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KINEMATICS ON THE GENERALIZED 3-SPACE

ABSTRACT

A motion in 3-dimensional generalized space defined by using a spatial curve, it is shown that, this motion is a homothetic motion. We investigate some properties of this motion and show that it has only one pole point at every instant t.

Keywords: Homothetic Motion, Hamilton Operator, Generalized Quaternion, Pole Point, Kinematic

GENELLEŞTİRİLMİŞ ÜÇ BOYUTLU UZAYIN KİNEMATİĞİ

ÖZ

Genelleştirilmiş bir üç boyutlu uzayda, hareket uzay eğrisi ile tanımlanan harekete homototik hareket denir. Bu çalışmada, homototik hareketin bazı özellikleri incelenmiş ve bir pole noktasına sahip olduğu yapılan çalışmada görülmüştür.

Anahtar Kelimeler: Homothetik Hareket, Hamilton Operatörü, Genelleştirilmiş Quaternion, Pole Noktası, Kinematik

1. INTRRODUCTION (GİRİŞ)

Kinematics is the study of classical mechanics to describe the motion of points, bodies (objects) and systems of bodies (groups of objects) without consideration of the causes of motion. It is used in astrophysics to describe the motion of celestial bodies and systems, in mechanical engineering, robotics and biomechanics to describe the motion of systems composed of joined parts (multi-link systems) such as an engine, a robotic arm or the skeleton of the human body [6]. One-parameter homothetic motion of a rigid body in $n-$dimensional Euclidean space is investigated in [1] and some of its properties are given by Hacisalihoglu [2], showing that the motion is regular and has one pole point at every instant t. The homothetic motions in E^4 via Hamilton operators is studied by Yayli [9]. In Lorentz 3-space, the properties of the homothetic motions is considered in [8].

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

In this paper, we introduce a generalized 3-space and investigate the homothetic motions and some of its properties in this space. Subsequently, with the aid of the Hamilton operators, we define a Hamilton motion in three-dimensional space $\mathrm{E}^3_{\alpha\beta}$. We demonstrate that this motion is a one-parameter homothetic motion. It is found that the Hamilton motion defined by regular curve of order r has only one pole point at every instant t. Finally, we give some examples for more clarification.

3. EXPERIMENTAL METHOD-PROCESS (DENEYSEL YÖNTEM-SÜREÇ)

In this section, we define a generalized inner product on a real 3-dimensional vector space and give a generalized 3-space.

Definition 1: Let $\vec{u}=(u_1,u_2,u_3),\ \vec{v}=(v_1,v_2,v_3)\in\mathrm{R}^3$. If $\alpha,\beta\in\mathrm{R}^+$, the generalized inner product of $\vec{u}$ and $\vec{v}$ is defined by

$$\langle\vec{u},\vec{v}\rangle=\alpha u_1v_1+\beta u_2v_2+\alpha\beta u_3v_3.$$

It could be written

$$\langle\vec{u},\vec{v}\rangle=u^T\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \alpha\beta\end{bmatrix}v=u^T G v.$$

Also, if $\alpha>0,\beta<0$, $\langle\vec{u},\vec{v}\rangle$ is called the generalized Lorentzian inner product. The vector space on R^3 equipped with the generalized inner product is called 3-dimensional generalized space and denoted by $\mathrm{E}^3_{\alpha\beta}$. The vector product in $\mathrm{E}^3_{\alpha\beta}$ is defined by

$$\vec{u}\times\vec{v}=\begin{vmatrix}\beta\vec{i} & \alpha\vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\beta(u_2v_3-u_3v_2)\vec{i}+\alpha(u_3v_1-u_1v_3)\vec{j}+\alpha\beta(u_1v_2-u_2v_1)\vec{k},$$

where $\vec{i}\times\vec{j}=\vec{k},\ \vec{j}\times\vec{k}=\beta\vec{i},\ \vec{k}\times\vec{i}=\alpha\vec{j}$.

Special Cases:

- If $\alpha=\beta=1,$ then $\mathrm{E}^3_{\alpha\beta}$ is an Euclidean 3-space $\mathrm{E}^3.$
- If $\alpha=1, \beta=-1,$ then $\mathrm{E}^3_{\alpha\beta}$ is a Minkowski 3-space E^3_1 .

