemann equivalent systems". The case $m=0$ yields Cauchy- Riemann equations.Then we give the relationship of $f: A \rightarrow C, f=P+iQ$ holomorphic (analytical) with Cauchy- Riemann equivalent systems ,and show that the derivative of f at $z=x+iy \in A$ is $f'(z)=$

$$\frac{1}{1+m^2}P_1(x,y,m)+\frac{m}{1+m^2}Q_1(x,y,m)+i\left[-\frac{m}{1+m^2}P_1(x,y,m)+\frac{1}{1+m^2}Q_1(x,y,m)\right]$$

$$=\frac{m}{1+m^2}P_2(x,y,m)+\frac{1}{1+m^2}Q_2(x,y,m)+i\left[-\frac{1}{1+m^2}P_2(x,y,m)+\frac{m}{1+m^2}Q_2(x,y,m)\right]$$

If we replace $m=0$ in this representation of derivative,we get the classical derivative form of $f'(z)$

Keywords: Analysis, Complex, Function, Holomorphic (Analytical), Equations

CAUCHY-RIEMANN DENKLEMLERİ VE HOLOMORF (ANALİTİK) FONKSİYONLAR ÜZERİNE

ÖZET

P ve Q fonksiyonları, iki reel değişkenli ve reel değerli ikı fonksiyon olsun. Bu çalışmada önce P ve Q fonksiyonları için " Cauchy-Riemann eşdeğer sistemleri " elde edilmiştir. Bu sistemin özel hali Cauchy-Riemann denklemlerini verir. Sonra da $f: A \rightarrow C, f=P+iQ$ fonksiyonun holomorf (analitik) olması Cauchy-Riemann eşdeğer sistemleri ile ilişkilendirilmiştir ve $z=x+iy \in A$ noktasında f nin türevinin $f'(z)=$

$$\frac{1}{1+m^2}P_1(x,y,m)+\frac{m}{1+m^2}Q_1(x,y,m)+i\left[-\frac{m}{1+m^2}P_1(x,y,m)+\frac{1}{1+m^2}Q_1(x,y,m)\right]$$

$$=\frac{m}{1+m^2}P_2(x,y,m)+\frac{1}{1+m^2}Q_2(x,y,m)+i\left[-\frac{1}{1+m^2}P_2(x,y,m)+\frac{m}{1+m^2}Q_2(x,y,m)\right]$$

olduğu gösterilmiştir. Türevin bu gösteriminde $m=0$ konursa türevin klasik türev biçimini elde ederiz.

Anahtar Kelimeler: Analiz, Kompleks, Fonksiyon, Holomorf (Analitik), Denklemler

1. INTRODUCTION (GİRİŞ)

In Complex analysis, Cauchy – Riemann equations are partial differential equations which satisfy the necessary and sufficient conditions for a function to be holomorphic on an open set [1, 4 and 5]. These equations were considered chronologically by D'Alembert, Euler (for holomorphic functions), Cauchy (for the theory of functions), and Riemann (for his Ph.D thesis) [4].

Let $P: IR^2 \to IR,\ P = P(x,y)$ and $Q: IR^2 \to IR,\quad Q = Q(x,y)$ be given. Cauchy – Riemann equations for these functions are the following partially derivative differantial equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \text{ and } \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \quad [1, 2, 3, 4, \text{ and } 5]. \quad (1)$$

sufficient that (1) is satisfied. The derivative $f'(z)$ of f is of Let P and Q be the real and imaginary parts, respectively, of a complex function $f(x+iy) = P(x,y) + iQ(x,y)$. Suppose that P and Q have continuos partial derivatives in an open set $A \subset C$. For $f = P + iQ$ to be holomorphic (analytical) it is necessary and the form

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} \quad [2,3 \text{ and } 5]. \quad (2)$$

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

A complex valued function of the form $f = P + iQ$ holomorphic (analytical) in a domain is closely related to Cauchy – Riemann equations (1). It is acrucial condition that in order for (1) to hold is that f is holomorphic (analytical) in that domain. We will form a system of equations equivalent to Cauchy – Riemann's by using wellknown fundamental concepts.

