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MATRIX ALGEBRAS IN $E^4_{\alpha\beta}$ AND THEIR APPLICATIONS

ABSTRACT

By Hamilton operators, generalized quaternions have been expressed in terms of 4×4 matrices. In this paper, geometric applications of these matrices in generalized 4-space $E^4_{\alpha\beta}$ are given. We also show that the set of these matrices with the group operation of matrix multiplication is Lie group of 6-dimension.

Keywords: De Moivre’s Formula, Homothetic Motion, Lie Group, Rotation, Matrix

$E^4_{\alpha\beta}$’DE MATRIS CEBİRİ VE UYGULAMALARI

ÖZET

Hamilton operatorleri ile bir gelişmiş kuaterniyon 4×4 matrisleri ile gösterilmiştir. Bu makalede matrislerin uygulamaları gelişmiş uzay’da verilmiştir. Ayrıca, bu matrislerin kümesi matris çarpım ile altı boyutlu bir Lie grubu oluşturulmuştur.

Anahtar Kelimeler: De Moivre’s Formülü, Homothetik Hareket, Lie Grubu, Dönme, Matris

1. INTRRODUCTION (GİRİŞ)

The quaternions are commonly used in physics, chemistry, robotics, mechanics and electronics. A brief introduction of the generalized quaternions is provided in [5], the subject which have investigated in algebra [6]. The generalized quaternion algebra is an associative and non-commutative 4-dimensional Clifford algebra. Recently, we have studied the generalized quaternion and some of their algebraic properties [1]. A matrix corresponding to Hamilton operators, defined for the generalized quaternions, determines a Homothetic motion and also can be used to described the rotation in 4-dimensional space in $E^4_{\alpha\beta}$. In addition, by De-Moivre's formula every power of this matrix is immediately obtained.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this work, after a review of some fundamental properties of the generalized quaternions, we study the applications of matrices corresponding to generalized quaternions. The set of these matrices with the group operation of matrix multiplication is Lie group of 6-dimension. Finally, we give some example for the purpose of more clarification.

3. EXPERIMENTAL METHOD-PROCESS (DENEYSEL ÇALIŞMA)

In this section, we define a new inner product and give a brief summary of the generalized quaternions.

- **Definition (Tanım) 1:** Let $\vec{u}=(u_1,u_2,u_3,u_4),\ \vec{v}=(v_1,v_2,v_3,v_4)\in \mathbb{R}^4.$ If $\alpha,\beta\in\mathbb{R}^+,$ the generalized inner product of $\vec{u}$ and $\vec{v}$ is defined by
$$\langle\vec{u},\vec{v}\rangle = u_1v_1+\alpha u_2v_2+\beta u_3v_3+\alpha\beta u_4v_4.$$

It could be written

$$\langle\vec{u},\vec{v}\rangle = u^T\begin{bmatrix}1&0&0&0\\0&\alpha&0&0\\0&0&\beta&0\\0&0&0&\alpha\beta\end{bmatrix}=u^T G v.$$

Also, if $\alpha>0,\beta<0$, $\langle\vec{u},\vec{v}\rangle$ is called the generalized Lorentz an inner product. The vector space on $\mathbb{R}^4$ equipped with the generalized inner product is called 4-dimensional generalized space and denoted by $E^4_{\alpha\beta}$.

- **Definition (Tanım) 2:** A matrix A is called a quasi-orthogonal matrix if $A^T\varepsilon A=\varepsilon$ and $\det A=1$ where

$$\varepsilon=\begin{bmatrix}1&0&0&0\\0&\alpha&0&0\\0&0&\beta&0\\0&0&0&\alpha\beta\end{bmatrix},$$

And $\alpha,\beta\in\mathbb{R}.$ the set of all quasi-orthogonal matrices, $QO(3),$ with the operation of matrix multiplication is called rotations group in 4-spaces $E^4_{\alpha\beta}$ [2].

