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A NOTE ON THE GROWTH OF POLYNOMIALS

ABSTRACT

Let z be a complex variable, p a complex polynomial, and let $M(p,R)=\max_{|z|=R}|p(z)|$, $M(p,1)=\max_{|z|=1}|p(z)|$. In this work, we investigate some new inequalities between $M(p,R)$ and $M(p^n,1)$ as well as between $M(p^n,R)$ and $M(p,1)$ where $n\geq 2$ is a natural number.

Keywords: Mathematicle Analysis, Complex Polynomials, Growth of Polynomials, Maximum Modulus Values, Inequalities

POLİNOMLARIN BÜYÜTÜLMESİ ÜZERİNE BAZI NOTLAR

ÖZET

z bir kompleks değişken, p bir kompleks polinom ve $n\geq 2$ bir doğal sayı olmak üzere, $M(p,R)=\max_{|z|=R}|p(z)|$, $M(p,1)=\max_{|z|=1}|p(z)|$ olsun. Bu çalışmada, $M(p,R)$ ve $M(p^n,1)$ arasında ve ayrıca $M(p^n,R)$ ve $M(p,1)$ arasında yeni eşitsizlikler araştırılmıştır.

Anahtar Kelimeler: Matematiksel Analiz, Kompleks Polinomlar Polinomların Büyütülmesi, Maksimum Modül Değerler, Eşitsizlikler

1. INTRODUCTION

Let C be the complex field, z a complex variable, and $p: C \to C$ an entire function. We set $M(p,1)=\max_{|z|=1}|p(z)|$ for $M(p,R)=\max_{|z|=R}|p(z)|$, where $R \geq 1$ (or $R \leq 1$) is a reel number. Theorem A is proved in [7].

Theorem A. If p is a polynomial of degree m satisfying $p(z) \neq 0$ for $|z|<1$, then for $R \geq 1$

$$M(p,R) \leq \frac{R^m+1}{2} M(p,1) \tag{1}$$

Theorem B. If p is a polynomial of degree m which does not vanish in the disk $|z|<1$, then for $R \geq 1$

$$M(p,R) \leq \frac{R^m+1}{2} M(p,1) - \left(\frac{R^m-1}{2}\right) \min_{|z|=1}|p(z)| \tag{2}$$

For a proof, see [2].

Lemma A. If p is a polynomial of degree m, having no zeros in $|z|<K$, $K \geq 1$, then

$$M(p,R) \leq \left(\frac{R+K}{1+K}\right)^m M(p,1)\ ,\ 1 \leq R \leq K^2 \tag{3}$$

For a proof, see [3].

Theorem C. If p is a polynomial of degree m which does not vanish in the disk $|z|<K$ where $K \geq 1$, then

$$M(p,R) \leq \frac{R^m+K^m}{1+K^m} M(p,1) \quad \text{for} \quad R \geq K^2 \tag{4}$$

For a proof, see [1].

Theorem D. If p is a polynomial of degree m, having all its zeros in $|z| \leq K$, $K>1$, then for $K<R<K^2$

$$M(p,R) \geq R^s\left(\frac{R+K}{1+K}\right) M(p,1) \tag{5}$$

s (< m) is the order of a possible zero of $p(z)$ at the origin. For a proof, see [9].

Lemma B. If p is a polynomial of degree m, having all its zeros in $|z| \leq K$, $K \leq 1$, then

$$M(p,R) \leq \left(\frac{R+K}{1+K}\right)^m M(p,1)\ ,\ K^2 \leq R \leq 1 \tag{6}$$

For a proof, see [8].

Now let 0 $<R<$ ∞, $1 \leq i \leq n$ $(n \geq 2$, a natural number) and $M_{f_i} = \underset{|z|=R}{Max}|f_i(z)|$. Let $f_i(z)=\prod_{j=1}^{d_i}(z-z_{ji})$ be a polynomial function where

$\left|z_{ji}\right| \le R$. The following theorems E and F are proved for 0 < R < ∞ in [4], and theorem E is proved for $R = 1$ in [6], respectively.