Definition 2: A matrix A is called a quasi-orthogonal matrix if $A^T\varepsilon A=\varepsilon$ and $\det A=1$ where

$$\varepsilon=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \alpha\beta\end{bmatrix},$$

and $\alpha,\beta\in\mathrm{R}.$ The set of all quasi-orthogonal matrices, $\mathrm{QO}(3),$ with the operation of matrix multiplication is called rotations group in 4-spaces $\mathrm{E}^4_{\alpha\beta}$ [4].

Generalized Quaternions: A generalized quaternion q is an expression of form

$$q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy the equalities

$$\vec{i}^{\,2}=-\alpha,\quad \vec{j}^{\,2}=-\beta,\quad \vec{k}^{\,2}=-\alpha\beta,$$

$$\vec{i}\vec{j}=\vec{\mathrm{k}}=-\vec{j}\vec{i},\quad \vec{j}\vec{k}=\beta\vec{i}=-\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i}=\alpha\vec{j}=-\vec{i}\vec{k},\qquad \alpha,\beta\in\mathrm{R}.$$

Special Cases:

- $\alpha=\beta=1,$ is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of real quaternions.
- $\alpha=1, \beta=-1,$ is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of split quaternions.
- $\alpha=1, \beta=0,$ is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of semi-quaternions[7].
- $\alpha=-1, \beta=0,$ is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of split semi-quaternions.
- $\alpha=0, \beta=0,$ is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of quasi-quaternions[3].

The set of all generalized quaternions are denoted by $\mathrm{H}_{\alpha\beta}.$ A generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q=a_0,$ and vector part $\vec{V}_q=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\in\mathrm{E}^3_{\alpha\beta}.$ Therefore, $\mathrm{H}_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis R and a 3-dimensional real linear space $\mathrm{E}^3_{\alpha\beta},$ so that, $\mathrm{H}_{\alpha\beta}=\mathrm{R}\oplus\mathrm{E}^3_{\alpha\beta}.$

If $q=(a_0,\vec{V}_q)$ and $p=(b_0,\vec{V}_p)$ are two quaternions, their sum is defined as

$$q+p=(a_0+b_0,\vec{V}_q+\vec{V}_p)$$

and their product (non-commutative) as

$$q \circ p = (a_0 b_0 - \langle \vec{V}_q, \vec{V}_p \rangle,\ a_0 \vec{V}_p + b_0 \vec{V}_q + \vec{V}_p \times \vec{V}_q),$$

here "<,>" and "×" are the inner and vector products in $E^3_{\alpha\beta}$, respectively. The conjugate quaternion of q is defined as $\bar{q} = (a_0, -\vec{V}_q)$ and the length or norm as $N_q = q \circ \bar{q} = \bar{q} \circ q = a_\circ^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 \in \mathbb{R}.$ Note that $N_{qp} = N_q N_p.$ Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \bar{q} / N_q.$ The generalized quaternion with a norm of one, $N_q = 1,$ is a unit generalized quaternion.

If a generalized quaternion is looked at as a four-dimensional vector, the generalized quaternion product can be described by a matrix-vector product as

$$qp = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Let q be a unit generalized quaternion, then $\overset{+}{h}_q : H_{\alpha\beta} \to H_{\alpha\beta}$ and $\overset{-}{h}_q : H_{\alpha\beta} \to H_{\alpha\beta}$ are defined as follows:

$$\overset{+}{h}_q(x) = q \circ x, \quad \overset{-}{h}_q(x) = x \circ q \qquad x \in H_{\alpha\beta}.$$

In both cases, considering $H_{\alpha\beta}$ to be $E^4_{\alpha\beta}$ spanned by the usual basic elements. We suspect that both these maps correspond to rotation, since it easy to show that they are norm and angle preserving. For example, considering the map $\overset{+}{h}_q$, we have already seen that if $x, y, q \in H_{\alpha\beta}$ and $N_q = 1,$ then

$$N_{qx} = N_q N_x = N_x.$$

The generalized Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \quad (1)$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \quad (2)$$

