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A COMPARSION AMONG HOMOTOPY PERTURBATION METHOD AND THE DECOMPOSITION METHOD WITH THE VARIATIONAL ITERATION METHOD FOR HELMHOLTZ EQUATION

ABSTRACT
In this article, we implement a relatively new numerical technique and we present a comparative study among Homotopy perturbation method and Adomian decomposition method, the variational iterational method. These methods in applied mathematics can be an effective procedure to obtain for approximate solutions. The study outlines the significant features of the three methods. The analysis will be illustrated by investigating the homogeneous Helmholtz equation model problem. This paper is particularly concerned a numerical comparison with the Adomian decomposition and Homotopy perturbation method, the variational iterational method the numerical results demonstrate that the new methods are quite accurate and readily implemented.
        Keywords: Helmholtz Equation, Decomposition Method,
                  Homotopy Perturbation Method,
                  Variational Iterational Method

HELMHOLTZ DENKLEMİ İÇİN HOMOTOPY PERTÜRBATİON, DECOMPOSİTİON METHOD VE VARİATİONAL İTERASYON YÖNTEMLERİ ARASINDA BİR KARŞILAŞTIRMA

ÖZET
Bu makalede, nispeten yeni bir nümerik teknik uyguladık ve Homotopy Perturbation Methodu, Adomian Decomposition Method, ve variational iteration metodu arasında mukayeseli bir çalışma sunduk. Uygulamalı matematikteki bu metodlar yaklaşık çözümler elde etmek için etkili bir yöntem olabilir. Çalışma üç metodun önemli özelliklerini ana hatları ile göstermektedir. Analizler, Helmholtz denkleminin model problemi incelenerek örneklendirilecektir. Bu makale özellikle Homotopy Perturbation Methodu, Adomian Decomposition Method, ve variational iteration metodunun nümerik bir karşılaştırması ile ilgilidir. Nümerik sonuçlar yeni metodularin oldukça doğru ve hızlı uygulanabilir olduğunu göstermektedir.
        Anahtar Kelimeler: Helmholtz Denklemi, Decomposition Methodu,
                          Homotopy Perturbation Methodu,
                          Variational İteration Method

1. INTRODUCTION (GİRİŞ)

Partial differential equations which arise in real-world physical problems are often too complicated to be solved exactly. And even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be diffucult to interpret the outcome. Very recently, some promising approximate analytical solutions are proposed, such as Exp-function method [1 ve 2], Adomian decomposition method [3, 4, 5, 6 and 7], variational iteration method [8, 9 and 10] and homotopy-perturbation method [11, 12, 13, 14, 15 and 16]. Other methods are reviewed in Refs. [17 and 18].

HPM is the most effective and convenient on efor both linear and nonlinear equations. This method does not depend on a small parameter. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". HPM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. HPM was first proposed by He [11] and was successfully applied to various engineering problems [19, 20 and 21].

Recently, VIM is applied for exact solutions of Helmholtz Equation [22, 23, 24, 25, 26 and 27]. The aim of this work is this work is to employ HPM and ADM to obtain the exact solutions for Helmholtz Equations and to compare the results with those of VIM. Different from ADM, where specific algorithms are usually used to determine the Adomian polynomials, HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one.

Two-dimensional Helmholtz equation has the following form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \rho u = 0 \qquad 0 \le x, y \le 1 \quad \text{or} \quad \nabla^2 u - \rho u = 0, \tag{1}$$

with the boundary and initial conditions

$$u(0, y) = \phi_1(y), \quad u_x(0, y) = \phi_2(y)$$
$$u(x,0) = \phi_3(x), \quad u_y(x,0) = \phi_4(y) \tag{2}$$

where $\phi_1(y), \phi_2(y), \phi_3(x), \phi_4(y)$ are given functions. These equations appear in such diverse phenomena as: elastic waves in solid including vibrating string, bars membranes, sound or acoustics, electromagnetic waves and nuclear reactors

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this article, we study two dimensional Helmoltz equation. We solved it with 3 processes and we have given results. We hope that this result will contribute in phsichs, mecanic and Nuclear studies.

3. NUMERICAL METHODS (SAYISAL METOTLAR)
3.1. Fundamentals of the Homotopy-Perturbation Method (Homotopy-Perturbation Metodunun Esasları)

To illustrate the basic ideas of this method, we consider the following equation [11]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{4}$$

where *A* is a general differential operator, *B* a boundary operator, *f* (r) a known analytical function and *Γ* is the boundary of the domain *Ω*.

