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On k-Generalized ψ-Hilfer Boundary Value Problems with Retardation and Anticipation

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Abstract

In this paper, we prove some existence and uniqueness results for a class of boundary valued problems for implicit nonlinear k-generalized ψ-Hilfer fractional differential equations involving both retarded and advanced arguments. Further, examples are given to illustrate the viability of our results.

Keywords: ψ-Hilfer fractional derivative k-generalized ψ-Hilfer fractional derivative Cauchy-type problem retarded arguments advanced arguments existence uniqueness.
2010 MSC: 34A08, 26A33, 34A12, 34B15, 34K37.

1. Introduction

In recent years, fractional calculus has proven to be a very valuable method for addressing the complexity structures from different branches of science and engineering. It concerns the generalization of the integer order differentiation and integration of a function to non-integer order, and its theory and application are solid and growing works [1, 2, 3, 7, 13, 14, 15, 20, 21, 22, 23, 24]. The authors of [16, 5, 6, 12] explored the existence, stability and uniqueness of solutions for various problems with fractional differential equation and inclusions concerning retarded or advanced arguments. In this paper, many of the properties of the special functions k-gamma and k-beta introduced by Diaz *et al.* in [10] are included. Several findings and generalizations for various fractional integrals and derivatives based on these properties can also be found in [17, 8, 18]. In [27], Sousa *et al.* introduce another so-called ψ-Hilfer fractional derivative with respect to

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another function and gave some important properties concerning this type of fractional operator. We direct readers to the papers [25, 26, 4] and the references therein for further results based on this operator.

Motivated by the above papers, we consider the boundary valued problem for the nonlinear implicit k-generalized ψ-Hilfer type fractional differential equation involving both retarded and advanced arguments,

$$\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)=f\left(t,x_t(\cdot),\left({}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)\right),\quad t\in(a,b], \tag{1}$$

$$\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+)+\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(b)=\alpha_3, \tag{2}$$

$$x(t)=\varpi(t),\quad t\in[a-\lambda,a],\ \lambda>0, \tag{3}$$

$$x(t)=\tilde{\varpi}(t),\quad t\in\left[b,b+\tilde{\lambda}\right],\ \tilde{\lambda}>0, \tag{4}$$

where ${}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}$ and $\mathcal{J}_{a+}^{k(1-\xi),k;\psi}$ are, respectively, the k-generalized ψ-Hilfer fractional derivative of order $\vartheta\in(0,k)$ and type $r\in[0,1]$ defined in Section 2, and k-generalized ψ-fractional integral of order $k(1-\xi)$ defined in [19], where $\xi=\frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k>0$, $f:[a,b]\times C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right)\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function, and $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ such that $\alpha_1+\alpha_2\neq 0$, and $\varpi(t)$ and $\tilde{\varpi}(t)$ are, respectively, continuous functions on $[a-\lambda,a]$ and $\left[b,b+\tilde{\lambda}\right]$. For each function x defined on $\left[a-\lambda,b+\tilde{\lambda}\right]$ and for any $t\in(a,b]$, we denote by x_t the element defined by

$$x_t(\tau)=x(t+\tau),\quad \tau\in\left[-\lambda,\tilde{\lambda}\right].$$

The paper is arranged as follows. In Section 2, some notations are introduced and we recall some preliminaries about the ψ-Hilfer fractional derivative, the functions k-Gamma and k-Beta, and some auxiliary results. Further, we give the definition of the k-generalized ψ-Hilfer type fractional derivative and some essential theorems and lemmas. In Section 3, we present two existence and uniqueness results for the problem (1)-(4) that are founded on the Banach contraction principle and Schauder's fixed point theorem. In the last section, we give two examples to illustrate the viability of our results.

2. Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this article. Let $0<a<b<\infty$, $J=[a,b]$, $\vartheta\in(0,k)$, $r\in[0,1]$, $k>0$ and $\xi=\frac{1}{k}(r(k-\vartheta)+\vartheta)$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_\infty=\sup\{|x(t)|:t\in J\}.$$

$AC^n(J,\mathbb{R})$ and $C^n(J,\mathbb{R})$ denote the spaces of n-times absolutely continuous and n-times continuously differentiable functions on J, respectively.