Theorem E. Let $d_1, d_2, ..., d_n$ be the degrees of polynomial functions $f_1, f_2, ..., f_n$ respectively. Then

$$M_{f_1 \cdot f_2 \cdots f_n} \ge k \;.\; M_{f_1} . M_{f_2} ... M_{f_n} \qquad (7)$$

where $k = (Sin\frac{2}{n}\frac{\pi}{8d_1})^{d_1} \;.\; (Sin\frac{2}{n}\frac{\pi}{8d_2})^{d_2} ... (Sin\frac{2}{n}\frac{\pi}{8d_n})^{d_n}$

Theorem F. Let $d_1, d_2, ..., d_n$ be the degrees of polynomial functions $f_1, f_2, ..., f_n$, respectively, which have the zero point as the multiple roots $r_1, r_2, ..., r_n$. Then

$$M_{f_1 \cdot f_2 \cdots f_n} \ge k_1 \;.\; M_{f_1} . M_{f_2} ... M_{f_n} \qquad (8)$$

where $k_1 = (Sin\frac{2}{n}\frac{\pi}{8(d_1 - r_1})^{d_1 - r_1} \;.\; (Sin\frac{2}{n}\frac{\pi}{8(d_2 - r_2)})^{d_2 - r_2} ... (Sin\frac{2}{n}\frac{\pi}{8(d_n - r_n})^{d_n - r_n}$.

2. RESEARCH SIGNIFICANCE

Let $p : C \to C$ be a polynomial function with a complex variable z. In the unit disc, we define $M(p,1) = \max_{|z|=1}|p(z)|$ for $M(p,R) = \max_{|z|=R}|p(z)|$, where $R \ge 1$ (or $R \le 1$) is a reel number. Inequalities between $M(p,R)$ and $M(p,1)$ are investigated in [1, 2, 3, 7, 8, 9].

For naturel number $n \ge 2$, the function $p^n : C \to C$ is also a polynomial function with complex variable z. In this work, we investigate inequalities between $M(p,R)$ and $M(p^n,1)$ and also $M(p^n,R)$ and $M(p,1)$, using inequalities between $M(p,R)$ and $M(p,1)$. Given [1, 2, 3, 7, 8, 9] and taking in account inequalities from [4] and [6].

3. ANALYTICAL STUDY

Our work is based on pure mathematics. Therefore, we deduce relations (formulas) and equations (analitical relations) by means of theorical methods, which are proof techniques. As usual, these methods in terms of hypothesies-conclusions [4].

4. NEW INEQUALITIES ON THE GROWTH OF POLYNOMIALS

Let p a polynomial; p^n is also a polynomial for $n \ge 2$, a natural number. Then some of the inequalities between $M(p^n,R)$ and $M(p^n,1)$ can be obtained from formulas (1)----(6). Similarly, some of the inequalities between $M(p^n,R)$ and $[M(p,R)]^n$ can be derived from (7) and (8). Inequalities between $M(p^n,1)$ and $M(p,R)$, and between $M(p^n,R)$ and $M(p,1)$ are investigated in the light of the following theorems.

Theorem 1. If p is a polynomial of degree m, $|z| < K$ and $p(z) \ne 0$ for $K \ge 1$, but all the zeros of $p(z)$ lie in $K \le |z| \le R$, then we have for $n \ge 2$ and $1 \le R \le K^2$

$$M(p,R) \le (\frac{K^2+R}{K})^m.(\frac{R+K}{1+K})^m.[M(p^n,1)]^{\frac{1}{n}} \quad (9)$$

Proof: Consider the polynomials $f_i(z) = c_i \prod_{j=1}^{m_i}(z-a_{ij})$ and the disc $D = \{ z \in C : |z| \le K^2 \}$ where $(1\le)$ $K \le |a_{ij}| \le R$ $(\le K^2)$ $(i = 1,2,...,n$; $1 \le j \le m_i)$ and $c_i \in C$ .Now for every $z \in \overline{D}$ we have $|f_i(z)| = |c_i|.\prod_{j=1}^{m_i}|z-a_{ij}| \le |c_i|.\prod_{j=1}^{m_i}(|z|+|a_{ij}|) \le |c_i|.\prod_{j=1}^{m_i}(K^2+R)$ and hence we find $|f_i(z)| \le |c_i|.(R+K^2)^{m_i}$ $(i = 1,2,...,n)$. Thus, in turn we get

$M(f_i,R) \le |c_i|.(R+K^2)^{m_i}$ $(i = 1,2,...,n)$ and

$$\prod_{i=1}^{n} M(f_i,R) \le (\prod_{i=1}^{n}|c_i|).(R+K^2)^{m_1+m_2+...+m_n} \quad (10)$$

On the other hand, since

$\left|\prod_{i=1}^{n} f_i(0)\right| = (\prod_{i=1}^{n}|c_i|).\prod_{j=1}^{m_1}|a_{1j}|.\prod_{j=1}^{m_2}|a_{2j}|...\prod_{j=1}^{m_n}|a_{nj}|$ by hypothesis we have

$$\left|\prod_{i=1}^{n} f_i(0)\right| \ge (\prod_{i=1}^{n}|c_i|).(K)^{m_1+m_2+...+m_n}$$