3. CAUCHY-RIEMANN EQUIVALENT SYSTEMS (CAUCHY-RIEMANN DENK SİSTEMLERİ)

Let us start with two results linking for a complex valued function $f = P + iQ$ to be holomorphic (analytical) to Cauchy – Riemann equations (1)

Definition 1 Let $A \subset C$ be an open set. Given a complex function $f: A \to C, w = f(z)$, f is said holomorphic (analytical) at $z_0 \in A$ if derivative $f'(z)$ exists for every $z \in B(z_0, r)$, where $B(z_0, r) \subset A$ is an open ball. f is called holomorphic (analytical) if it is on the set A.

Proposition 1 [2, 3, 4 and 5]

Let $f = P + iQ$ be holomorphic (analytical) on open set A. Then both the partials of both $P and Q$

exist on A and for any $z = x + iy \in A$

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \text{ and } \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Proposition 2: [2, 3, and 5]

Let $f = P + iQ : A \to C$ be a function such that the partials $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ all exist on A , are continous and satisfy, Cauchy – Riemann Equations. Then f is holomorphic (analytical) on A and $f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}$.

Now let $A \subset C$ be an open set,and $f(x+iy) = P(x,y) + iQ(x,y) : \; A \to C$ a complex valued function. Assuring that $m \neq 1$, $m \in IR$ and $\Delta x \neq 0, \Delta y \neq 0$ we define the functions P_1, Q_1, P_2, Q_2 as follows:

$$P_1(x,y,m) = \lim_{\Delta x \to 0} \frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x},$$
$$Q_1(x,y,m) = \lim_{\Delta x \to 0} \frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x},$$
$$P_2(x,y,m) = \lim_{\Delta y \to 0} \frac{P(x+m\Delta y, y+\Delta y) - P(x,y)}{\Delta y}, \qquad (3)$$
$$Q_2(x,y,m) = \lim_{\Delta y \to 0} \frac{Q(x+m\Delta y, y+\Delta y) - Q(x,y)}{\Delta y}.$$

Theorem 1. Let $A \subset C$ be an open set and $f : A \to C$, $f(x+iy) = P(x,y) + iQ(x,y)$ be a holomorphic(analytical) complex valued function on A. Then there exist functions P_1, Q_1, P_2, Q_2 on A and for every $z = x+iy \in A$ the folloving system holds:

$$\frac{1}{1+m^2}P_1(x,y,m) + \frac{m}{1+m^2}Q_1(x,y,m) = \frac{m}{1+m^2}P_2(x,y,m) + \frac{1}{1+m^2}Q_2(x,y,m) \text{ and}$$
$$-\frac{m}{1+m^2}P_1(x,y,m) + \frac{1}{1+m^2}Q_1(x,y,m) = -\frac{1}{1+m^2}P_2(x,y,m) + \frac{m}{1+m^2}Q_2(x,y,m) \quad (4)$$

In case $m = 0$, we obtain Proposition 1 [2, 3, 4, and 5].

Proof: Let $\Delta z = \Delta x + i\Delta y$ ($\Delta x, \Delta y \in IR$). By hypothesis,we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} =$$

$$\lim_{\Delta x + i\Delta y \to 0}\left[\frac{P(x+\Delta x, y+\Delta y) - P(x,y)}{\Delta x + i\Delta y} + i\frac{Q(x+\Delta x, y+\Delta y) - Q(x,y)}{\Delta x + i\Delta y}\right]$$ [2, 3, and 5].

We will consider two cases:

(i) Let $\Delta y = m\Delta x$ with $, m \neq 1, m \in IR$. In this case, we have

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ (\Delta y = m\Delta x)}} \left[\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x + im\Delta x} + i\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x + im\Delta x} \right]$$

$$= \frac{1}{1+m^2} \lim_{\Delta x \to 0} \frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{m}{1+m^2} \lim_{\Delta x \to 0} \frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}$$

$$+i[-\frac{m}{1+m^2} \lim_{\Delta x \to 0} \frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{1}{1+m^2} \lim_{\Delta x \to 0} \frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]. \quad (5)$$

Then each of terms in (5) exists, so $P_1(x,y,m)$ and $Q_1(x,y,m)$ have a meaning by (3).