Let $C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right)$, $\mathcal{C}=C([a-\lambda,a],\mathbb{R})$ and $\tilde{\mathcal{C}}=C\left(\left[b,b+\tilde{\lambda}\right],\mathbb{R}\right)$ be the spaces endowed, respectively, with the norms

$$\|x\|_{[-\lambda,\tilde{\lambda}]}=\sup\left\{|x(t)|:t\in\left[-\lambda,\tilde{\lambda}\right]\right\},$$

$$\|x\|_{\mathcal{C}}=\sup\{|x(t)|:t\in[a-\lambda,a]\},$$

and

$$\|x\|_{\tilde{\mathcal{C}}}=\sup\left\{|x(t)|:t\in\left[b,b+\tilde{\lambda}\right]\right\}.$$

Consider the weighted Banach space

$$C_{\xi,k;\psi}(J)=\left\{x:(a,b]\to\mathbb{R}:t\to\Psi_{\xi}^{\psi}(t,a)x(t)\in C(J,\mathbb{R})\right\},$$

where $\Psi_\xi^\psi(t,a) = (\psi(t)-\psi(a))^{1-\xi}$, with the norm

$$\|x\|_{C_{\xi,k;\psi}} = \sup_{t\in J}\left|\Psi_\xi^\psi(t,a)x(t)\right|,$$

and

$$C^n_{\xi,k;\psi}(J) = \left\{x \in C^{n-1}(J) : x^{(n)} \in C_{\xi,k;\psi}(J)\right\}, n \in \mathbb{N},$$
$$C^0_{\xi,k;\psi}(J) = C_{\xi,k;\psi}(J),$$

with the norm

$$\|x\|_{C^n_{\xi,k;\psi}} = \sum_{i=0}^{n-1}\|x^{(i)}\|_\infty + \|x^{(n)}\|_{C_{\xi,k;\psi}}.$$

Next, we consider the Banach space

$$\mathbb{F} = \Big\{x : \left[a-\lambda, b+\tilde{\lambda}\right] \to \mathbb{R} : x|_{[a-\lambda,a]} \in \mathcal{C},\, x|_{[b,b+\tilde{\lambda}]} \in \tilde{\mathcal{C}} \text{ and } x|_{(a,b]} \in C_{\xi,k;\psi}(J)\Big\},$$

with the norm

$$\|x\|_{\mathbb{F}} = \max\left\{\|x\|_{\mathcal{C}}, \|x\|_{\tilde{\mathcal{C}}}, \|x\|_{C_{\xi,k;\psi}}\right\}.$$

Consider the space $X^p_\psi(a,b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those real-valued Lebesgue measurable functions g on $[a,b]$ for which $\|g\|_{X^p_\psi} < \infty$, where the norm is defined by

$$\|g\|_{X^p_\psi} = \left(\int_a^b \psi'(t)|g(t)|^p dt\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[a,b]$ such that ψ' is continuous on $[a,b]$ with $\psi(0)=0$. In particular, when $\psi(x)=x$, the space $X^p_\psi(a,b)$ coincides with the $L_p(a,b)$ space.

Definition 2.1. *([10]) The k-gamma function is defined by*

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \alpha > 0.$$

When $k \to 1$ then $\Gamma(\alpha) = \Gamma_k(\alpha)$, and some other useful relations are $\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)$, $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$ and $\Gamma_k(k) = \Gamma(1) = 1$. Furthermore, the k-beta function is defined as,

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt$$

so that $B_k(\alpha,\beta) = \frac{1}{k}B\left(\frac{\alpha}{k},\frac{\beta}{k}\right)$ and $B_k(\alpha,\beta) = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$.

Now, we give all the definitions to the different fractional operators used throughout this paper.

Definition 2.2. *([19]) (k-generalized ψ-fractional integral) Let $g \in X^p_\psi(a,b)$ and $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty,\infty)$, $\psi(t) > 0$ be an increasing function on $(a,b]$ and $\psi'(t) > 0$ be continuous on (a,b), and $\vartheta > 0$. The generalized k-fractional integral operators of a function f (left-sided and right-sided) of order ϑ are defined by*

$$\mathcal{J}^{\vartheta,k;\psi}_{a+}g(t) = \int_a^t \bar{\Psi}^{k,\psi}_\vartheta(t,s)\psi'(s)g(s)ds,$$
$$\mathcal{J}^{\vartheta,k;\psi}_{b-}g(t) = \int_t^b \bar{\Psi}^{k,\psi}_\vartheta(s,t)\psi'(s)g(s)ds,$$