- **Definition (Tanım) 3:** A generalized quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

Where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2} = -\alpha, \quad \vec{j}^{\,2} = -\beta, \quad \vec{k}^{\,2} = -\alpha\beta,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k}, \qquad \alpha, \beta \in \mathrm{R}.$$

The set of all generalized quaternions is denoted by $\mathrm{H}_{\alpha\beta}$. We express the basic operations in the $\vec{i}, \vec{j}, \vec{k}$ form. The addition becomes as

$$(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= (a_0 + b_0) + (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$$

and the multiplication as

$$(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3)$$
$$+ (a_1b_0 + a_0b_1 - \beta a_3b_2 + \beta a_2b_3)\vec{i}$$
$$+ (a_2b_0 + \alpha a_3b_1 + a_0b_2 - \alpha a_1b_3)\vec{j}$$
$$+ (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\vec{k}.$$

Given $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_0 is called the *scalar part* of q, denoted by $S(q) = a_0$,

and $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is called the *vector part* of q, denoted by

$$\vec{V}(q) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

The *conjugate* of q is

$$\overline{q} = a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

The *norm* of q is

$$N_q = \overline{q}q = q\overline{q} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2.$$

The *inverse* of q with $N_q \neq 0$, is

$$q^{-1} = \frac{1}{N_q}\overline{q}.$$

Clearly $qq^{-1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}$. Note also that $\overline{qp} = \overline{p}\,\overline{q}$ and $(qp)^{-1} = p^{-1}q^{-1}$ [1].

- **Definition (Tanım) 4:** A Lie group is a group G, equipped with a manifold structure such that the group operations

$$\text{Mult: } G \times G \to G, \ (g_1, g_2) \to g_1g_2$$

$$\text{Inv: } G \to G, \ g \to g^{-1} \ \text{ are smooth.}$$

For example, the general linear group

$$\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathrm{R}) : \det A \neq 0\}$$

is an open subset of $\mathrm{Mat}_n(\mathrm{R})$, hence a sub manifold, and the smoothness of group multiplication follows since the product map for $\mathrm{Mat}_n(\mathrm{R})$, is obviously smooth[4].

- **Definition (Tanım) 5:** Left multiplication by a generalized quaternion q is a linear map

$$\overset{+}{h}_q(x) = qx, \qquad x \in \mathrm{H}_{\alpha\beta},$$

from the quaternions into the quaternions, as is right multiplication,

$$\overset{-}{h}_q(x) = xq \qquad x \in \mathrm{H}_{\alpha\beta}.$$

Since these multiplications are linear maps from four dimensional vector space into itself, we can find a matrix representation of each.

The Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \qquad (1)$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \qquad (2)$$

- **Theorem (Teorem) 1:** If q and p are two real quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in equations (1) and (2), respectively, then the following identities hold:

1. $q = p \Leftrightarrow \overset{+}{H}(q) = \overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q) = \overset{-}{H}(p).$
2. $\overset{+}{H}(q+p) = \overset{+}{H}(q) + \overset{+}{H}(p), \quad \overset{-}{H}(q+p) = \overset{-}{H}(q) + \overset{-}{H}(p).$
3. $\overset{+}{H}(\lambda q) = \lambda \overset{+}{H}(q), \; \overset{-}{H}(\lambda q) = \lambda \overset{-}{H}(q).$
4. $\overset{+}{H}(qp) = \overset{+}{H}(q)\overset{+}{H}(p), \quad \overset{-}{H}(qp) = \overset{-}{H}(p)\overset{-}{H}(q).$
5. $\overset{+}{H}(q^{-1}) = \left[\overset{+}{H}(q)\right]^{-1}, \quad \overset{-}{H}(q^{-1}) = \left[\overset{-}{H}(q)\right]^{-1}, \quad (N_q)^2 \neq 0.$
6. $\overset{+}{H}(\bar{q}) = \left[\overset{+}{H}(q)\right]^{T}, \quad \overset{-}{H}(\bar{q}) = \left[\overset{-}{H}(q)\right]^{T}.$

7. $\det\left[\overset{+}{H}(q)\right]=(N_q)^2,\quad \det\left[\overset{-}{H}(q)\right]=(N_q)^2.$

8. $tr\left[\overset{+}{H}(q)\right]=4a_0,\quad tr\left[\overset{-}{H}(q)\right]=4a_0.$

Proof: The proof can be found in [1].

