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New presentations of real numbers with *k*-Lucas numbers

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Abstract

In this paper, we first obtain a series of k-Lucas numbers using k-Lucas numbers. We give new presentations of any real number u ≠ 0 using this obtained k-Lucas series and show polynominal repsesentations that every nonzero real number can be uniquely represented as the sum of the squares of consecutive k-Lucas numbers. To do this, we give new presentation theorems for any real number using k-Lucas series. Finally, to support these theorems, we give examples where we obtain the roots of the polynomial representations of a selected real number u ≠ 0, as well as the values representing the first ten prime numbers corresponding to a chosen k-Lucas polynomial.

Keywords: *k-Lucas number, k-Lucas series, presentation of real numbers.*

Reel sayıların *k*-Lucas sayıları ile yeni temsilleri

Öz

Bu çalışmada, ilk önce k-Lucas sayıların özelliklerini kullanarak k-Lucas sayılarının bir serisini elde ediyoruz. Daha sonra elde ettiğimiz bu k-Lucas serisini kullanarak herhangi bir u ≠ 0 reel sayısının yeni temsillerini elde ediyor ve her reel sayının ardışık k-Lucas sayılarının karelerinin toplamı olarak bir tek şekilde temsil edilebilir olduklarına yönelik polinom temsillerini gösteriyoruz. Bunu yapmak için k-Lucas serisini kullanarak herhangi bir reel sayı için yeni temsil teoremleri veriyoruz. Son olarak ise bu teoremleri desteklemek amacıyla seçilen özel bir u ≠ 0 gerçek sayısının polinom temsillerinin köklerini ve ayrıca seçilen bir k-Lucas polinomuna karşılık gelen ilk on asal sayıyı temsil eden değerlerini elde ettiğimiz örnekler veriyoruz.

Anahtar kelimeler: *k-Lucas sayıları, k-Lucas serileri, gerçek sayıların temsilleri.*

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1. Introduction

In recent years, many interesting properties of classical Fibonacci numbers, classical Lucas numbers and their generalizations have been investigated by researchers. Applications of such kind theoretical studies have appeared almost every field of science and art. For example, see [1-7]. The classical Fibonacci $\{F_n\}_{n \in N}$ and Lucas $\{L_n\}_{n \in N}$ sequences are defined as, respectively,

$F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and

$L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$;

where F_n and L_n denotes the n-th Fibonacci and Lucas numbers, respectively. Besides the usual Fibonacci and Lucas numbers, many kind of generalizations of these numbers have been presented in literature (see for instence, [1-8]). For example in [7], Hoggatt used the k-Fibonacci sequence to generalize Zeckendorf's theorem. Afterwards, Özgür and Uçar studied new presentations for real numbers with k-Fibonacci sequence in [8].

In [9] Falcon has created the k-Lucas numbers as an extension of the k-Fibonacci numbers [1-2] and proved some of its many properties. Later, some studies were carried out using k-Lucas numbers in [10-11].

Definition 1.1 For any integer number $k > 0$, the k-th Lucas sequence, say $\{L_{k,n}\}_{n \in N}$ is defined recurrently by

$L_{k,0} = 2, L_{k,1} = k$ and $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $n \geq 1$ (see [3]). (1)

As particular cases

if $k = 1$, we obtain the Lucas sequence $L_1 = \{L_{1,n}\} = \{2,1,3,4,7,11,18,29,\ldots\}$,

if $k = 2$, we obtain the Pell-Lucas sequence $L_2 = \{L_{2,n}\} = \{2,2,6,14,34,82,198,478,\ldots\}$,

if $k = 3$, we obtain the 3-Lucas sequence $L_3 = \{L_{3,n}\} = \{2,3,11,36,119,393,1298, 4287,..\}$.

From defnition of the k-Lucas numbers, the first eight of them are presented in the following Table 1.

Table 1. k-Lucas sayıları.