A can be divided into two parts which are *L* and *N*, where *L* is linear and *N* is nonlinear. Eq. (3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega,$$
(5)

Homotopy perturbation structure is shown as follows:

$$H(v, p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0,$$
(6)

where

$$v(r, p) : \Omega \times [0,1] \rightarrow \Re.$$
(7)

In Eq. (6), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (6) can be written as a power series in *p*, as following:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ...$$
(8)

and the best approximation for solution is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ...$$
(9)

The above convergence is discussed in [11].

3.2. Adomian Decomposition Method (Adomian Ayrışım Metodu)

The decomposition series method does not require this discretization and resulting massive computation. In this work we apply the second method to obtain analytic and approximate solutions of the equation from (1), and using the decomposition method, the equation (1) is approximated by the operators in the following form

$$L_x u = f(x, y) - \rho u - L_y u$$
(10)

where L_x and L_y symbolize $\dfrac{\partial^2}{\partial x^2}$ *and* $\dfrac{\partial^2}{\partial y^2}$, respectively. Assuming the invers of the operator L_x^{-1} exists, and it can conveniently be integrated with respect to x from 0 to x, i.e, $L_x^{-1} = \displaystyle\int_0^x \int_0^x (.)dxdx$, then applying the inverse operator L_x^{-1} to (10) yields

$$L_x^{-1} L_x u = L_x^{-1} f(x, y) - L_x^{-1} \rho u - L_x^{-1} L_y u$$
(11)

Therefore, it follows that

$$u(x, y) = u(0, y) + x u_x(0, y) + L_x^{-1} f(x, y) - L_x^{-1} \rho u - L_x^{-1} L_y u$$
(12)

The zeroth components is obtained, by using the initial condition, as

$$u_0 = u(x, y) + x u_x(0, y) + L_x^{-1}(f(x, y))$$
(13)

Which is defined by all terms that arise from the initial conditions. Thus, the unknown function u(x,y) is computed in terms of the components defined by the decomposition series given as

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$$
(14)

The remaining components u_n(x,y), n≥1, can be completely determined such that each term is computed by using the previous term. Since u_0 is know,

$$u_1 = -L_x^{-1}(\rho u_0) - L_x^{-1} L_y u_0,$$

$$u_2 = -L_x^{-1}(\rho u_1) - L_x^{-1}L_y u_1 ,$$

$$. \mathbb{N}$$

$$u_n = -L_x^{-1}(\rho u_{n-1}) - L_x^{-1}L_y u_{n-1} . \tag{15}$$

As a results, the series solutions is given by

$$u(x,y) = u_0 - \sum_{n=1}^{\infty}\left\{L_x^{-1}(\rho u_{n-1}) + L_x^{-1}L_y u_{n-1}\right\} , \tag{16}$$

where L_x^{-1} is the previously given integration operator. The solution u(x,y) must satisfy the requirements imposed by the initial conditions.

The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques. Moreover, the proposed method does not need discretization of the problem to obtain numerical results. We can evaluate the approximate solution ϕ_γ, by using the γ - term approximation. That is,

$$\phi_\gamma = \sum_{k=0}^{\gamma-1} u_k(x,y) \tag{17}$$

Where the components are produce as

$$\phi_1 = u_0(x,y)$$

$$\phi_2 = u_0(x,y) + u_1(x,y)$$

$$\phi_3 = u_0(x,y) + u_1(x,y) + u_2(x,y)$$

$$. . . \mathbb{N}$$

$$\phi_\gamma = u_0(x,y) + u_1(x,y) + u_2(x,y) + ... + u_{\gamma-1}(x,y)$$

3.3. The Variational Iterational Method
(Variyasyonel İterasyon Metodu)

Consider the differential equation

$$L_x u + Nu = g(t) . \tag{18}$$

Where L is a linear operator, N is a non-linear operator and g(t) is a known and Nonline analytical function. Ji Huan He has modified the above method into an iteration method[2,7] in the following way:

$$u_{n+1} = u_n + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi , \tag{19}$$

where λ is a general Lagrange's multipler, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u} = 0$.