A direct consequence of the above operators is the following identities:

$$\overset{+}{H}(1) = I_4,\ \overset{+}{H}(\vec{\mathrm{i}}) = E_1,\ \overset{+}{H}(\vec{\mathrm{j}}) = E_2,\ \overset{+}{H}(\vec{\mathrm{k}}) = E_3,$$

and

$$\overset{-}{H}(1) = I_4,\ \overset{-}{H}(\vec{\mathrm{i}}) = F_1,\ \overset{-}{H}(\vec{\mathrm{j}}) = F_2,\ \overset{-}{H}(\vec{\mathrm{k}}) = F_3,$$

where I_4 is a 4×4 identity matrix. Note, that the properties of the E_n and F_n $(n=1,2,3)$ are identical to those of generalized quaternionic unit $\vec{i},\vec{j},\vec{k}$. Since $\overset{+}{H}$ and $\overset{-}{H}$ are linear, it follows that;

$$\begin{aligned}\overset{+}{H}(q) &= a_0\overset{+}{H}(1)+a_1\overset{+}{H}(\vec{i})+a_2\overset{+}{H}(\vec{j})+a_3\overset{+}{H}(\vec{k})\\ &= a_0I_4+a_1E_1+a_2E_2+a_3E_3,\end{aligned}$$

and

$$\begin{aligned}\overset{-}{H}(q) &= a_0\overset{-}{H}(1)+a_1\overset{-}{H}(\vec{i})+a_2\overset{-}{H}(\vec{j})+a_3\overset{-}{H}(\vec{k})\\ &= a_0I_4+a_1F_1+a_2F_2+a_3F_3.\end{aligned}$$

Using the definitions of $\overset{+}{H}$ and $\overset{-}{H}$, the multiplication of the two generalized quaternions q and p is given by

$$q\circ p=\overset{+}{H}(q)p=\overset{-}{H}(p)q.$$

Theorem 1: If q and p are two generalized quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in equations (1) and (2), respectively, then the following identities hold:

- $q=p \Leftrightarrow \overset{+}{H}(q)=\overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q)=\overset{-}{H}(p).$
- $\overset{+}{H}(q+p)=\overset{+}{H}(q)+\overset{+}{H}(p),\quad \overset{-}{H}(q+p)=\overset{-}{H}(q)+\overset{-}{H}(p).$
- $\overset{+}{H}(\lambda q)=\lambda\overset{+}{H}(q),\ \overset{-}{H}(\lambda q)=\lambda\overset{-}{H}(q).$
- $\overset{+}{H}(q\circ p)=\overset{+}{H}(q)\overset{+}{H}(p),\quad \overset{-}{H}(q\circ p)=\overset{-}{H}(p)\overset{-}{H}(q).$
- $\overset{+}{H}(q^{-1})=\left[\overset{+}{H}(q)\right]^{-1},\quad \overset{-}{H}(q^{-1})=\left[\overset{-}{H}(q)\right]^{-1},\quad (N_q)^2\neq 0.$
- $\overset{+}{H}(\overline{q})=\left[\overset{+}{H}(q)\right]^{T},\quad \overset{-}{H}(\overline{q})=\left[\overset{-}{H}(q)\right]^{T}.$
- $\det\left[\overset{+}{H}(q)\right]=(N_q)^2,\quad \det\left[\overset{-}{H}(q)\right]=(N_q)^2.$
- $tr\left[\overset{+}{H}(q)\right]=4a_0,\quad tr\left[\overset{-}{H}(q)\right]=4a_0.$

Proof: The proof can be found in [4].

4. HOMOTHETIC MOTIONS AT $E^3_{\alpha\beta}$ ($E^3_{\alpha\beta}$'DA HOMOTHETIC HAREKETLER)

In three-dimensional generalized space $E^3_{\alpha\beta}$, 1-parameter homothetic motions of a body are generated by transformation

$$\begin{bmatrix}Y\\1\end{bmatrix}=\begin{bmatrix}hA & C\\ 0 & 1\end{bmatrix}\begin{bmatrix}X\\1\end{bmatrix},$$

where A is a 3×3 quasi-orthogonal matrix and h is homothetic scalar. The matrix $B=hA$ is called a homothetic matrix and Y,X and C are 3×1 real matrices. The homothetic scalar h and the elements of A and C are continuously differentiable functions of a real parameter t. Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space $R_\circ$

