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Note on a time fractional diffusion equation with time dependent variables coefficients

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Abstract

In this short paper, we study time fractional diffusion equations with time-dependent coefficients. The derivative operator that appears in the main equation is Riemann-Liouville. The main purpose of the paper is to prove the existence of a global solution. Due to the nonlocality of the derivative operator, we cannot represent the solution directly when the coefficient depends on time. Using some new transformations and techniques, we investigate the global solution. This paper can be considered as one of the first results on the topic related to problems with time-dependent coefficients. Our main tool is to apply Fourier analysis method and combine with some estimates of Mittag-Lefler functions and some Sobolev embeddings.

Keywords: Fractional diffusion equation; Riemman-Liouville, regularity
2010 MSC: 35R11, 35B65, 26A33.

1. Introduction

Nowadays, when studying some physical models or natural phenomena, it is found that there are some diffusion models that describe more closely to reality than fractional derivatives than other models with the classical derivative. Fractional calculus has many important applications in many different fields of science and engineering, such as in biological population models, fluid mechanics, electrical and electromagnetic networks, electrochemical, optical and viscosity [10, 11]. As far as we know, there are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, Griinwald-Letnikov, Marchand, etc. Some works are attracting the attention of the community, for example [4, 5, 6, 25, 26, 27, 28, 21, 22, 23]. and the references therein. Although most of them have been

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extensively studied, most mathematicians are interested and studied the two derivative Caputo derivative and Riemann-Liouville.

In this note, we consider the fractional diffusion equation

$$\begin{cases} \mathbf{D}^{\alpha}_{0+}u + a(t)(-\Delta)^{\beta}u = G(u), \quad (x,t) \in \Omega \times (0,T), \\ u = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ t^{1-\alpha}u|_{t=0} = \psi(x). \end{cases} \tag{1}$$

where $\mathbf{D}^{\alpha}_{0+}v$ denotes a Riemann–Liouville fractional derivative of v with order α, $0 < \alpha \leq 1$. It is defined by

$$\mathbf{D}^{\alpha}_{0+}v(t) = \frac{d}{dt}\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}v(r)dr\Big), \tag{2}$$

and $\mathbf{D}^{\alpha}_{0+}v(t) =: \dfrac{d}{dt}v(t)$ if $\alpha = 1$.

The equation described above involves viscous terms, appearing in many application models, such as phase transition, biochemistry, plasma turbulence [17], fractal geometry [19], and single-molecular protein dynamics [18]. And some other applications can be found in the following references, see [29, 30, 31, 32, 33, 34, 35, 36**?** , 37, 38]. Under ideal conditions, the coefficients of thermal conductivity a are usually constant and constant. However, when the process is disturbed by external factors and because of the presence of memory, the coefficient a will often depend on both time and space. That is also the reason we choose model (1) for this study. To the best of our knowledge, there is not yet or very little work related to Problem (1) with non-constant coefficients.

We can refer the reader to some interesting papers on fractional diffusion equations, for example, [1, 2]. Several other models related to our problem where the Riemann-Liouville derivative appears on the left-hand side have also been investigated by [15, 3] and therein references. We now mention to the recent paper [3] where the authors studied the backward problem as follows

$$\mathbf{D}^{\alpha}_{0+}u - u_{xx} = F(x,t,u).$$

To the best of our knowledge, there are not any result concerning on Problem (1). Our present paper is the first result on this topic.

Our main goal in this note is to provided the global existence and uniqueness of the mild solution for Problem (1). The regularity estimates for the mild solution are established in some various spaces. To overcome these difficulties, we learned a very interesting technique in recent articles [7, 8].

There are difficulties when studying models with time-dependent coefficients. For simplicity, we discuss the difficulty even if the simple nonlinear function G on the right hand side of the main equation of eq1 coincides with the zero function.

- First, when $\alpha = 1$, we still get the solution by the explicit formula when solving first differential equation $y'(t) - a(t)y(t) = 0$. However, when we use the Rieman-Liouville derivative, it is very difficult to obtain an explicit solution for the first order fractional differential equation $\mathbf{D}^{\alpha}_{0+}y(t) - a(t)y(t) = 0$. To overcome this difficulty, we need to use a transformation so that the left side of the new equation appears a constant coefficient.