It is obvious now that the main steps of He's variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution

$$u = \lim u_n, for(n \to \infty) . \tag{20}$$

In other words, correction functional (19) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

To give a clear overview of the methodology, we consider several examples in the following section.

4. EXPERIMENTAL EVALUATIONS (DENEYSEL DEĞERLENDİRMELER)

In this section, we apply the above described methods (Homotopy perturbation method and Adomian's decomposition method, variational iteration method) on some examples so that the comparisons are made numerically. We wish to emphasize that the purpose of the comparison is only to give the reader insight into the relative efficiencies of the three methods and not definitive comparisons. Also we re-emphasize that the basic intends of new method is user convenience and easy as opposed to speed of computation. In order to verify numerically whether the proposed methodology lead to higher accuracy chose two example which were selected to show the computational accuracy

Example 1: For comparison purposes, we consider a Helmholtz equation model problem in order to illustrate the technique discussed above. This problem is as follows;

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u(x,y) = 0 , \tag{21}$$

with in initial conditions

$$u_0(x,y) = (1+x)y + xCosh(y) \tag{22}$$

In order to solve Eq.(1), using HPM, we can construct a homotopy for this equation

$$(1-p)\left[\ddot{Y} - \ddot{u}_0\right] + p\left[\ddot{Y} + Y'' - Y\right] = 0 , \tag{23}$$

where $\ddot{Y} = \dfrac{\partial^2 Y}{\partial x^2}$, $Y'' = \dfrac{\partial^2 Y}{\partial y^2}$ and $p \in [0,1]$. With initial approximation $Y_0 = u_0 = (1+x)y + xCosh(y)$, suppose the solution of Eq. (23) has the form:

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + \ldots = \sum_{n=0}^{\infty} p^n Y_n(x,t) . \tag{24}$$

Then, substituting Eq. (24) into Eq. (23), and rearranging based on powers of p -terms, we obtain:

$$p^0 : \ddot{Y}_0 - \ddot{u}_0 = 0, \tag{25}$$

$$p^1 : \ddot{Y}_1 + \ddot{u}_0 + Y_0'' - Y_0 = 0, \tag{26}$$

$$p^2 : \ddot{Y}_2 + Y_1'' - Y_1 = 0, \tag{27}$$

$$p^3 : \ddot{Y}_3 + Y_2'' - Y_2 = 0, \tag{28}$$

$$p^4 : \ddot{Y}_4 + Y_3'' - Y_3 = 0, \tag{29}$$

$$\mathbb{N}$$

with solving Eqs. (25)-(29)

$$Y_0 = (1+x)y + xCosh(y) , \tag{30}$$

$$Y_1 = \left(\frac{x^2}{2} + \frac{x^3}{6}\right)y , \tag{31}$$

$$Y_2 = \left(\frac{x^4}{24} + \frac{x^5}{120} \right) y \, , \tag{32}$$

$$Y_3 = \left(\frac{x^6}{720} + \frac{x^7}{5040} \right) y \, , \tag{33}$$

$$Y_4 = \left(\frac{x^8}{40320} + \frac{x^9}{362880} \right) y \, , \tag{34}$$

$$\mathsf{N}$$

the above terms of the series (23) could calculated. When we consider the series (23) with the terms (30)-(34) and suppose $p=1$, we obtain approximation solution of Eq. (22) as following:

$$u(x,t) = (1+x)y + xCosh(y) + \left(\frac{x^2}{2} + \frac{x^3}{6} \right) y + \left(\frac{x^4}{24} + \frac{x^5}{120} \right) y + \left(\frac{x^6}{720} + \frac{x^7}{5040} \right) y +$$

$$+ \left(\frac{x^8}{40320} + \frac{x^9}{362880} \right) y + \mathsf{L} \tag{35}$$

As a result, the components $Y_0, Y_1, Y_2, \ldots$ are identified and the series solution thus entirely determined.