- **Theorem (Teorem) 2:** The map

$$\psi:(\mathrm{H}_{\alpha\beta},+,.)\rightarrow(\mathrm{M}_{(4,\mathrm{R})},\oplus,\otimes)$$

defined as

$$\psi(a_0+a_1\vec{\mathrm{i}}+a_2\vec{\mathrm{j}}+a_3\vec{\mathrm{k}})\mapsto\begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

is an isomorphism of algebras.

Proof: See [7] for a similar proof.

- **Theorem (Teorem) 3:** Let

$$\Omega=\left\{A=\begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & -\beta x_3 & \beta x_2 \\ x_2 & \alpha x_3 & x_0 & -\alpha x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix}: x_i\in\mathbb{R},\ 1\le i\le 4\right\}.$$

Then Ω is a differentiable manifold.

Proof: Let us consider the following function:

$$f:\Omega\rightarrow\mathrm{E}^4_{\alpha\beta}$$
$$A\rightarrow f(A)=(x_0,x_1,x_2,x_3),$$

f is one-to-one and on to function, and since $f(\Omega)=\mathrm{E}^4_{\alpha\beta}$ then $f(\Omega)$ is open set. Furthermore, since x_i, $i=1,2,3,4$ are continuously, and then f,f^{-1} are continuously functions. $\{(f,\Omega)\}$ is a differentiable atlas with one chart, so Ω is a differentiable manifold.

- **Theorem (Teorem) 4:** Under matrix multiplication, $\Omega^*=\Omega-\{0\}$ is a Lie group of dimension 6.

Proof: Ω^* under matrix multiplication is a matrix group. Also, Ω^* is a sub manifold of Ω. Furthermore, the group operations

$$\text{Mult}:\Omega^*\times\Omega^*\rightarrow\Omega^*,\quad (A_1,A_2)\rightarrow A_1A_2$$
$$\text{Inv}:\ \Omega^*\rightarrow\Omega^*,\ A\rightarrow A^{-1}$$

are obviously smooth.

Let us find the left algebra, i.e., the tangent space at the unit elementary, $T_e(\Omega^*)$.Let us consider the map $\Phi : E^4_{\alpha\beta} - \{0\} \to \Omega^*$, defined by

$$\Phi(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & -\beta x_3 & \beta x_2 \\ x_2 & \alpha x_3 & x_0 & -\alpha x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix}.$$

For point $p = (1,0,0,0)$, $\Phi(p) = e = I_4$ is identity element of Ω^*.

- **Theorem (Teorem) 5:** Consider the map $\Phi_*|_p : T_p E^4_{\alpha\beta} \to T_{\Phi(p)} \Omega^*$. This map is one-to-one.

Proof: If we show that $\Phi_*(V_p) = 0 \Rightarrow V_p = 0$ then Theorem is proved. For every $V_p \in T_p E^4_{\alpha\beta}$, we have $V_p = a_1 \partial/\partial x_1 + a_2 \partial/\partial x_2 + a_3 \partial/\partial x_3 + a_4 \partial/\partial x_4$, so

$$\Phi_*|_p(V_p) = \begin{bmatrix} V_p[x_1] & V_p[-\alpha x_2] & V_p[-\beta x_3] & V_p[-\alpha\beta x_4] \\ V_p[x_2] & V_p[x_1] & V_p[-\beta x_4] & V_p[\beta x_3] \\ V_p[x_3] & V_p[\alpha x_4] & V_p[x_1] & V_p[-\alpha x_2] \\ V_p[x_4] & V_p[-x_3] & V_p[x_2] & V_p[x_1] \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & -\alpha a_2 & -\beta a_3 & -\alpha\beta a_4 \\ a_2 & a_1 & -\beta a_4 & \beta a_3 \\ a_3 & \alpha a_4 & a_1 & -\alpha a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix} = [0],$$