$L_{k,0}=2$
$L_{k,1}=k$
$L_{k,2}=k^2+2$
$L_{k,3}=k^3+3k$
$L_{k,4}=k^4+4k^2+2$
$L_{k,5}=k^5+5k^3+5k$
$L_{k,6}=k^6+6k^4+9k^2+2$
$L_{k,7}=k^7+7k^5+14k^3+7k$

In this paper, we prove that every real number can be uniquely represented as the sum of the squares of consecutive k-Lucas numbers. To do this, we give new presentation theorems for any real number $u \neq 0$ using k-Lucas series. Especially, we have found infinitely many presentations for any real number. As an application, we present a new infinite family of complex series for $1/\pi$ (see [12] for more details).

2. New representation theorems

We begin with the following theorem.

Theorem 2.1 Let $k > 0$ be any real number, $r \geq 0$ be an integer and $L_{k,n}$ be the k-th Lucas number. Then we have

$$\sum_{n=1}^{\infty} \frac{1}{L_{k,n+r} L_{k,n+r+2}} = \frac{1}{k L_{k,r+1} L_{k,r+2}}. \tag{2}$$

Proof. Using (1) we obtain

$$\frac{1}{L_{k,n+r} L_{k,n+r+2}} = \frac{1}{\left(k L_{k,n+r+1}\right)\left(L_{k,n+r}\right)} - \frac{1}{\left(k L_{k,n+r+1}\right)\left(L_{k,n+r+2}\right)} \tag{3}$$

and then we find

$$\begin{aligned} S_n &= \sum_{l=1}^{n} \frac{1}{L_{k,l+r} L_{k,l+r+2}} \\ &= \sum_{l=1}^{n} \left(\frac{1}{\left(k L_{k,l+r+1}\right)\left(L_{k,l+r}\right)} - \frac{1}{\left(k L_{k,l+r+1}\right)\left(L_{k,l+r+2}\right)} \right) \\ &= \frac{1}{k L_{k,r+1} L_{k,r+2}} - \frac{1}{k L_{k,r+n+1} L_{k,r+n+2}}. \end{aligned}$$

So we have

$$\sum_{n=1}^{\infty} \frac{1}{L_{k,n+r} L_{k,n+r+2}} = \frac{1}{k L_{k,r+1} L_{k,r+2}}.$$

Theorem 2.2 For any real number $u \neq 0$ and any positive integer r, there exists unique real number k satisfying the following equation

$$\sum_{n=1}^{\infty} \frac{1}{L_{k,n+r} L_{k,n+r+2}} = \frac{1}{u}. \tag{4}$$

Proof. Let $u > 0$ be any real number. From Theorem 2.1 we know the following equation

$$\sum_{n=1}^{\infty} \frac{1}{L_{k,n+r} L_{k,n+r+2}} = \frac{1}{k L_{k,r+1} L_{k,r+2}}.$$

Then we get

$$\frac{1}{k L_{k,r+1} L_{k,r+2}} = \frac{1}{u}$$

and so we get the following equation

$$kL_{k,r+1}L_{k,r+2} - u = 0. \tag{5}$$

Using $L_{k;r+2}=kL_{k;r+1} + L_{k,r}$, we obtain

$$\begin{aligned} kL_{k,r+1}L_{k,r+2} &= kL_{k,r+1}\left(kL_{k,r+1} + L_{k,r}\right) \\ &= k^2\left(L_{k,r+1}\right)^2 + \left(kL_{k,r+1}.L_{k,r}\right). \end{aligned} \tag{6}$$

Then if the expression $L_{k,r+1} = kL_{k,r} + L_{k,r-1}$ is substituted in the equation (6) and if the operations are continued in a similar way

$$kL_{k,r+1}L_{k,r+2} = k^2\left(L_{k,1}^2 + L_{k,2}^2 + L_{k,3}^2 + \ldots + L_{k,r}^2 + L_{k,r+1}^2\right). \tag{7}$$

Let us consider following polynomial

$$f_r(k) = k^2\left(L_{k,1}^2 + L_{k,2}^2 + L_{k,3}^2 + \ldots + L_{k,r}^2 + L_{k,r+1}^2\right) - u. \tag{8}$$

Applying the Descartes rule of signs, $f_r(k)$ has unique positive real zero, say k_0, and then we have equation (4) for k_0. If $u < 0$, then for the real number $a = |u|$ we get

$$\sum_{n=1}^{\infty}\frac{1}{L_{k,n+r}L_{k,n+r+2}} = \frac{1}{a}$$

and hence

$$-\sum_{n=1}^{\infty}\frac{1}{L_{k,n+r}L_{k,n+r+2}} = \frac{1}{u}.$$

Thus, for all real numbers different from, there exists unique k such that (4) holds.