- The technique of evaluating and proving global solutions is inherently difficult math. To overcome this difficulty, we use the Lemma derived from the work [24].

This article is organized as follows. Section 2 gives some preliminary and mild solution. In Section 3, we deal with the global existence for Problem (1).

2. Preliminaries

Let us recall that the spectral problem

$$\begin{cases}\big(-\Delta\big)^{\beta} e_n(x)=\lambda_n^{\beta} e_n(x), & x\in\Omega,\ \beta\in(0,1),\\ e_n(x)=0, & x\in\partial\Omega,\end{cases}$$

admits a family of eigenvalues

$$0<\lambda_1\le\lambda_2\le\cdots \quad\text{with}\quad \lambda_n\to\infty \quad\text{for}\quad n\to\infty,$$

and the corresponding eigenfunctions $e_n\in H_0^1(\Omega)$.

Definition 2.1. Consider the Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}$$

($z\in\mathbb{C}$), for $\alpha>0$ and $\beta\in\mathbb{R}$. When $\beta=1$, it is abbreviated as $E_\alpha(z)=E_{\alpha,1}(z)$. We call to mind the following lemmas (see for example [9]. We have the following lemma which useful for next proof.

Lemma 2.1. *Let $0<\alpha<1$. Then the function $z\mapsto E_{\alpha,\alpha}(z)$ has no negative root. Moreover, there exists a constant $\overline{C}_\alpha$ such that*

$$0\le E_{\alpha,\alpha}(-z)\le\frac{\overline{C}_\alpha}{1+z},\quad z>0. \tag{3}$$

For positive number $r\ge0$, we also define the Hilber scale space

$$\mathbb{H}^\sigma(\Omega)=\Big\{\psi\in L^2(\Omega):\sum_{n=1}^{\infty}\lambda_n^{2\sigma}\langle\psi,e_n\rangle^2<+\infty\Big\}, \tag{4}$$

with the following norm $\big\|\psi\big\|_{\mathbb{H}^\sigma(\Omega)}=\Big(\sum_{n=1}^{\infty}\lambda_n^{2\sigma}\langle\psi,e_n\rangle^2\Big)^{\frac12}\cdot$ First we state the following lemma which will be useful in our main results (this lemma can be found in [24], Lemma 8, page 9).

Lemma 2.2. *Let $a>-1$, $b>-1$ such that $a+b\ge-1$, $\theta>0$ and $t\in[0,T]$. For $\mu>0$, the following limit holds*

$$\lim_{\mu\to\infty}\Big(\sup_{t\in[0,T]}t^\theta\int_0^1 r^a(1-r)^b e^{-\mu t(1-r)}\mathrm{d}r\Big)=0.$$

Lemma 2.3. *For $\alpha\in(0,1)$ and $\theta>-1$. Then we have*

$$E_{\alpha,\alpha}(-y)=\alpha\int_0^\infty r\Phi_\alpha(r)e^{-yr}dr. \tag{5}$$

Moreover, we have the following equality

$$\Phi_\alpha(r)\ge0,\quad\forall r\ge0,\quad \textit{and}\quad \int_0^\infty r^\theta\Phi_\alpha(r)dr=\frac{\Gamma(\theta+1)}{\Gamma(\theta\alpha+1)},\quad\forall\theta>-1. \tag{6}$$