Then $a_1 = a_2 = a_3 = a_4 = 0$. So, Φ_* an injective map. On the other hand, $\dim T_p E^4_{\alpha\beta} = \dim T_e \Omega^* = 4$, thus, Φ_* is a linear isomorphism. Since every linear isomorphism maps any basis of space to another one. So we determine the basis of space $T_{\Phi(p)} \Omega^*$.

It is obviously that $T_p E^4_{\alpha\beta} = Sp\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4\}$. We find the image of this basis under the map Φ_*.

$$\Phi_*|_p\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_1} \begin{bmatrix} x_1 & -\alpha x_2 & -\beta x_3 & -\alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & \beta x_3 \\ x_3 & \alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_*|_p\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_2} \begin{bmatrix} x_1 & -\alpha x_2 & -\beta x_3 & -\alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & \beta x_3 \\ x_3 & \alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Phi_*|_p\left(\frac{\partial}{\partial x_3}\right)=\frac{\partial}{\partial x_3}\begin{bmatrix} x_1 & -\alpha x_2 & -\beta x_3 & -\alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & \beta x_3 \\ x_3 & \alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\Phi_*|_p\left(\frac{\partial}{\partial x_4}\right)=\frac{\partial}{\partial x_4}\begin{bmatrix} x_1 & -\alpha x_2 & -\beta x_3 & -\alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & \beta x_3 \\ x_3 & \alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

So we have

$$\mathrm{T}_e\,\Omega^*=\mathrm{Sp}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right\}.$$

- **Theorem (Teorem) 6:** (De Moivre's formula) Let matrix

$$A=\begin{bmatrix} \cos\theta & -\alpha u_1\sin\theta & -\beta u_2\sin\theta & -\alpha\beta u_3\sin\theta \\ u_1\sin\theta & \cos\theta & -\beta u_3\sin\theta & \beta u_2\sin\theta \\ u_2\sin\theta & \alpha u_3\sin\theta & \cos\theta & -\alpha u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix}, \qquad (3)$$

correspond to generalized quaternion q. The n-th power of the matrix A reads as

$$A^n=\begin{bmatrix} \cos n\theta & -\alpha u_1\sin n\theta & -\beta u_2\sin n\theta & -\alpha\beta u_3\sin n\theta \\ u_1\sin n\theta & \cos n\theta & -\beta u_3\sin n\theta & \beta u_2\sin n\theta \\ u_2\sin n\theta & \alpha u_3\sin n\theta & \cos n\theta & -\alpha u_1\sin n\theta \\ u_3\sin n\theta & -u_2\sin n\theta & u_1\sin n\theta & \cos n\theta \end{bmatrix}.$$

Proof: The proof is easily followed by induction on n.

- **Example (Örnek) 1:** Let $q=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}}\right)=\cos\frac{\pi}{3}+\vec{u}\sin\frac{\pi}{3}$ be a unit generalized quarternion. The matrix corresponding to this quaternion is

$$A=\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{\alpha}}{2} & -\frac{\sqrt{\beta}}{2} & -\frac{\sqrt{\alpha\beta}}{2} \\ \frac{1}{2\sqrt{\alpha}} & \frac{1}{2} & -\frac{\sqrt{\beta}}{2\sqrt{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{2\sqrt{\beta}} & \frac{\sqrt{\alpha}}{2\sqrt{\beta}} & \frac{1}{2} & -\frac{\sqrt{\alpha}}{2} \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{2\sqrt{\beta}} & \frac{1}{2\sqrt{\alpha}} & \frac{1}{2} \end{bmatrix},$$

every power of this matrix with the aid of Theorem 6 is found to be expressible similarly, for example, *28-this*