with $k>0$ *and* $\bar{\Psi}^{k,\psi}_{\vartheta}(t,s)=\dfrac{(\psi(t)-\psi(s))^{\frac{\vartheta}{k}-1}}{k\Gamma_k(\vartheta)}$. *Also in [18], Nápoles Valdés defined more generalized fractional integral operators by*

$$\mathcal{J}^{\vartheta,k;\psi}_{G,a+}g(t)=\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{\psi'(s)g(s)ds}{G(\psi(t)-\psi(s),\frac{\vartheta}{k})},$$
$$\mathcal{J}^{\vartheta,k;\psi}_{G,b-}g(t)=\frac{1}{k\Gamma_k(\vartheta)}\int_t^b\frac{\psi'(s)g(s)ds}{G(\psi(s)-\psi(t),\frac{\vartheta}{k})},$$

where $G(z,\vartheta)\in AC[a,b]$.

Theorem 2.3. *([18]) Let* $g:[a,b]\to\mathbb{R}$ *be an integrable function, and take* $\vartheta>0$ *and* $k>0$. *Then* $\mathcal{J}^{\vartheta,k;\psi}_{G,a+}g$ *exists for all* $t\in[a,b]$.

Theorem 2.4. *([18]) Let* $g\in X^p_\psi(a,b)$ *and take* $\vartheta>0$ *and* $k>0$. *Then* $\mathcal{J}^{\vartheta,k;\psi}_{G,a+}g\in C([a,b],\mathbb{R})$.

Lemma 2.5. *Let* $\vartheta>0$, $r>0$ *and* $k>0$. *Then, we have the following semigroup properties given by*

$$\mathcal{J}^{\vartheta,k;\psi}_{a+}\mathcal{J}^{r,k;\psi}_{a+}f(t)=\mathcal{J}^{\vartheta+r,k;\psi}_{a+}f(t)=\mathcal{J}^{r,k;\psi}_{a+}\mathcal{J}^{\vartheta,k;\psi}_{a+}f(t)$$

and

$$\mathcal{J}^{\vartheta,k;\psi}_{b-}\mathcal{J}^{r,k;\psi}_{b-}f(t)=\mathcal{J}^{\vartheta+r,k;\psi}_{b-}f(t)=\mathcal{J}^{r,k;\psi}_{b-}\mathcal{J}^{\vartheta,k;\psi}_{b-}f(t).$$

Proof. By [27, Lemma 1] and the property of the k-gamma function, for $\vartheta>0$, $r>0$ and $k>0$, we get

$$\begin{aligned}\mathcal{J}^{\vartheta,k;\psi}_{a+}\mathcal{J}^{r,k;\psi}_{a+}f(t)&=\frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I^{\frac{\vartheta}{k};\psi}_{a+}I^{\frac{r}{k};\psi}_{a+}f(t)\\&=\frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2k^{\frac{\vartheta}{k}-1}\Gamma(\frac{\vartheta}{k})k^{\frac{r}{k}-1}\Gamma(\frac{r}{k})}I^{\frac{\vartheta}{k};\psi}_{a+}I^{\frac{r}{k};\psi}_{a+}f(t)\\&=\frac{1}{k^{\frac{\vartheta+r}{k}}}I^{\frac{\vartheta+r}{k};\psi}_{a+}f(t)\\&=\mathcal{J}^{\vartheta+r,k;\psi}_{a+}f(t),\end{aligned}$$

where $I^{\vartheta;\psi}_{a+}$ is the fractional integral defined in [27]. We have also,

$$\begin{aligned}\mathcal{J}^{\vartheta,k;\psi}_{a+}\mathcal{J}^{r,k;\psi}_{a+}f(t)&=\frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I^{\frac{\vartheta}{k};\psi}_{a+}I^{\frac{r}{k};\psi}_{a+}f(t)\\&=\frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I^{\frac{r}{k};\psi}_{a+}I^{\frac{\vartheta}{k};\psi}_{a+}f(t)\\&=\mathcal{J}^{r,k;\psi}_{a+}\mathcal{J}^{\vartheta,k;\psi}_{a+}f(t).\end{aligned}$$

□

Lemma 2.6. *Let* $\vartheta,r>0$ *and* $k>0$. *Then, we have*

$$\mathcal{J}^{\vartheta,k;\psi}_{a+}\bar{\Psi}^{k,\psi}_{r}(t,a)=\bar{\Psi}^{k,\psi}_{\vartheta+r}(t,a)$$

and

$$\mathcal{J}^{\vartheta,k;\psi}_{b-}\bar{\Psi}^{k,\psi}_{r}(b,t)=\bar{\Psi}^{k,\psi}_{\vartheta+r}(b,t).$$