3. Main results

Theorem 3.1. *Let G be such that*

$$\big\|G(u) - G(v)\big\|_{\mathbb{H}^{\theta}(\Omega)} \le K^* \big\|u - v\big\|_{\mathbb{H}^{\nu}(\Omega)}, \tag{7}$$

where $0 \le \nu - \theta < \alpha$ and $0 < \beta < \alpha$. Let us assume that $|1 - a(t)| \le Ct^{\delta}$ for any $\delta > \max\left(\frac{\nu-\theta-\alpha}{2}, \frac{\beta-\alpha}{2}\right)$. Let us choose ε such that $\max\left(\frac{\nu-\theta}{\alpha}, \frac{\beta}{\alpha}\right) < \varepsilon < \frac{\alpha+2\delta}{\alpha}$. Then problem (1) has a unique solution $u \in \mathbf{X}_{b,p}((0,T];\mathbb{H}^{\nu}(\Omega))$ for p enough large. Here

$$0 < b < \min\Big(\frac{1}{2}, \frac{\alpha - \alpha\varepsilon}{2}, \alpha\varepsilon + 1 - \alpha\Big). \tag{8}$$

Proof. Let us define the space $\mathbf{X}_{b,p}((0,T];\mathbb{H}^{\nu}(\Omega))$ denotes the weighted space of all functions $v \in L^{\infty}((0,T];\mathbb{H}^{\nu}(\Omega))$ such that

$$\|f\|_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^{\nu}(\Omega))} := \sup_{t\in(0,T]} t^b e^{-pt} \|f(t,\cdot)\|_{\mathbb{H}^{\nu}(\Omega)} < \infty,$$

where $p > 0$. Let us first to give the explicit formula of the mild solution of Problem (1). It is obvious and not difficult to transform problem (1) into the following problem

$$\begin{cases} \mathbf{D}^{\alpha}_{0+} u + (-\Delta)^{\beta} u = G(u) + (1 - \varphi(x,t))\,(-\Delta)^{\beta} u, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ t^{1-\alpha} u|_{t=0} = \psi(x). \end{cases} \tag{9}$$

For convenience, we denote by

$$F(u(x,t)) = G(u(x,t)) + (1 - \varphi(x,t))\,(-\Delta)^{\beta} u(x,t).$$

The separation of variables helps us to yield the solution of (1) which is defined by Fourier series

$$u(x,t) = \sum_{n\in\mathbb{N}} \left(\int_{\Omega} u(x,t) e_n(x) dx\right) e_n(x), \quad u_n(t) = \int_{\Omega} u(x,t) e_n(x) dx.$$

It becomes to the fractional ordinary differential equation

$$\mathbf{D}^{\alpha}_{0+}\left(\int_{\Omega} u(x,t) e_n(x) dx\right) + \lambda_n^{\beta}\left(\int_{\Omega} u(x,t) e_n(x) dx\right) = \int_{\Omega} F(u(x,t)) e_n(x) dx.$$

Let $\psi = t^{1-\alpha} u|_{t=0}$. Then we get the following identity

$$\begin{aligned} \int_{\Omega} u(x,t) e_n(x) dx &= \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^{\beta} t^{\alpha}\right)\left(\int_{\Omega} \psi(x) e_n(x) dx\right) \\ &\quad + \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^{\beta}(t-z)^{\alpha}\right)\left(\int_{\Omega} F(u(x,z)) e_n(x) dx\right) dz. \end{aligned} \tag{10}$$

Using (5), We represent the Mittag-Lefler function by the indefinite integral form of the Wright function by the following equality

$$E_{\alpha,\alpha}\left(-\lambda_n^{\beta} t^{\alpha}\right) = \alpha \int_0^{\infty} r \Phi_{\alpha}(r) e^{-\lambda_n^{\beta} t^{\alpha} r} dr, \quad t > 0. \tag{11}$$

We substitute this expression in (10) to get it immediately

$$
\begin{aligned}
&\int_\Omega u(x,t)e_n(x)dx \\
&= \alpha\Gamma(\alpha)t^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta t^\alpha r}dr\right)\left(\int_\Omega \psi(x)e_n(x)dx\right) \\
&+ \alpha\int_0^t (t-z)^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right)\left(\int_\Omega G(u(x,z))e_n(x)dx\right)dz \\
&+ \alpha\int_0^t (t-z)^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right)\left(\int_\Omega (1-\varphi(x,z))(-\Delta)^\beta u(x,z)e_n(x)dx\right)dz. \qquad (12)
\end{aligned}
$$