$$A^{28}=\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{\alpha}}{2} & \frac{\sqrt{\beta}}{2} & \frac{\sqrt{\alpha\beta}}{2} \\ -\frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} & \frac{\sqrt{\beta}}{2\sqrt{\alpha}} & -\frac{\sqrt{\beta}}{2} \\ -\frac{1}{2\sqrt{\beta}} & -\frac{\sqrt{\alpha}}{2\sqrt{\beta}} & -\frac{1}{2} & \frac{\sqrt{\alpha}}{2} \\ -\frac{1}{2\sqrt{\alpha\beta}} & \frac{1}{2\sqrt{\beta}} & -\frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} \end{bmatrix}.$$

- **Definition (Tanım) 6: (Euler's formula)** Let

$$A=\begin{bmatrix} 0 & -\alpha u_1 & -\beta u_2 & -\alpha\beta u_3 \\ u_1 & 0 & -\beta u_3 & \beta u_2 \\ u_2 & \alpha u_3 & 0 & -\alpha u_1 \\ u_3 & -u_2 & u_1 & 0 \end{bmatrix},$$

be a real matrix. One immediately finds $A^2=-I_4$. We have a natural generalization of Euler's formula for matrix A;

$$\begin{aligned} e^{A\theta} &= I_4 + A\theta + \frac{(A\theta)^2}{2!} + \frac{(A\theta)^3}{3!} + \frac{(A\theta)^4}{4!} + \dots \\ &= I_4(1-\frac{\theta^2}{2!}+\frac{\theta 4}{4!}-\dots)+A(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots) \\ &= I_4\cos\theta + A.\sin\theta \\ &= I_4\cos\theta + \begin{bmatrix} 0 & -\alpha u_1 & -\beta u_2 & -\alpha\beta u_3 \\ u_1 & 0 & -\beta u_3 & \beta u_2 \\ u_2 & \alpha u_3 & 0 & -\alpha u_1 \\ u_3 & -u_2 & u_1 & 0 \end{bmatrix}.\sin\theta \\ &= \begin{bmatrix} \cos\theta & -\alpha u_1\sin\theta & -\beta u_2\sin\theta & -\alpha\beta u_3\sin\theta \\ u_1\sin\theta & \cos\theta & -\beta u_3\sin\theta & \beta u_2\sin\theta \\ u_2\sin\theta & -\alpha u_3\sin\theta & \cos\theta & -\alpha u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

Let $q=\cos\theta+\vec{u}\sin\theta$ be a unit generalized quaternion. The matrix associated with this quaternion q is of the form (3). In a more general case, we substitute the matrix (3) by

$$A=\begin{bmatrix} \cos(\theta+2k\pi) & -\alpha u_1\sin(\theta+2k\pi) & -\beta u_2\sin(\theta+2k\pi) & -\alpha\beta u_3\sin(\theta+2k\pi) \\ u_1\sin(\theta+2k\pi) & \cos(\theta+2k\pi) & -\beta u_3\sin(\theta+2k\pi) & \beta u_2\sin(\theta+2k\pi) \\ u_2\sin(\theta+2k\pi) & -\alpha u_3\sin(\theta+2k\pi) & \cos(\theta+2k\pi) & -\alpha u_1\sin(\theta+2k\pi) \\ u_3\sin(\theta+2k\pi) & -u_2\sin(\theta+2k\pi) & u_1\sin(\theta+2k\pi) & \cos(\theta+2k\pi) \end{bmatrix},$$

where $k\in \mathbb{Z}$. The equation $x^n=A$ has *n* roots, and they are as follows

$$A_k^{\frac{1}{n}}=\begin{bmatrix}\cos(\frac{\theta+2k\pi}{n}) & -\alpha u_1\sin(\frac{\theta+2k\pi}{n}) & -\beta u_2\sin(\frac{\theta+2k\pi}{n}) & -\alpha\beta u_3\sin(\frac{\theta+2k\pi}{n})\\ u_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -\beta u_3\sin(\frac{\theta+2k\pi}{n}) & \beta u_2\sin(\frac{\theta+2k\pi}{n})\\ u_2\sin(\frac{\theta+2k\pi}{n}) & -\alpha u_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -\alpha u_1\sin(\frac{\theta+2k\pi}{n})\\ u_3\sin(\frac{\theta+2k\pi}{n}) & -u_2\sin(\frac{\theta+2k\pi}{n}) & u_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n})\end{bmatrix}.$$