(ii) Let $\Delta x = m\Delta y$ with $, m \neq 1, m \in IR$ .Then we get

$$f'(z) = \lim_{\substack{\Delta y \to 0 \\ (\Delta x = m\Delta y)}} \left[\frac{P(x+m\Delta y, y+\Delta y) - P(x,y)}{m\Delta y + i\Delta y} + i\frac{Q(x+m\Delta y, y+\Delta y) - Q(x,y)}{m\Delta y + i\Delta y} \right]$$

$$= \frac{m}{1+m^2} \lim_{\Delta y \to 0} \frac{P(x+m\Delta y, y+\Delta y) - P(x,y)}{\Delta y} + \frac{1}{1+m^2} \lim_{\Delta y \to 0} \frac{Q(x+m\Delta y, y+\Delta y) - Q(x,y)}{\Delta y}$$

$$+i[-\frac{1}{1+m^2} \lim_{\Delta y \to 0} \frac{P(x+m\Delta y, y+\Delta y) - P(x,y)}{\Delta y} + \frac{m}{1+m^2} \lim_{\Delta y \to 0} \frac{Q(x+m\Delta y, y+\Delta y) - Q(x,y)}{\Delta y}]. \quad (6)$$

Each term in (6) have asense,and so by (3) $P_2(x,y,m)$ and $Q_2(x,y,m)$ exists.The equality of (5) and (6) gives the system (4). Now, if we take $m = 0$ in (3), we obtain succeptively

$$P_1(x,y,0) = \lim_{\Delta x \to 0} \frac{P(x+\Delta x, y) - P(x,y)}{\Delta x} = P_x(x,y) ,$$

$$Q_1(x,y,0) = \lim_{\Delta x \to 0} \frac{Q(x+\Delta x, y) - Q(x,y)}{\Delta x} = Q_x(x,y) , \quad (7)$$

$$P_2(x,y,0) = \lim_{\Delta y \to 0} \frac{P(x, y+\Delta y) - P(x,y)}{\Delta y} = P_y(x,y) ,$$

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$$Q_2(x,y,0) = \lim_{\Delta y \to 0} \frac{Q(x, y+\Delta y) - Q(x,y)}{\Delta y} = Q_y(x,y) .$$

Then Theorem 1 reduces to Proposition 1 [2, 3, 4, and 5] and the system (4) reduces to Cauchy-Riemann equations (1).

Definition 2: System (4) is called "Cauchy-Riemann equivalent systems".

Corollary 1: For $f = P + iQ$ to be holomorphic (analytical) at $z = x + iy \in A$ it is necessary and sufficient that functions P_1, Q_1, P_2, Q_2 exist for $z = x + iy \in A$ and Cauchy-Riemann Equivalent Systems are satisfied.
Proof: It suffices to take the counter-positive of Theorem 1.

Theorem 2. Let $A \subset C$ be an open setand $f : A \to C$, $f(x+iy) = P(x,y) + iQ(x,y)$ be given. Suppose that P_1, Q_1, P_2, Q_2 exist and continuous on A ,and Cauchy-Riemann Equivalent Systems are satisfied. Then f is holomorphic (analytical) on the set A and $f'(z) =$

$$\frac{1}{1+m^2}P_1(x,y,m) + \frac{m}{1+m^2}Q_1(x,y,m) + i\left[-\frac{m}{1+m^2}P_1(x,y,m) + \frac{1}{1+m^2}Q_1(x,y,m)\right]$$

$$= \frac{m}{1+m^2}P_2(x,y,m) + \frac{1}{1+m^2}Q_2(x,y,m) + i\left[-\frac{1}{1+m^2}P_2(x,y,m) + \frac{m}{1+m^2}Q_2(x,y,m)\right] \quad (8)$$