The mild solution of Problem (9) is given by

$$
\begin{aligned}
&u(x,t) \\
&= \sum_{n\in\mathbb{N}}\left(\int_\Omega u(x,t)e_n(x)dx\right)e_n(x) \\
&= \alpha\Gamma(\alpha)t^{\alpha-1}\sum_{n\in\mathbb{N}}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta t^\alpha r}dr\right)\left(\int_\Omega \psi(x)e_n(x)dx\right)e_n(x) \\
&+ \alpha\sum_{n\in\mathbb{N}}\left(\int_0^t (t-z)^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right)\left(\int_\Omega G(u(x,z))e_n(x)dx\right)dz\right)e_n(x) \\
&+ \alpha\sum_{n\in\mathbb{N}}\left(\int_0^t (t-z)^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right)\left(\int_\Omega (1-\varphi(x,z))(-\Delta)^\beta u(x,z)e_n(x)dx\right)dz\right)e_n(x).
\end{aligned}
\qquad (13)
$$

Set the following function $\mathcal{B}\varphi(x,t) = \mathcal{B}_0(x,t) + \mathcal{B}^*(t)\varphi + \mathcal{B}^{**}(t)\varphi$. Here we define the following operators

$$
\mathcal{B}_0(x,t) = \alpha\Gamma(\alpha)t^{\alpha-1}\sum_{n\in\mathbb{N}}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta t^\alpha r}dr\right)\left(\int_\Omega \psi(x)e_n(x)dx\right)e_n(x). \qquad (14)
$$

$$
\mathcal{B}^*(t)\varphi = \alpha\sum_{n\in\mathbb{N}}\left(\int_0^t (t-z)^{\alpha-1}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right)\left(\int_\Omega G(\varphi(x,z))e_n(x)dx\right)dz\right)e_n(x), \qquad (15)
$$

and

$$
\begin{aligned}
\mathcal{B}^{**}(t)\varphi = \alpha\sum_{n\in\mathbb{N}}\Bigg(\int_0^t (t-z)^{\alpha-1}&\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta (t-z)^\alpha r}dr\right) \\
&\left(\int_\Omega (1-\varphi(x,z))(-\Delta)^\beta u(x,z)e_n(x)dx\right)dz\Bigg)e_n(x). \qquad (16)
\end{aligned}
$$

First, we estimate the term

$$
J_1 = \int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta t^\alpha r}dr.
$$

Using the inequality $e^{-y} \le C_\varepsilon y^{-\varepsilon}$, we find that $e^{-\lambda_n^\beta t^\alpha r} \le C_\varepsilon(\lambda_n^\beta t^\alpha r)^\varepsilon$ which allows us to obtain that

$$
J_1 \le C_\varepsilon t^{-\varepsilon\alpha}\lambda_n^{-\varepsilon\alpha}\left(\int_0^\infty r^{1-\varepsilon}\Phi_\alpha(r)dr\right). \qquad (17)
$$

Since $0 < \varepsilon < 2$, we know that $\int_0^\infty r^{1-\varepsilon}\Phi_\alpha(r)dr$ is convergent and also is equal to $\frac{\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}$. Hence, we find that

$$J_1 \le C_\varepsilon \frac{\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)} t^{-\varepsilon\alpha}\lambda_n^{-\varepsilon\alpha}. \tag{18}$$

Let us take any $\varphi, \widetilde{\varphi} \in \mathbb{H}^\nu(\Omega)$.

Step 1. Estimate of the term $\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|_{\mathbb{H}^\nu(\Omega)}$.