For $k=0$, the first root is

$$A_0^{\frac{1}{n}}=\begin{bmatrix}\cos(\frac{\theta}{n}) & -\alpha u_1\sin(\frac{\theta}{n}) & -\beta u_2\sin(\frac{\theta}{n}) & -\alpha\beta u_3\sin(\frac{\theta}{n})\\ u_1\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) & -\beta u_3\sin(\frac{\theta}{n}) & \beta u_2\sin(\frac{\theta}{n})\\ u_2\sin(\frac{\theta}{n}) & -\alpha u_3\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) & -\alpha u_1\sin(\frac{\theta}{n})\\ u_3\sin(\frac{\theta}{n}) & -u_2\sin(\frac{\theta}{n}) & u_1\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n})\end{bmatrix},$$

and for $k=1$, the second root is

$$A_1^{\frac{1}{n}}=\begin{bmatrix}\cos(\frac{\theta+2\pi}{n}) & -\alpha u_1\sin(\frac{\theta+2\pi}{n}) & -\beta u_2\sin(\frac{\theta+2\pi}{n}) & -\alpha\beta u_3\sin(\frac{\theta+2\pi}{n})\\ u_1\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n}) & -\beta u_3\sin(\frac{\theta+2\pi}{n}) & \beta u_2\sin(\frac{\theta+2\pi}{n})\\ u_2\sin(\frac{\theta+2\pi}{n}) & -\alpha u_3\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n}) & -\alpha u_1\sin(\frac{\theta+2\pi}{n})\\ u_3\sin(\frac{\theta+2\pi}{n}) & -u_2\sin(\frac{\theta+2\pi}{n}) & u_1\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n})\end{bmatrix}.$$

Similarly, for $k=n-1$, we obtain the *n-th* root.

Some relations between the powers of matrices associated with a generalized quaternion is sketched in the following Theorem.

- **Theorem (Teorem) 7:** Let q be a unit generalized quaternion with the polar form $q=\cos\theta+\vec{u}\sin\theta$. And let $m=\frac{2\pi}{\theta}\in Z^{+}-\{1\}$ and the matrix A correspond to q. Then $n\equiv p(\mathrm{mod}\, m)$ is true if and only if $A^{n}=A^{p}$.

Proof: The proof follows easily from the induction on *n*.

- **Example (Örnek) 2:**
Let $q=-\frac{1}{2}+\frac{1}{2}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})=\cos\frac{2\pi}{3}+\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}}).\sin\frac{2\pi}{3}$ be a unit generalized quarternion. The matrix corresponding to this quaternion is

$$A=\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{\alpha}}{2} & -\frac{\sqrt{\beta}}{2} & -\frac{\sqrt{\alpha\beta}}{2} \\ \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} & -\frac{\sqrt{\beta}}{2\sqrt{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{2\sqrt{\beta}} & \frac{\sqrt{\alpha}}{2\sqrt{\beta}} & -\frac{1}{2} & -\frac{\sqrt{\alpha}}{2} \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{2\sqrt{\beta}} & \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} \end{bmatrix},$$