In order to solve this equation by using the decomposition method, we simply take the equation in an operator form.
Following the same manner as given by equation (13) to find the zeroth component of u_0 as

$$u_0 = u(0,y) + u_x(0,y) + L_x^{-1} f(x,y) \tag{36}$$

$$u_0 = y + xy + x\cosh(y) = (1+x)y + x\cosh(y)$$

and remaining components u_1, u_2, u_3, etc. We are computed by a recursive scheme directly by hand using (15). Some of the symbolically computed components are as follows

$$u_1 = L_x^{-1} u_0 - L_x L_y u_0$$

$$= \int_0^x \int_0^x [(1+x)y + x\cosh y] dx dx - \int_0^x \int_0^x [\frac{\partial^2}{\partial y^2}(1+x)y + x\cosh y] dx dx$$

$$= y\left(\frac{x^2}{2} + \frac{x^3}{3!} \right) + \frac{x^3}{3!}\cosh y - \frac{x^3}{3!}\cosh(y) \tag{37}$$

$$= y\left(\frac{x^2}{2!} + \frac{x^3}{3!} \right)$$

$$u_2 = L_x^{-1} u_1 - L_x^{-1} L_y u_1 = y\left(\frac{x^4}{4!} + \frac{x^5}{5!} \right)$$

$$u_3 = L_x^{-1} u_2 - L_x^{-1} L_y u_2 = y\left(\frac{x^6}{6!} + \frac{x^7}{7!} \right)$$

$$u_4 = L_x^{-1} u_3 - L_x^{-1} L_y u_3 = y\left(\frac{x^8}{8!} + \frac{x^9}{9!} \right)$$

In this manner, four component of the decomposition series were obtained of which u(x,y) was evaluated to have the following expansion.

$$u(x, y) = u_0 + u_1 + u_2 + u_3 + u_4 + \mathrm{K}$$

$$u(x, y) = y \cdot \left\{ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \Lambda \right\} + x \cosh(y)$$

Continuing the expansion to the last term, it may be proved that the section of the decomposition series (14) as

$$u(x, y) = y \exp(x) + x \cosh(y)$$

In order to solve (21) equation by using the variational iteration method, we simply take the equation in an operator form.

$$L_x u + N u = g(t) \, .$$

and

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda_1 \left(\frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} + \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial \tau^2} - \tilde{u}_n(x, \tau) \right) d\tau, \tag{38}$$

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda_2 \left(\frac{\partial^2 \tilde{u}_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi, y)}{\partial y^2} - \tilde{u}_n(\xi, y) \right) d\xi, \tag{39}$$

where $\lambda_i (i = 1,2)$ are general Lagrange multipliers [1] ,and can be identified optimally via the variational theory $[1, 8-15]$, u_0 is an initial approximation or trial- function with possible unknowns, $\tilde{u}_n$ are considered as restricted variations $[2, 7-10]$,ie. $\delta \tilde{u} = 0$.

Making the correction functional (39) stationary, noticing that $\delta \tilde{u} = 0$,

$$\delta u_{n+1}(x, y) = \delta u_n(x, y) + \delta \int_0^x \lambda_2 \left(\frac{\partial^2 u_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi, y)}{\partial y^2} - \tilde{u}_n(\xi, y) \right) d\xi, \tag{40}$$

$$\delta u_{n+1}(x, y) = \delta u_n(x, y) + \delta \int_0^x \lambda_2 \left(\frac{\partial^2 u_n(\xi, y)}{\partial \xi^2} \right) d\xi, \tag{41}$$

$$\delta u_{n+1}(x, y) = \delta u_n(x, y)(1 - \lambda'_2(t)) + \delta \left(\frac{\partial u_n(\xi, y)}{\partial \xi^2} \lambda_2(\xi) \right) + \int_0^x \delta u_n(\xi, y) - \lambda''_2(\xi) d\xi = 0,$$

yields the following stationary conditions

$$\delta u_n : 1 - \lambda'_2(\xi) = 0 \, ,$$

$$\delta \frac{\partial u_n}{\partial \xi} : \lambda_2(\xi) = 0 \, ,$$

$$\delta u_n : \lambda''_2(\xi) = 0$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda_2(\xi) = \xi - x$$

and by the same manipulation ,we have $\lambda_1(\tau) = \tau - y$.