Proof : Theorem 2 reduces to Proposition 2 for $m = 0$ [see 2, 3, and 5]. Therefore we suppose $m \neq 0, m \neq 1, m \in IR$ and $\Delta x \neq 0$. Then the following calculation is clear:

$$\Delta x\,[\ \frac{1}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{m}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$
$$+ m\Delta x\,[\frac{m}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} - \frac{1}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$

$$= P(x+\Delta x, y+m\Delta x) - P(x,y) \quad (9)$$

and

$$\mathrm{i}\,\Delta x\,[-\frac{m}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{1}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$
$$+\mathrm{i}\, m\Delta x\,[\frac{1}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{m}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$

$$= \mathrm{i}[Q(x+\Delta x, y+m\Delta x) - Q(x,y)]. \quad (10)$$

Thus from (9) and (10) we obtain

$$(\Delta x + \mathrm{i}\, m\Delta x)\,[\frac{1}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{m}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$

$$+ (-m\Delta x + i\Delta x)\,[-\frac{m}{1+m^2}\frac{P(x+\Delta x, y+m\Delta x) - P(x,y)}{\Delta x} + \frac{1}{1+m^2}\frac{Q(x+\Delta x, y+m\Delta x) - Q(x,y)}{\Delta x}]$$

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$$= P(x+\Delta x, y+m\Delta x) - P(x,y) + \mathrm{i}[Q(x+\Delta x, y+m\Delta x) - Q(x,y)]. \quad (11)$$

Divide both sides of (11) by $\Delta x + i\, m\Delta x$ and pass $\Delta x \to 0$. This implies

$$f'(z)=\frac{1}{1+m^2}P_1(x,y,m)+\frac{m}{1+m^2}Q_1(x,y,m)+i\left[-\frac{m}{1+m^2}P_1(x,y,m)+\frac{1}{1+m^2}Q_1(x,y,m)\right] \quad (12)$$

by(3)and definition $f'(z)$.Since Cauchy-Riemann Equivalent Systems are satisfied, we get (8) from (12).

Similarly, for $\Delta y \neq 0$; $m \neq 0, m \neq 1, m \in IR$ we obtain $f'(z)=$

$$\frac{m}{1+m^2}P_2(x,y,m)+\frac{1}{1+m^2}Q_2(x,y,m)+i\left[-\frac{1}{1+m^2}P_2(x,y,m)+\frac{m}{1+m^2}Q_2(x,y,m)\right] \quad (13)$$

and hence by the same argument, (8) is found from (13).

Corollary 2: Let $A \subset C$ be an open set and $f: A \to C$, $f(x+iy)=P(x,y)+iQ(x,y)$ be a complex valued function and P_1, Q_1, P_2, Q_2 continuous on A.Then for f to be holomorphic (analytical) on A it is necessary and sufficient that Cauchy-Riemann Equivalent Systems are satisfied.

Proof: It follows from Theorem 1-2.

4. APPLICATIONS (UYGULAMALAR)

Application 1. We show that the function $f: C \to C, f(z)=\bar{z}^2$ is not holomorphic(not analytical) at any point.

Solution 1. Let $f(x+iy)=P(x,y)+iQ(x,y)$.Then we easily see that $P(x,y)=x^2-y^2$, $Q(x,y)=-2xy$,

$P_1(x,y,m)=2x-2my$, $P_2(x,y,m)=2mx-2y$, $Q_1(x,y,m)=-2mx-2y$, $Q_2(x,y,m)=-2x-2my$.Hence, the following are clear :

$$\frac{1}{1+m^2}P_1(x,y,m)+\frac{m}{1+m^2}Q_1(x,y,m)=\frac{2x-2m^2x-4my}{1+m^2},$$

$$\frac{m}{1+m^2}P_2(x,y,m)+\frac{1}{1+m^2}Q_2(x,y,m)=\frac{-2x+2m^2x-4my}{1+m^2},$$

$$-\frac{m}{1+m^2}P_1(x,y,m)+\frac{1}{1+m^2}Q_1(x,y,m)=\frac{-4mx-2y+2m^2y}{1+m^2},$$

$$-\frac{1}{1+m^2}P_2(x,y,m)+\frac{m}{1+m^2}Q_2(x,y,m)=\frac{-4mx+2y-2m^2y}{1+m^2}.$$

$$\frac{1}{1+m^2}P_1(0,0,m)+\frac{m}{1+m^2}Q_1(0,0,m)+i\left[-\frac{m}{1+m^2}P_1(0,0,m)+\frac{1}{1+m^2}Q_1(0,0,m)\right]=0$$

Cauchy-Riemann Equivalent Systems (4) for $z=0=0+i0$.