The square roots of the matrix A can be calculated as follows:

$$A_k^{\frac{1}{2}}=\begin{bmatrix} \cos(\frac{2k\pi+2\pi/3}{2}) & -\alpha u_1\sin(\frac{2k\pi+2\pi/3}{2}) & -\beta u_2\sin(\frac{2k\pi+2\pi/3}{2}) & -\alpha\beta u_3\sin(\frac{2k\pi+2\pi/3}{2}) \\ u_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -\beta u_1\sin(\frac{2k\pi+2\pi/3}{2}) & \beta u_2\sin(\frac{2k\pi+2\pi/3}{2}) \\ u_2\sin(\frac{2k\pi+2\pi/3}{2}) & \alpha u_3\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -\alpha u_1\sin(\frac{2k\pi+2\pi/3}{2}) \\ u_3\sin(\frac{2k\pi+2\pi/3}{2}) & -u_2\sin(\frac{2k\pi+2\pi/3}{2}) & u_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) \end{bmatrix}$$

The first root for $k=0$ is

$$A_0^{\frac{1}{2}}=\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{\alpha}}{2} & -\frac{\sqrt{\beta}}{2} & -\frac{\sqrt{\alpha\beta}}{2} \\ \frac{1}{2\sqrt{\alpha}} & \frac{1}{2} & -\frac{\sqrt{\beta}}{2\sqrt{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{2\sqrt{\beta}} & \frac{\sqrt{\alpha}}{2\sqrt{\beta}} & \frac{1}{2} & -\frac{\sqrt{\alpha}}{2} \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{2\sqrt{\beta}} & \frac{1}{2\sqrt{\alpha}} & \frac{1}{2} \end{bmatrix},$$

and the second one for $k=1$ is

$$A_1^{\frac{1}{2}}=\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{\alpha}}{2} & \frac{\sqrt{\beta}}{2} & \frac{\sqrt{\alpha\beta}}{2} \\ -\frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} & \frac{\sqrt{\beta}}{2\sqrt{\alpha}} & -\frac{\sqrt{\beta}}{2} \\ -\frac{1}{2\sqrt{\beta}} & -\frac{\sqrt{\alpha}}{2\sqrt{\beta}} & -\frac{1}{2} & \frac{\sqrt{\alpha}}{2} \\ -\frac{1}{2\sqrt{\alpha\beta}} & \frac{1}{2\sqrt{\beta}} & -\frac{1}{2\sqrt{\alpha}} & -\frac{1}{2} \end{bmatrix}.$$

Also, it is easy to see that $A_0^{\frac{1}{2}}+A_1^{\frac{1}{2}}=0.$

From the Theorem 7, with $m=\frac{2\pi}{2\pi/3}=3$, we get

$$A=A^4=A^7=A^{10}=\dots$$
$$A^2=A^5=A^8=A^{11}=\dots$$
$$A^3=A^6=A^9=A^{12}=\dots=I_4.$$

- **Case (Durum) 2:** Let α be a positive number and β be a negative number. In this case, the Theorems 6 holds.

In following Theorem, we show how unit quaternions can be used to described the rotation in 4-space $E^4_{\alpha\beta}$.

- **Theorem (Teorem) 8:** Let q be a unit generalized quaternion. Matrices generated by operators $\overset{+}{H}$ and $\overset{-}{H}$ are quasi-orthogonal matrices, *i.e.*

i) $\left[\overset{+}{H}(q)\right]^T \varepsilon \overset{+}{H}(q) = \varepsilon,$

ii) $\left[\overset{-}{H}(q)\right]^T \varepsilon \overset{-}{H}(q) = \varepsilon, \varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$

- **Corollary (Sonuç) 1:** Let $q = \cos\theta + \vec{u}\sin\theta$ be a unit generalized quaternion. Then the generalized Hamilton operators $\overset{+}{h}_q$ and $\overset{-}{h}_q$ represent rotations of x in $E^4_{\alpha\beta}$.

The angle of rotation (using $\overset{+}{h}_q$) is easily determined. This is the angle ω between x and qx :

$$\cos\omega = \frac{S(x(\overline{qx}))}{\sqrt{N_x}\sqrt{N_{qx}}}$$

$$= \frac{S(x(\overline{x}\overline{q}))}{N_x\sqrt{N_q}} = \frac{S(q)}{\sqrt{N_q}} = S(q) = \cos\theta.$$

Therefore that the angle of rotation ω is the angle of q.