As a result, we obtain the following iteration formulae in y - and x- directions:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\tau - y)\left(\frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - u_n(x,\tau)\right) d\tau,$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\tau - y)\left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n(\xi,y)\right) d\xi, \qquad (42)$$

Now we begin with an arbitary initial approximation : $u_0 = A + Bx,$ where A and B are constants in x to be determined by using the initial conditions (22),thus we have

$$u_0 = y(1+x) + x(\cosh(y)).$$

By the above variational iteration formula in x-direction (42), we can obtain following result:

$$u_1(x,y) = u_0(e,y) + \int_0^x (\xi - x)\left(\frac{\partial^2 u_0(\xi,x)}{\partial \xi^2} + \frac{\partial^2 u_0(\xi,x)}{\partial y^2} - u_0(\xi,x)\right) d\xi, \qquad (43)$$

$$u_1(x,y) = y(1+x) + x(\cosh(y))$$

$$\int_0^x (\xi - x)\left(\frac{\partial^2 y(1+\xi) + \xi(\cosh(y))}{\partial \xi^2} + \frac{\partial^2 y(1+\xi) + \xi(\cosh(y))}{\partial y^2} - (y(1+\xi) + \xi(\cosh(y)))\right) d\xi,$$

thus we have

$$u_1(x,y) = y(1+x) + x\cosh(y) - y\left(-\frac{1}{2}x^2 - \frac{1}{6}x^3\right).$$

Continuing in this manner, we obtain the first few components of $u_n(x,y)$

$$u_0 = y(1+x) + x(\cosh(y)),$$

$$u_1(x,y) = y(1+x) + x\cosh(y) + y\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right),$$

$$u_2(x,y) = y(1+x) + x\cosh(y) + y\left(\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right),$$

$$\Lambda$$

and so on ,in the same manner the rest of components of the iteration formula (42) were obtained using the Mathematica Package.

$$u(x,y) = y\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \Lambda\right\} + x\cosh(y)$$

The Solution $u(x,y)$ in a closed form is

$$u(x,y) = y\exp(x) + x\cosh(y).$$

The approximation can also be obtained by y-direction or by alternating use of x-and y-directions iterations formulae.

Example 2:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - 5u(x,y) = 0, \qquad (44)$$

with in initial conditions

$$u(0,y) = 0 , \quad u_x(0,y) = 3\sinh(2y) \qquad (45)$$

In the same way as Example 1 ,using (13) to find the zeroth component of u_0 as

$$u_0 = 3x \sinh(2y) \qquad (46)$$

and remaining components u_1, u_2, u_3, etc. We are computed by a recursive scheme directly by hand using (15). Some of the symbolically computed components are as follows

$$u_1 = -5L_x^{-1}u_0 - L_x L_y u_0 = \frac{(3x)^3}{3!}\sinh(2y), \qquad (47)$$

$$u_2 = -5L_x^{-1}u_1 - L_x^{-1}L_y u_1 = \frac{(3x)^5}{5!}\sinh(2y),$$

$$u_3 = -5L_x^{-1}u_2 - L_x^{-1}L_y u_2 = \frac{(3x)^7}{7!}\sinh(2y),$$

$$\mathsf{N}$$

In this manner, four component of the decomposition series were obtained of which u(x,y) was evaluated to have the following expansion.

$$u(x,y) = u_0 + u_1 + u_2 + u_3 + u_4 + \mathrm{K}$$

$$u(x,y) = 3x\sinh(2y) - \frac{(3x)^3}{3!}\sinh(2y) + \frac{(3x)^5}{5!}\sinh(2y) - \frac{(3x)^7}{7!}\sinh(2y) + ...$$

Continuing the expansion to the last term, it may be proved that the section of the decomposition series (14) as

$$u(x,y) = \sin(3x)\sinh(2y)$$

which is exactly the same as those by homotopy perturpation method.

In order to solve (44) equation by using the variational iteration method, its correction variational functionals in y-and x-directions can be expressed, respectively, as follows:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda_1 \left(\frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial \tau^2} - 5\tilde{u}_n(x,\tau) \right) d\tau, \qquad (48)$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda_2 \left(\frac{\partial^2 \tilde{u}_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi,y)}{\partial y^2} - 5\tilde{u}_n(\xi,y) \right) d\xi, \qquad (49)$$

Following the same procedure as the previous example. The Lagrange multipliers, therefore, can be readily identified,

$$\lambda_1(\tau) = \tau - y , \ \lambda_2(\xi) = \xi - x$$

As a result, we obtain the following iteration formulae in y -and x-directions:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\tau - y) \left(\frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - u_n(x,\tau) \right) d\tau,$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\tau - y) \left(\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,y)}{\partial y^2} - u_n(\xi,y) \right) d\xi, \qquad (50)$$