Using Parseval's equality, we obtain that

$$\begin{aligned}
&\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&= \alpha^2 \sum_{n\in\mathbb{N}} \lambda_j^{2\nu} \Bigg(\int_0^t (t-z)^{\alpha-1} \left(\int_0^\infty r\Phi_\alpha(r) e^{-\lambda_n^\beta (t-z)^\alpha r} dr \right) \\
&\qquad \left(\int_\Omega \left(G(\varphi(x,z)) - G(\widetilde{\varphi}(x,z))\right) e_n(x) dx \right) dz \Bigg)^2 \\
&\le \left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)} \right)^2 \sum_{n\in\mathbb{N}} \lambda_n^{2\nu-2\alpha\varepsilon} \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left(\int_\Omega \left(G(\varphi(x,z)) - G(\widetilde{\varphi}(x,z))\right) e_n(x)dx \right) dz \right)^2.
\end{aligned} \tag{19}$$

We continue to use Hölder inequality to obtain that

$$\begin{aligned}
&\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left(\int_\Omega \left(G(\varphi(x,z)) - G(\widetilde{\varphi}(x,z))\right) e_n(x)dx \right) dz \right)^2 \\
&\le \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} dz \right) \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left(\int_\Omega \left(G(\varphi(x,z)) - G(\widetilde{\varphi}(x,z))\right) e_n(x)dx \right)^2 \right).
\end{aligned} \tag{20}$$

From two above observations, we get that

$$\begin{aligned}
&\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le \frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon} \left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)} \right)^2 \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left\| G(\varphi(.,z)) - G(\widetilde{\varphi}(.,z)) \right\|^2_{\mathbb{H}^{\nu-\alpha\varepsilon}(\Omega)} dz \\
&\le \frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon} \left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)} \right)^2 \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left\| G(\varphi(.,z)) - G(\widetilde{\varphi}(.,z)) \right\|^2_{\mathbb{H}^{\theta}(\Omega)} dz,
\end{aligned} \tag{21}$$

where we note that $\nu - \alpha\varepsilon \le \theta$. Based on the globally Lipschitz of G, we bound the integral term on the right hand side of (21) as follows

$$\begin{aligned}
&\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left\| G(\varphi(.,z)) - G(\widetilde{\varphi}(.,z)) \right\|^2_{\mathbb{H}^{\theta}(\Omega)} dz \\
&\le K^* \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} \left\| \varphi(.,z) - \widetilde{\varphi}(.,z) \right\|^2_{\mathbb{H}^{\nu}(\Omega)} dz.
\end{aligned} \tag{22}$$

From two observation, we derive that

$$
\begin{aligned}
& t^{2b}e^{-2pt}\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
& \le K^* \frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 t^{2b}e^{-2pt}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}\left\|\varphi(.,z)-\widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)}dz \\
& \le K^* \frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 \\
& t^{2b}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}z^{2b}e^{-2pz}\left\|\varphi(.,z)-\widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)}dz.
\end{aligned} \tag{23}
$$

Let us continue to treat the integral term. Indeed, we derive that

$$
\begin{aligned}
& \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}z^{2b}e^{-2pz}\left\|\varphi(.,z)-\widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)}dz \\
& \le \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}dz\right)\left\|\varphi-\widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))}.
\end{aligned} \tag{24}
$$

From two above observation, we find that

$$
\begin{aligned}
& t^{2b}e^{-2pt}\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
& \le \overline{K}(T,\alpha,\varepsilon)t^{2b}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}dz\right)\left\|\varphi-\widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))}
\end{aligned} \tag{25}
$$

where $\overline{K}(T,\alpha,\varepsilon) = K^*\frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2$.

Step 2. Estimate of the term $\left\|\mathcal{B}^{**}(t)\varphi - \mathcal{B}^{**}(t)\widetilde{\varphi}\right\|_{\mathbb{H}^\nu(\Omega)}$.

Using Parseval's equality, we obtain that

$$
\begin{aligned}
& \left\|\mathcal{B}^{**}(t)\varphi - \mathcal{B}^{**}(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
& \le \left(\frac{C_\varepsilon \Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 \\
& \sum_{n\in\mathbb{N}} \lambda_n^{2\nu-2\alpha\varepsilon}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}\left(\int_\Omega \Big((1-a(z))\left((-\Delta)^\beta\varphi(x,z)-(-\Delta)^\beta\widetilde{\varphi}(x,z)\right)\Big)e_n(x)dx\right)dz\right)^2.
\end{aligned} \tag{26}
$$