and the fixed space R, respectively. At the initial time $t=t_{\circ}$, we consider the coordinate systems of $R_{\circ}$ and R are coincident.
To avoid the case of affine transformation we assume that

$$h(t)\neq cons.$$

and to avoid the case of a pure translation or a pure rotation, we also assume that

$$\frac{d}{dt}(hA)\neq 0,\quad \frac{d}{dt}(C)\neq 0.$$

If we differentiate of $Y=BX+C$ with respect to and t yields

$$\dot{Y}=\dot{B}X+\dot{C}+B\dot{X}$$

where $V_r=B\dot{X}$ is the relative velocity, $V_s=\dot{B}X+\dot{C}$ is the sliding velocity and $V_a=\dot{Y}$ is called absolute velocity of point of x moving system. So we can give the following Theorem.

Theorem 2. In three-dimensional generalized space, for one-parameter homothetic motion, absolute velocity vector of moving system of a point X at time t is the sum of the sliding velocity vector and relative velocity vector of that point.

Theorem 3. The homothetic motions of generalized space $E^3_{\alpha\beta}$ are regular motions.

Proof: Equation $B=hA$ by differentiation with respect to t gives

$$\dot{B}=\dot{h}A+\dot{A}h$$

or

$$\dot{B}=h(\frac{\dot{h}}{h}A+\dot{A}).$$

We may write $\det\dot{B}=h^n.\det(\frac{\dot{h}}{h}A+\dot{A}).$ It is obviously, for any t, $\det\dot{B}\neq 0.$

5. POLE POINTS AND POLE CURVES OF THE MOTION (POLE NOKTALARI VE HAREKETİN POLE EĞRİLERİ)

To find the pole points, we have to solve the equation

$$\dot{B}X+\dot{C}=0. \qquad (3)$$

Any solution of the equation (3) is a pole point of the motion at that instant in $R_{\circ}$. Since $\dot{B}$ is regular, the equation (3) has only one solution, *i.e.*, $X_{\circ}=(-\dot{B})^{-1}\dot{C}=0$ at every instant t. This pole point in the fixed system is

$$X=B(-\dot{B})^{-1}\dot{C}+C.$$

Theorem 4. During the homothetic motion of generalized space of 3-dimensions, there is a unique instantaneous pole point at every time t.

6. HAMILTON MOTIONS IN GENERALIZED 3-SPACE (GENELLEŞTİRİLMİŞ ÜÇ BOYUTLU UZAY İÇİNDEKİ HAMILTON HAREKETLERİ)

Let us consider the curve $\gamma: I\subset \mathbb{R}\to E^4_{\alpha\beta}$ defined by

$$\gamma(t)=(a_{\circ}(t),a_1(t),a_2(t),a_3(t)), \qquad (4)$$

for every $t\in I.$ We suppose that $\gamma(t)$ is a differentiable curve of order r which does not pass through the origin.
Also, the map F_{γ} acting on a pure quaternion ω :

$$F_\gamma : \mathrm{K} \to \mathrm{K}\,,\quad F_\gamma(\omega) = \gamma \circ \omega \circ \overline{\gamma}, \qquad (5)$$

where $\overline{\gamma}$ is conjugate of the γ. We put $F_\gamma(\omega) = \omega'$.Using the definition of $\overset{+}{H}$ and $\overset{-}{H}$ the equation (4) is written as

$$\omega' = \overset{+}{H}(\gamma)\overset{-}{H}(\gamma^*)\omega.$$

From (1) and (2), we obtain

$$\overset{+}{H}(\gamma)\overset{-}{H}(\gamma^*) = \begin{bmatrix} a_\circ^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 & 0 & 0 & 0 \\ 0 & a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2 & 2\beta(a_1a_2 - a_0a_3) & 2\beta(\alpha a_1a_3 + a_0a_2) \\ 0 & 2\alpha(a_1a_2 + a_0a_3) & a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2 & 2\alpha(\beta a_2a_3 - a_0a_1) \\ 0 & 2(\alpha a_1a_3 - a_0a_2) & 2(a_0a_1 + \beta a_2a_3) & a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \end{bmatrix}.$$