Now we begin with an arbitary initial approximation : $u_0 = A + Bx,$ where A and B are constants in x to be determined by using the initial conditions (45),thus we have

$$u_0 = 3x\sinh(2y) \ .$$

By the above variational iteration formula in x-direction (49), we can obtain following result:
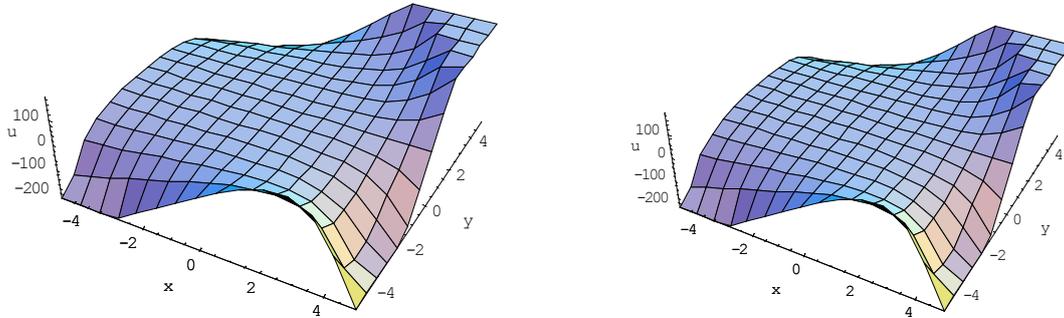
$$u_1 = \sinh(2y)\left(3x - \frac{(3x)^3}{3!}\right),$$

$$u_2 = \sinh(2y)\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!}\right),$$

$$\vdots$$

and so on ,in the same manner the rest of components of the iteration formula (42) were obtained using the Mathematica Package.

$$u(x,y) = 3x\sinh(2y) - \frac{(3x)^3}{3!}\sinh(2y) + \frac{(3x)^5}{5!}\sinh(2y) - \frac{(3x)^7}{7!}\sinh(2y) + ...$$

The Solution $u(x,y)$ in a closed form is

$$u(x,y) = \sin(3x)\sinh(2y) .$$

(a):Exact solution (Tam çözüm) (b):Approximation solution (Yaklaşık çözüm)

Figure 1a. The numerical results for Y_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(21) by means of HPM

(Şekil 1a. HPM yoluyladenklem 21'in başlangıç şartı alındığında y=0.05 U(x,Y)nın analitik çözümüyle Y4 için elde edilen sayısal sonuçların karşılaştırılması)
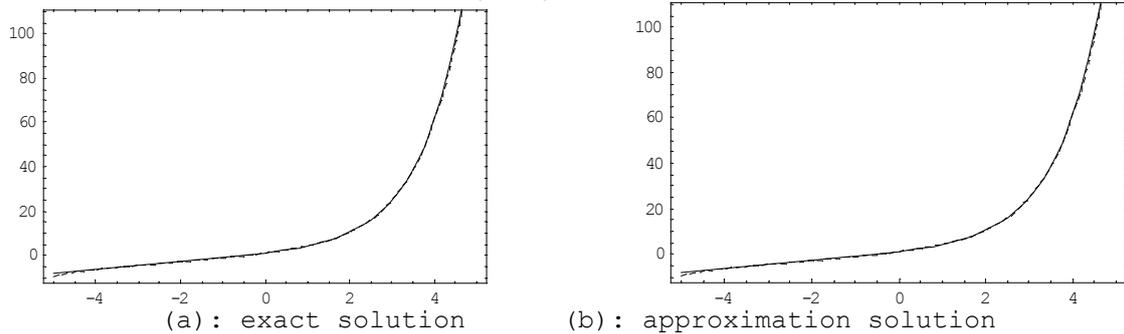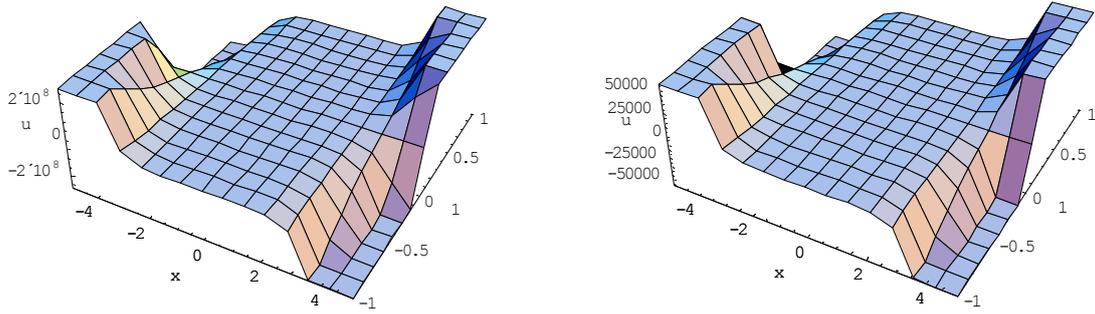