Then we find by the Maximum Modulus Principle [5]

$$M(\prod_{i=1}^{n} f_i,R) \ge (\prod_{i=1}^{n}|c_i|).(K)^{m_1+m_2+...+m_n} \quad (11)$$

We can write (10) and (11)

$$\frac{M(\prod_{i=1}^{n} f_i,R)}{\prod_{i=1}^{n} M(f_i,R)} \ge \frac{(K)^{m_1+m_2+...+m_n}}{(K^2+R)^{m_1+m_2+...+m_n}} \quad , \quad (12)$$

from which we have

$$\prod_{i=1}^{n} M(f_i,R) \le (\frac{K^2+R}{K})^{m_1+m_2+...+m_n} . M(\prod_{i=1}^{n} f_i,R) \quad (13)$$

Now insert $f_i(z) = p(z)$ and $m_i = m$ $(i = 1,2,...,n)$ into (13) to get

$$[M(p,R)]^n \le \left(\frac{K^2+R}{K}\right)^{nm} M(p^n,R) \quad (14)$$

On the other hand, we have from formula (3) in Lemma A for $n \ge 2$

$$M(p^n,R) \le \left(\frac{R+K}{1+K}\right)^{nm} M(p^n,1) , \quad 1 \le R \le K^2 \quad (15)$$

Finally, from (14) and (15) we can write

$$[M(p,R)]^n \le (\frac{K^2+R}{K})^{nm}.(\frac{R+K}{1+K})^{nm}.M(p^n,1)$$

which gives us desired (9).

Theorem 2. If p is a polynomial of degree m and $p(z)\neq 0$ for $|z|<1$, but all the zeros of $p(z)$ are in $1\leq |z|\leq R$, then we have for $n\geq 2$

$$M(p,R)\leq (2R)^m.\left[\left(\frac{R^{nm}+1}{2}\right)M(p^n,1)\right]^{\frac{1}{n}} \tag{16}$$

Proof: We obtain by formulas (1) in Theorem A $M(p^n,R)\leq \frac{R^{nm}+1}{2}\ M(p^n,1)$ .Following the proof style in Theorem 1, we obtain $[M(p,R)]^n\leq (2R)^{nm}.M(p^n,R)$.

From formulas (2) in Theorem B we can state the following:

Corollary 1. If p is a polynomial of degree m and $p(z)\neq 0$ for $|z|<1$, but all the zeros of $p(z)$ belong to $1\leq |z|\leq R$, then we have for $n\geq 2$

$$M(p,R)\leq (2R)^m.\left[(\frac{R^{nm}+1}{2})M(p^n,1)-(\frac{R^{nm}-1}{2}).\min_{|z|=1}|p^n(z)|\right]^{\frac{1}{n}}. \tag{17}$$

Theorem 3. If p is a polynomial of degree m which does not vanish in the disk $|z|<K$ where $K\geq 1$, but all the zeros of $p(z)$ are in $K\leq |z|\leq K^2$ then we have for $n\geq 2$ and $R\geq K^2$

$$M(p,R)\leq (\frac{K^2+R}{K})^m.\left(\frac{R^{nm}+K^{nm}}{1+K^{nm}}\right)^{\frac{1}{n}}.[M(p^n,1)]^{\frac{1}{n}} \tag{18}$$

Proof: We have $M(p^n,R)\leq \left(\frac{R^{nm}+K^{nm}}{1+K^{nm}}\right)^{\frac{1}{n}}.M(p^n,1)$ from formulas (4) in Theorem C where $n\geq 2$ and $R\geq K^2$. Furthermore, since the hypothesis of Theorem 2 are satisfied, we arrive (18) by taking in account formula (14).

Corollary 2: If p is a polynomial of degree m which does not vanish in the disk $|z|<K$ where $K\geq 1$, but all the zeros of $p(z)$ are in $K^2\leq |z|\leq R$, then we have for $n\geq 2$

$$M(p,R)\leq (\frac{2R}{K^2})^m.\left(\frac{R^{nm}+K^{nm}}{1+K^{nm}}\right)^{\frac{1}{n}}.[M(p^n,1)]^{\frac{1}{n}} \tag{19}$$

Proof: One can see that $[M(p,R)]^n\leq \left(\frac{2R}{K^2}\right)^{nm}.M(p^n,R)$ and then it suffices to consider formula (1.4) in Theorem C.

Corollary 3: If p is a polynomial of degree m which does not vanish in the disk $|z|<K$ where $K\geq 1$, but all the zeros of $p(z)$ are in $K\leq |z|\leq R$, then we have for $n\geq 2$ and $R\geq K^2$

$$M(p,R) \leq (\frac{2R}{K})^m.\left(\frac{R^{nm}+K^{nm}}{1+K^{nm}}\right)^{\frac{1}{n}}.[M(p^n,1)]^{\frac{1}{n}} \qquad (20)$$

Proof: It suffices to show $[M(p,R)]^n \leq \left(\frac{2R}{K}\right)^{nm}.M(p^n,R)$ and consider formula (4) in Theorem C. □.