We continue to use Hölder inequality to obtain that

$$
\begin{aligned}
& \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}\left(\int_\Omega \Big((1-a(z))\left((-\Delta)^\beta\varphi(x,z)-(-\Delta)^\beta\widetilde{\varphi}(x,z)\right)\Big)e_n(x)dx\right)dz\right)^2 \\
& \le \left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}dz\right)\left[\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}\left(\int_\Omega \Big((1-a(z))\left((-\Delta)^\beta\varphi(x,z)-(-\Delta)^\beta\widetilde{\varphi}(x,z)\right)\Big)e_n(x)dx\right)^2 dz\right].
\end{aligned} \tag{27}
$$

From two above observations and noting that $\left\|\Delta^\beta v\right\|_{\mathbb{H}^s(\Omega)} = \left\|v\right\|_{\mathbb{H}^{s+\beta}(\Omega)}$, we get that the following estimate

$$\begin{aligned}
&\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le \frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon}\left|1-a(z)\right|^2 \left\|(-\Delta)^\beta\varphi(.,z) - (-\Delta)^\beta\widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^{\nu-\alpha\varepsilon}(\Omega)} dz \\
&\le C\frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 \int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta}\left\|\varphi(.,z) - \widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^{\nu-\alpha\varepsilon+\beta}(\Omega)} dz. \qquad (28)
\end{aligned}$$

Based on some previous evaluations and notice that $\nu - \alpha\varepsilon + \beta \le \nu$, we derive that

$$\begin{aligned}
&t^{2b}e^{-2pt}\left\|\mathcal{B}^{**}(t)\varphi - \mathcal{B}^{**}(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le K^*\frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 t^{2b}e^{-2pt}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta}\left\|\varphi(.,z) - \widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)} dz \\
&\le K^*\frac{T^{\alpha-\alpha\varepsilon}}{\alpha-\alpha\varepsilon}\left(\frac{C_\varepsilon\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2 \\
&t^{2b}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}z^{2b}e^{-2pz}\left\|\varphi(.,z) - \widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)} dz. \qquad (29)
\end{aligned}$$

Let us continue to treat the integral term. Indeed, we derive that

$$\begin{aligned}
&t^{2b}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}z^{2b}e^{-2pz}\left\|\varphi(.,z) - \widetilde{\varphi}(.,z)\right\|^2_{\mathbb{H}^\nu(\Omega)} dz \\
&\le t^{2b}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}dz\right)\left\|\varphi - \widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))}. \qquad (30)
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
&t^{2b}e^{-2pt}\left\|\mathcal{B}^{**}(t)\varphi - \mathcal{B}^{**}(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le \overline{K}(T,\alpha,\varepsilon)t^{2b}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}dz\right)\left\|\varphi - \widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))}. \qquad (31)
\end{aligned}$$

Combining (25) and (31), we derive that

$$\begin{aligned}
&t^{2b}e^{-2pt}\left\|\mathcal{B}(t)\varphi - \mathcal{B}(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le 2t^{2b}e^{-2pt}\left\|\mathcal{B}^*(t)\varphi - \mathcal{B}^*(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} + 2t^{2b}e^{-2pt}\left\|\mathcal{B}^{**}(t)\varphi - \mathcal{B}^{**}(t)\widetilde{\varphi}\right\|^2_{\mathbb{H}^\nu(\Omega)} \\
&\le 2\overline{K}(T,\alpha,\varepsilon)t^{2b}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}dz\right)\left\|\varphi - \widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))} \\
&+ 2\overline{K}(T,\alpha,\varepsilon)t^{2b}\left(\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}dz\right)\left\|\varphi - \widetilde{\varphi}\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))}. \qquad (32)
\end{aligned}$$