Corollary 2.3 For any positive integer r, there exists unique positive real number k satisfying the following equation

$$\sum_{n=1}^{\infty}\frac{1}{L_{k,n+r}L_{k,n+r+2}} = \frac{1}{\pi}. \tag{9}$$

Thus, we have an infinite family of real series converging to $\frac{1}{\pi}$.

Theorem 2.4 Let $u \neq 0$ be any real number. For any positive integer r, there exists unique positive real number k such that we have following polynomial representation for u

$$u = k^2\left(L_{k,1}^2 + L_{k,2}^2 + L_{k,3}^2 + \ldots + L_{k,r}^2 + L_{k,r+1}^2\right). \tag{10}$$

Proof. The proof follows easily from the proof of Theorem 2.2.

Thus, we have an infinite family of polynomials representing any nonzero reel number u.

Example 2.5 Let $u=\sqrt{29}$ and $r = 3$. From (5) we have $kL_{k,4}L_{k,5}-\sqrt{29}=0$. Hence we have the equation $k\left(k^4+4k^2+2\right)\left(k^5+5k^3+5k\right)-\sqrt{29}=0$. The unique positive root of this equation is $0,51718$. Thus we have

$$\sum_{n=1}^{\infty}\frac{1}{L_{0,51718\ ,n+3}L_{0,51718\ ,n+5}}=\frac{1}{\sqrt{29}}.$$

Now we restrict our attention to polynomial representation and the equation (10), so let u be any nonzero real number. By Theorem 2.4, for any positive integer r, there exists unique positive real number k such that we have following polynomial representation for u

$$u=k^2\left(L_{k,1}^2+L_{k,2}^2+L_{k,3}^2+...+L_{k,r}^2+L_{k,r+1}^2\right)$$

and

$$u_r(k)=k^2\left(L_{k,1}^2+L_{k,2}^2+L_{k,3}^2+...+L_{k,r}^2+L_{k,r+1}^2\right)=k^2\sum_{n=0}^{r}\left(L_{k,n+1}^2\right).$$

Example 2.6 For the prime number $u = 83$, we have following Table 2.

Table 2. k values for $u=83$.

u	r	$u_r(k)$	k
83	1	$k^2\left(k^2+\left(k^2+2\right)^2\right)$	1,72729
83	2	$k^2\left(k^2+\left(k^2+2\right)^2+\left(k^3+3k\right)^2\right)$	1,28274
83	3	$k^2\left(k^2+\left(k^2+2\right)^2+\left(k^3+3k\right)^2+\left(k^4+4k^2+2\right)^2\right)$	1,02008
83	4	$k^2\left(k^2+\left(k^2+2\right)^2+\left(k^3+3k\right)^2+\left(k^4+4k^2+2\right)^2+\left(k^5+5k^3+5k\right)^2\right)$	0,86015

So we have seen that one of the presentations representing any prime looks like very simple, $u_1(k)=k^2\left(k^2+\left(k^2+2\right)^2\right)$. Now we focus on the case $r = 1$. We consider the following example.

Example 2.7 Let $r = 1$ be fixed. Then we have $u_1(k)=k^2\left(k^2+\left(k^2+2\right)^2\right)$. In the following Table 3 we can see the values of k such that the corresponding polynomial $p=u_1(k)=k^2\left(k^2+\left(k^2+2\right)^2\right)$ represents the first ten primes.

Table 3. *k* values for *p* values.

p	*k*
2	*0,58559*
3	*0,67860*
5	*0,80690*
7	*0,89742*
11	*1,02823*
13	*1,07871*
17	*1,16640*
19	*1,19806*
23	*1,26066*
29	*1,33890*

Notice that k is not an integer for the first ten primes.
As a result, we show that every nonzero real number can be uniquely represented as the sum of the squares of consecutive k-Lucas numbers.

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