Theorem 4. If p is a polynomial of degree m and all its zeros are in $K \leq 1$, $|z| \leq K$, then we have for $n \geq 2$ and $K^2 \leq R \leq 1$

$$M(p,R) \leq (Sin\frac{2}{n}\frac{\pi}{8m})^{-m}.(\frac{R+K}{1+K})^m.[M(p^n,1)]^{\frac{1}{n}} \qquad (21)$$

Proof: We can write from formula (6) in Lemma B for $n \geq 2$

$$M(p^n,R) \leq \left(\frac{R+K}{1+K}\right)^{nm} M(p^n,1)\text{ , } K^2 \leq R \leq 1 \qquad (22)$$

Moreover, replace $f_i(z)=p(z)$ and $d_i=m$ ($i=1,2,...,n$) in formula (7) in Theorem E to get

$$M(p^n,R) \geq \left(Sin\frac{2}{n}\frac{\pi}{8m}\right)^{nm}.[M(p,R)]^n \quad (K^2 \leq R \leq 1) \qquad (23)$$

Then we can write from (2.14) and (2.15)

$$\left(Sin\frac{2}{n}\frac{\pi}{8m}\right)^{nm}.[M(p,R)]^n \leq M(p^n,R) \leq (\frac{R+K}{1+K})^{nm}.M(p^n,1)\text{ ,}$$

which yields formula (21).

Theorem 5. If p is a polynomial of degree m and all its zeros are in $|z| \leq K$, $K>1$, then we have for $n \geq 2$ and $K<R<K^2$

$$[M(p^n,R)]^{\frac{1}{n}} \geq (Sin\frac{2}{n}\frac{\pi}{8m})^{m-s}.R^s(\frac{R+K}{1+K})M(p,1) \qquad (24)$$

s (< m) is the order of a possible zero of $p(z)$ at origin.

Proof: Substitute $f_i(z)=p(z)$ and $d_i-r_i=m-s$ ($i=1,2,...,n$) in formula (8) in Theorem F to obtain

$$M(p^n,R) \geq \left(Sin\frac{2}{n}\frac{\pi}{8(m-s)}\right)^{n(m-s)}.[M(p,R)]^n \qquad (25)$$

On the other hand, we have from formula (5) in Theorem D for $n \geq 2$

$$[M(p,R)]^n \geq R^{ns}.(\frac{R+K}{1+K})^n.[M(p,1)]^n \qquad (26)$$

Thus from (25) and (26) we can write

$$M(p^n,R) \geq \left(Sin\frac{2}{n}\frac{\pi}{8(m-s)}\right)^{n(m-s)}.R^{ns}.(\frac{R+K}{1+K})^n.[M(p,1)]^n\text{ ,}$$

and this inequalities poduce us formula (24).

Theorem 6. If p is a polynomial of degree m, having all its zeros in $|z| \leq K$, $K<1$, then we have for $n \geq 1$ and $K^2 \leq R \leq 1$

$$\left[M(p^n,R)\right]^{\frac{1}{n}} \leq \left(\frac{R+K}{1+K}\right)^m M(p,1) \tag{27}$$

Proof: We can write $M(p^n,R) \leq \left(\frac{R+K}{1+K}\right)^{nm} .M(p^n,1)$ from formula (6) in Lemma B. However, by the Maximum Principle, we have $M(p^n,1) \leq \left[M(p,1)\right]^n$.

5. DISCUSSIONS, CONCLUSION AND RECOMMENDATIONS

Formulas (1),(2),(3),(4),(5),(6)are found by considering the zeros of polynomial p in some circular regions. Polynomial p^n ($n \geq 2$) does not ocur in any of those formulas. Given polynomial p^n ($n \geq 2$), its degree is nm whenever the degree of p is m. Thus, similar formulas to (1)-(6) can be obtained between $M(p^n,R)$ and $M(p^n,1)$. However, it may not the case $M(p^n,R)$ and $M(p,1)$ or $M(p,R)$ and $M(p^n,1)$ is in question. Inequalities between then are expressed in formulas (9),(16),(17),(18),(19),(20),(21)and(24)by using inequalities in [4] and [6]. We emphasize on the fact that those inequalities are not generalization of inequalities (1)-(6) although Formula (27) is a generalization of Formula (6); note that Formula (6) follows from (27) for $n=1$.

One may investigate similar inequalities for a hyperbolic region or full hyperbolic regions.

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