(a): exact solution   (b): approximation solution

Figure 1b. The plots of the numerical results for Y_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(21) by means of HPM

Şekil 1b. HPM yoluyla denklem 21'in başlangıç şartı alındığında y=0.05 $u(x,y)$ için analitik çözümüyle Y4 için elde edilen sayısal sonuçların çizilmesi

Table 1. Absolute Error For the Example 1 using the HPM method
(Tablo 1. HPM metodu kullanarak örnek 1 için hatalar)

y	x			
	0.2	0.4	0.6	0.8
0.1	2.94209E-15	2.99827 E-12	1.76194E-10	3.18942E-09
0.2	5.82867E-15	5.99654 E-12	3.52387E-10	6.37885E-09
0.3	8.65974E-15	8.99469 E-12	5.28581E-10	9.56827E-09
0.4	1.16573E-15	1.19931 E-12	7.04775E-10	1.27577E-09

(a):Exact solution (Tam çözüm) (b):Approximation solution (Yaklaşık çözüm)

Figure 2a. The numerical results for u_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(44) by means of ADM.

Şekil 2a. ADM yoluyla denklem 44'ün başlangıç şartı ile y=0.05 alındığında $u(x,y)$ için analitik çözümüyle U4 için elde edilen sayısal sonuçların karşılaştırılması
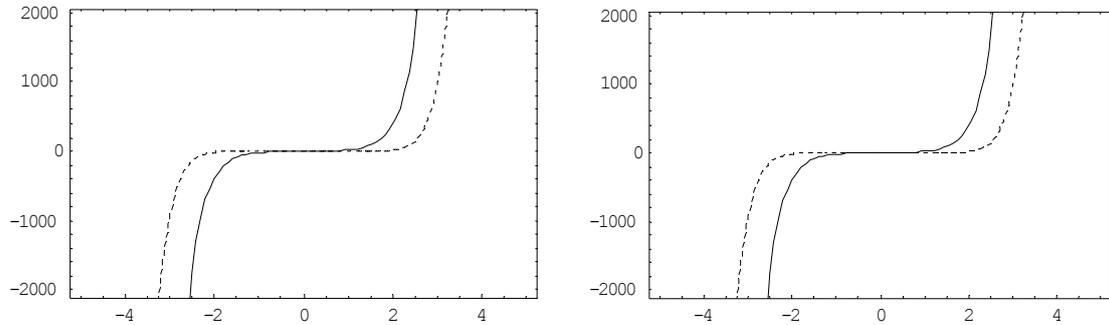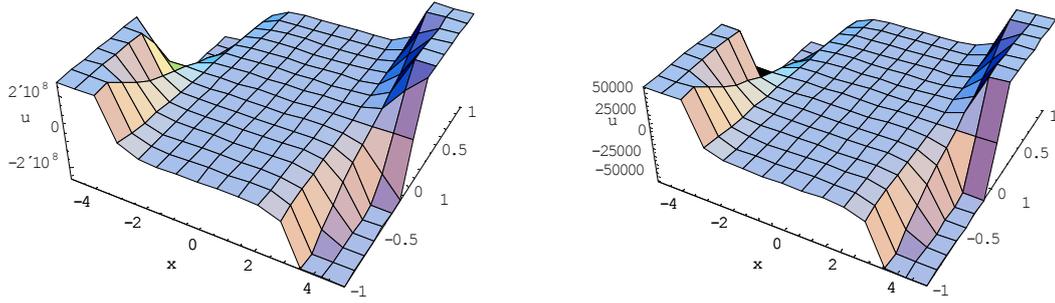

(a):Exact solution (Tam çözüm) (b):Approximation solution (Yaklaşık çözüm)

Figure 2b. The plots of the numerical results for u_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(44) by means of ADM