- **Example (Örnek) 3:** Let $q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, 0)$ be a unit generalized quaternion and $\alpha, \beta > 0$. The matrix corresponding to this quaternion is

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{\alpha} & -\sqrt{\beta} & 0 \\ \frac{1}{\sqrt{\alpha}} & \sqrt{2} & 0 & \sqrt{\beta} \\ \frac{1}{\sqrt{\beta}} & 0 & \sqrt{2} & -\sqrt{\alpha} \\ 0 & -\frac{1}{\sqrt{\beta}} & \frac{1}{\sqrt{\alpha}} & \sqrt{2} \end{bmatrix}.$$

A is a quasi-orthogonal matrix and therefore it represents a rotation in 4-space $E^4_{\alpha\beta}$.

In following, we show how matrices corresponding to generalized quaternion can be used to described the homothetic motion 4-space $E^4_{\alpha\beta}$. Let us consider the following curve:

$$a : I \subset R \to E^4_{\alpha\beta}$$

defined by $a(t) = (a_0(t), a_1(t), a_2(t), a_3(t))$ for every $t \in I$.

We suppose that the unit velocity curve $a(t)$ is differentiable regular curve of order r. The operator B called the Hamiltonian operator, corresponding to $a(t)$ is defined by the following matrix;

$$B = \overset{+}{H}[a(t)] = \begin{bmatrix} a_0(t) & -\alpha a_1(t) & -\beta a_2(t) & -\alpha\beta a_3(t) \\ a_1(t) & a_0(t) & -\beta a_3(t) & \beta a_2(t) \\ a_2(t) & \alpha a_3(t) & a_0(t) & -\alpha a_1(t) \\ a_3(t) & -a_2(t) & a_1(t) & a_0(t) \end{bmatrix}.$$

- **Definition (Tanım) 9.** The 1-parameter Hamilton motions of a body in $E^4_{\alpha\beta}$ are generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

or equivalently

$$Y = BX + C. \quad (4)$$

Here $B = \overset{+}{H}[a(t)]$ and Y, X and C are $n \times 1$ real matrices. Y and X correspond to the position vectors of the same point *P*.

- **Theorem (Teorem) 9:** The Hamilton motion determined by equation (4) is a homothetic motion in $E^4_{\alpha\beta}$.

Proof: We suppose that length of $a(t)$ is not zero, so the matrix B can be represented as

$$B = h \begin{bmatrix} \frac{a_0(t)}{h} & -\frac{\alpha a_1(t)}{h} & -\frac{\beta a_2(t)}{h} & -\frac{\alpha\beta a_3(t)}{h} \\ \frac{a_1(t)}{h} & \frac{a_0(t)}{h} & -\frac{\beta a_3(t)}{h} & \frac{\beta a_2(t)}{h} \\ \frac{a_2(t)}{h} & \frac{\alpha a_3(t)}{h} & \frac{a_0(t)}{h} & -\frac{\alpha a_1(t)}{h} \\ \frac{a_3(t)}{h} & \frac{-a_2(t)}{h} & \frac{a_1(t)}{h} & \frac{a_0(t)}{h} \end{bmatrix}$$

where $h : I \subset R \to R$,

$$t \to h(t) = \|\alpha(t)\| = \sqrt{a_0^2(t) + \alpha a_1^2(t) + \beta a_2^2(t) + \alpha\beta a_3^2(t)}. .$$

So, we find $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, thus B is a homothetic matrix and equation (4) determines a homothetic motion. For detailed information about the homothetic motions; we refer the reader to [2].

4. CONCLUSION (SONUÇ)

With the aid of the Hamilton operators, generalized quaternions have been expressed in terms of 4×4 matrices. In this paper, algebraic properties and geometric applications of these matrices in generalized 4-space $E^4_{\alpha\beta}$ are studied. Also, it is shown that the set of these matrices with the group operation of matrix multiplication is a Lie group of 6-dimension and its Lie algebra is found.

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