Set $z = ts$, we get that

$$t^{2b}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{-2b}e^{-2p(t-z)}dz = t^{\alpha-\alpha\varepsilon}\int_0^1 (1-z')^{\alpha-\alpha\varepsilon-1}(z')^{-2b}e^{-2pt(1-z')}dz'$$

and

$$t^{2b}\int_0^t (t-z)^{\alpha-1-\alpha\varepsilon} z^{2\delta-2b}e^{-2p(t-z)}dz = t^{\alpha-\alpha\varepsilon}\int_0^1 (1-z')^{\alpha-\alpha\varepsilon-1}(z')^{2\delta-2b}e^{-2pt(1-z')}dz'.$$

Applying Lemma (2.2) and noting the condition $\alpha-\alpha\varepsilon>0,\ \alpha-\alpha\varepsilon-1>-1,\ -2b>-1,\ \alpha-\alpha\varepsilon-1-2b>-1,$ $2\delta-2b>-1,$ we find that two following equality

$$\lim_{p\to\infty}\left(\sup_{t\in[0,T]} t^{\alpha-\alpha\varepsilon}\int_0^1 (1-z')^{\alpha-\alpha\varepsilon-1}(z')^{-2b}e^{-2pt(1-z')}dz'\right)=0. \tag{33}$$

and

$$\lim_{p\to\infty}\left(\sup_{t\in[0,T]} t^{\alpha-\alpha\varepsilon}\int_0^1 (1-z')^{\alpha-\alpha\varepsilon-1}(z')^{2\delta-2b}e^{-2pt(1-z')}dz'\right)=0. \tag{34}$$

By combining (32) and (33) and (34), we deduce that $\mathcal{B}$ is a contraction on the space $\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))$ if p enough large. If $\varphi=0$ then $\mathcal{B}(t)\varphi=\mathcal{B}_0(x,t)$. Then, from the fact that $\nu-\alpha\varepsilon\le\mu$, we get the following estimate

$$\begin{aligned}
\left\|\mathcal{B}(.)\varphi\right\|^2_{\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))} &= \sup_{t\in(0,T]} t^{2b}e^{-2pt}\|\mathcal{B}_0(.,t)\|^2_{\mathbb{H}^\nu(\Omega)}\\
&\le \sup_{t\in(0,T]} t^{2b}e^{-2pt}\alpha^2\Big|\Gamma(\alpha)\Big|^2 t^{2\alpha-2}\sum_{n\in\mathbb{N}}\left(\int_0^\infty r\Phi_\alpha(r)e^{-\lambda_n^\beta t^\alpha r}dr\right)^2\left(\int_\Omega \psi(x)e_n(x)dx\right)^2\\
&\le \left(C_\varepsilon\frac{\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2\alpha^2\Big|\Gamma(\alpha)\Big|^2 t^{2b-2\varepsilon\alpha+2\alpha-2}\sum_{n\in\mathbb{N}}\lambda_n^{2\nu}\lambda_n^{-2\varepsilon\alpha}\left(\int_\Omega \psi(x)e_n(x)dx\right)^2\\
&\le \left(C_\varepsilon\frac{\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2\alpha^2\Big|\Gamma(\alpha)\Big|^2 T^{2b-2\varepsilon\alpha+2\alpha-2}\|\psi\|^2_{\mathbb{H}^{\nu-\epsilon\alpha}(\Omega)}\\
&\le \left(C_\varepsilon\frac{\Gamma(2-\varepsilon)}{\Gamma(\alpha+1-\alpha\varepsilon)}\right)^2\alpha^2\Big|\Gamma(\alpha)\Big|^2 T^{2b-2\varepsilon\alpha+2\alpha-2}\|\psi\|^2_{\mathbb{H}^{\nu}(\Omega)}.
\end{aligned} \tag{35}$$

where we note that $b+\alpha\le\varepsilon\alpha+1$. The above inequality implies that $\mathcal{B}(.)\varphi\in\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))$. By using Banach fixed point theorem, we can deduce that Problem (1) has a unique solution in the space $\mathbf{X}_{b,p}((0,T];\mathbb{H}^\nu(\Omega))$. □

4. Conclusion

In this paper, we try to consider the time-fractional problem with time dependents coefficients. This is a difficult problem. We obtain the existence and uniqueness of the global solution for our problem. Our main techniques are based on some previous techniques as in [7, 8].

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