Şekil 2b. ADM yoluyla denklem 44'ün başlangıç şartı y=0.05 alındığında $u(x,y)$ için analitik çözüm ve U4 için elde edilen sayısal sonuçların çizilmesi

Table 2. Absolute error for the Example 2 using the ADM method
(Tablo 2. ADM metodu kullanarak örnek 1 için hatalar)

y	x 0.2	0.4	0.6	0.8
0.1	0.01449E-15	0.11625 E-12	0.39629E-10	0.96448E-09
0.2	0.02957E-15	0.23717 E-12	0.80848E-10	1.96767E-09
0.3	0.04584E-15	0.36761 E-12	1.25313E-10	3.04982E-09
0.4	0.06395E-15	0.51281 E-12	1.74807E-10	4.25438E-09

(a):Exact solution (Tam çözüm) (b):Approximation solution (Yaklaşık çözüm)
Figure 3a. The numerical results for u_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(44) by means of VIM.
Şekil 3a. VIM yoluyla denklem 44'ün başlangıç şartı ile y=0.05 alındığında $u(x,y)$ için analitik çözümüyle U4 için elde edilen sayısal sonuçların karşılaştırılması
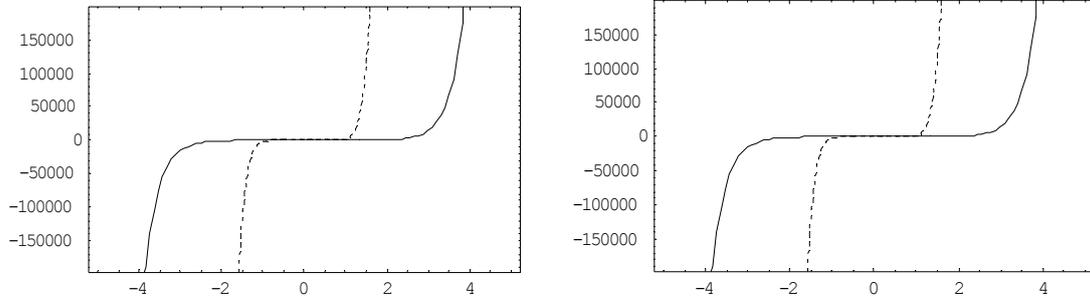
(a):Exact solution (Tam çözüm) (b):Approximation solution (Yaklaşık çözüm)
Figure 3b. The plots of the numerical results for u_4 in comparison with the analytic solution $u(x,y)$ when y=0.05 with initial condition of Eq.(44) by means of VIM
Şekil 3b. VIM yoluyla denklem 44'ün başlangıç şartı y=0.05 alındığında $u(x,y)$ için analitik çözüm ve U4 için elde edilen sayısal sonuçların çizilmesi

Table 3. Absolute error for the example 2 using the VIM method
(Tablo 3. ADM metodu kullanarak örnek 2 için hatalar)

y	x 0.2	0.4	0.6	0.8
0.1	0.07356E-15	0.30242 E-12	0.53881E-10	-4.93566E-09
0.2	0.15008E-15	0.61698 E-12	1.09926E-10	-10.0694E-09
0.3	0.23226E-15	0.95630 E-12	1.70381E-10	-15.6073E-09
0.4	0.32449E-15	1.33401 E-12	2.37675E-10	-21.7715E-09

5. COMPARISON AMONG HPM, VIM AND ADM
(HPM, VIM VE ADM METODLARININ KARŞILAŞTIRILMASI)
It can be seen from the examples studied, that:
- Comparison among HPM, VIM and ADM shows that although the results of these methods when applied to the Helmholtz equation are the same, HPM does not require specific algorithms and complex calculations, such as ADM or construction of correction functionals using general Lagrange multipliers, such as VIM and is much easier and more convenient than ADM and VIM.
- HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one. But in nonlinear problems, we encounter difficulties to calculate the so-called Adomian polynomials, when using ADM. Also, optimal identification of Lagrange multipliers via the variational theory can be difficult in VIM.

6. CONCLUSION (SONUÇ)
In this Letter, we have successfully developed HPM and ADM to obtain the exact solutions of Helmholtz equation. The results are then compared with those of VIM. It is apparently seen that these methods are very powerful and efficient techniques for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solutions. The solutions obtained show that the results of these methods are in agreement but HPM is an easy and convenient one.

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