This simplifies to

$$\overset{+}{H}(\gamma)\overset{-}{H}(\gamma^*) = \begin{bmatrix} h' & 0 \\ 0 & B \end{bmatrix},$$

where $h' = N_\gamma = a_\circ^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$ and

$$B = \left[b_{ij}\right]_{3\times 3} = \begin{bmatrix} a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2 & 2\beta(a_1a_2 - a_0a_3) & 2\beta(\alpha a_1a_3 + a_0a_2) \\ 2\alpha(a_1a_2 + a_0a_3) & a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2 & 2\alpha\,(\beta a_2a_3 - a_0a_1) \\ 2(\alpha a_1a_3 - a_0a_2) & 2(a_0a_1 + \beta a_2a_3) & a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \end{bmatrix}.$$

For the matrix B, we have $B^T \varepsilon B = h'^2 I_3$ and $\det B = h'^3$.

The $1-$parameter Hamilton motions of a body in generalized $3-$space are generated by transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_\circ \\ 1 \end{bmatrix} \qquad (6)$$

where B is the above matrix. $X, X_\circ$ and C are 3×1 real matrices. B and C are continuously differentiable functions of a real parameter t. X and $X_\circ$ correspond to the position vectors of the same point P.

Theorem 5. The Hamilton motion determined by the equation (6) is a homothetic motion in $\mathrm{E}^3_{\alpha\beta}$.

Proof: The matrix B can be represented as

$$B = h\begin{bmatrix} \frac{b_{11}}{h} & \frac{b_{12}}{h} & \frac{b_{13}}{h} \\ \frac{b_{21}}{h} & \frac{b_{22}}{h} & \frac{b_{23}}{h} \\ \frac{b_{31}}{h} & \frac{b_{32}}{h} & \frac{b_{33}}{h} \end{bmatrix} = hA,$$

where $h : I \subset \mathrm{R} \to \mathrm{R}$,

$$t \to h(t) = a_\circ^2(t) + \alpha a_1^2(t) + \beta a_2^2(t) + \alpha\beta a_3^2(t).$$

So, we find $A \in QO(3)$ and $h \in \mathrm{R}.$. Thus B is a homothetic matrix and the equation (6) determines a homothetic motion.

Special case: If $\alpha = \beta = 1$, then Theorem 5 holds for real quaternions, see [5].

Example 1. Let $\gamma: I \subset \mathrm{R} \to \mathrm{E}^4_{\alpha\beta}$ be a curve given by

$$t \to \gamma(t) = (\cos t, t, \sin t, 0),$$

for every $t \in I.\ \gamma(t)$ is a differentiable regular of order r. Since, $\gamma(t)$ does not pass though the origin, the matrix B can be represented as

$$B = \begin{bmatrix} \cos^2 t + \alpha t^2 - \beta \sin^2 t & 2\beta t \sin t & 2\beta(\cos t \sin t) \\ 2\alpha t \sin t & \cos^2 t - \alpha t^2 + \beta \sin^2 t & 2\alpha(-t\cos t) \\ -2(\cos t \sin t) & 2(t \cos t) & \cos^2 t - \alpha t^2 - \beta \sin^2 t \end{bmatrix}$$

$$= (\cos^2 t + \alpha t^2 + \beta \sin^2 t) A,$$

where $h(t) = (\cos^2 t + \alpha t^2 + \beta \sin^2 t)$, $A \in \mathrm{QO}(3)$. Thus, B is a homothetic matrix and it determines a homothetic motion in E^3.

7. CONCLUSION (SONUÇ)

In this paper, we introduce a generalized 3-space and investigate the homothetic motions and some of its properties in this space. Subsequently, with the aid of the Hamilton operators, we define a Hamilton motion in three-dimensional space $\mathrm{E}^3_{\alpha\beta}$. We demonstrate that this motion is a one-parameter homothetic motion. It is found that the Hamilton motion defined by regular curve of order r has only one pole point at every instant t. Finally, we give some examples for more clarification. A motion in 3-dimensional generalized space defined by using a spatial curve, it is shown that, this motion is a homothetic motion. We investigate some properties of this motion and show that it has only one pole point at every instant t.

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