Functions P_1, Q_1, P_2, Q_2 are continuous at $(0,0)$, and thus Theorem 2 we get $f'(0)=0$.But the system (4) is not satisffied for all $z=x+iy \neq 0$. So from Theorem 1 $f'(z)$ $(z \neq 0)$ does not exist. Therefore f is not holomorphic (not analytical) according to Definition 1.

Application 2. We show that the function $f: C \to C, f(z)=z^2$ is holomorphic(analytical) on C.

Solution 2. Let $f(x+iy) = P(x,y) + iQ(x,y)$. Then it is immediatly seen that $P(x,y) = x^2 - y^2$, $Q(x,y) = 2xy$, $P_1(x,y,m) = 2x - 2my$, $P_2(x,y,m) = 2mx - 2y$, $Q_1(x,y,m) = 2mx + 2y$, $Q_2(x,y,m) = 2x + 2my$. Then we find

$$\frac{1}{1+m^2}P_1(x,y,m) + \frac{m}{1+m^2}Q_1(x,y,m) = \frac{2x+2m^2x}{1+m^2}, \quad (14)$$

$$\frac{m}{1+m^2}P_2(x,y,m) + \frac{1}{1+m^2}Q_2(x,y,m) = \frac{2x+2m^2x}{1+m^2}, \quad (15)$$

$$-\frac{m}{1+m^2}P_1(x,y,m) + \frac{1}{1+m^2}Q_1(x,y,m) = \frac{2y+2m^2y}{1+m^2}, \quad (16)$$

$$-\frac{1}{1+m^2}P_2(x,y,m) + \frac{m}{1+m^2}Q_2(x,y,m) = \frac{2y-2m^2y}{1+m^2}. \quad (17)$$

From (14) and (15) the first equation of system (4) is satisffied for all $(x,y) \in IR^2$, by means of (16) and (17) the second equation of (4) is also satisfied. On the other hand, it is obvious that the functions P_1, Q_1, P_2, Q_2 are continuous at every $(x,y) \in IR^2$. Thus f is holomorphic(analytical) on C in view of Theorem 2, and we find

$$f'(z) = \frac{2x+2mx^2}{1+m^2} + i\frac{2y+2m^2y}{1+m^2} = \frac{(2x+i2y)}{1+m^2}(1+m^2) = 2z .$$

5. CONCLUSION AND DISCUSSION (SONUÇ VE TARTIŞMA)

Consider the notion of limit for functions with two real variables. Cauchy-Riemann Equivalent Systems (4) are the equations obtained as the limit by first approaching to the point (0,0) on the lines $\Delta y = m\Delta x, (m \neq 1, m \in IR)$ and then on the lines $\Delta x = m\Delta y, (m \neq 1, m \in IR)$. On the other hand, for $m = 0$ the systems (4) give Cauchy-Riemann equations (1). As limits, these equations are calculated to be the limit at the point (0,0) by approaching first on ox-axis and then on oy-axis.

If we thing of the counter-positive Corollary 1,we can say this :"A function that does not satisfy Cauchy-Riemann Equivalent Systems(4)at a point $z = x + iy \in A$ is not holomorphic(naturally, f has no derivative at z).

In the proof of the derivative $f'(z)$ of the form (8) we did need the Mean Value Theorem for functions with two variables. However the given conditions fulfill those of Mean Value Theorem. If we remove the continuous clause for the functions P_1, Q_1, P_2, Q_2, then it will be impossible to realize Theorem 2 for $m = 0$. Besides, we would not take in consideration the continuity of derivative function f'.

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