Proof. By Definition 2.2 and using the change of variable $\mu = \dfrac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$, where $t > a$, we get

$$\begin{aligned}\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) &= \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,s)\psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)ds \\ &= \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,a)\left[1-\frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}\right]^{\frac{\vartheta}{k}-1}\psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)ds \\ &= \bar{\Psi}_\vartheta^{k,\psi}(t,a)\bar{\Psi}_r^{k,\psi}(t,a)\int_0^1 [1-\mu]^{\frac{\vartheta}{k}-1}\mu^{\frac{r}{k}-1}d\mu.\end{aligned}$$

Using the Definition 2.1 of the k-beta function and the relation with the gamma function, we have

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(t,a).$$

□

Theorem 2.7. *Let $0 < a < b < \infty, \vartheta > 0, 0 \le \xi < 1$, $k > 0$ and $x \in C_{\xi,k;\psi}(J)$. If $\dfrac{\vartheta}{k} > 1-\xi$, then*

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = \lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = 0.$$

Proof. $x \in C_{\xi,k;\psi}(J)$ means that $\Psi_\xi^\psi(t,a)x(t) \in C(J,\mathbb{R})$. Then there exists a positive constant R such that for $t \in (a,b]$ we have

$$|\Psi_\xi^\psi(t,a)x(t)| < R,$$

thus,

$$|x(t)| < R\Gamma_k(k\xi)|\bar{\Psi}_{k\xi}^{k,\psi}(t,a)|. \tag{5}$$

Now, we apply the operator $\mathcal{J}_{a+}^{\vartheta,k;\psi}(\cdot)$ on both sides of Equation (5), and we use Lemma 2.6, so that we have

$$\begin{aligned}\left|\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t)\right| &< R\Gamma_k(k\xi)\left|\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_{k\xi}^{k,\psi}(t,a)\right| \\ &= R\Gamma_k(k\xi)\bar{\Psi}_{\vartheta+k\xi}^{k,\psi}(t,a).\end{aligned}$$

Then, we have the right-hand side approaches zero, as $x \to a$, and

$$\lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = 0.$$

□

We are now able to define the k-generalized ψ-Hilfer derivative as follows.

Definition 2.8. *(k-generalized ψ-Hilfer derivative) Let $n-1 < \dfrac{\vartheta}{k} \le n$ with $n \in \mathbb{N}$, $J = [a,b]$ an interval such that $-\infty \le a < b \le \infty$ and $g,\psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \ne 0$, for all $t \in J$. The k-generalized ψ-Hilfer fractional derivatives (left-sided and right-sided) ${}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}(\cdot)$ and ${}_k^H\mathcal{D}_{b-}^{\vartheta,r;\psi}(\cdot)$ of a function g of order ϑ and type $0 \le r \le 1$, with $k > 0$ are defined, respectively, by*

$${}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}g(t) = \left(\mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)$$

$$= \left(\mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)$$

and

$$\begin{aligned}{}_{k}^{H}\mathcal{D}_{b-}^{\vartheta,r;\psi}g(t) &= \left(\mathcal{J}_{b-}^{r(kn-\vartheta),k;\psi}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\left(k^{n}\mathcal{J}_{b-}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t)\\ &= \left(\mathcal{J}_{b-}^{r(kn-\vartheta),k;\psi}(-1)^{n}\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{b-}^{(1-r)(kn-\vartheta),k;\psi}g\right)\right)(t),\end{aligned}$$

where $\delta_{\psi}^{n} = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}$.

Lemma 2.9. *Let* $t > a$, $0 < \frac{\vartheta}{k} < 1, 0 \le r \le 1, k > 0$. *Then for* $0 < \xi < 1; \xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, *we have*

$$\left[{}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}\left(\Psi_{\xi}^{\psi}(s,a)\right)^{-1}\right](t) = 0.$$

Proof. From Definitions 2.2 and 2.8, we have

$$\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}k\left(\Psi_{\xi}^{\psi}(t,a)\right)^{-1} = \int_{a}^{t} k\bar{\Psi}_{kX}^{k,\psi}(t,s)\left(\Psi_{\xi}^{\psi}(s,a)\right)^{-1}\psi'(s)ds,$$

where $X = \frac{1}{k}(1-r)(k-\vartheta)$. Now, we make the change of the variable by $\mu = \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ to obtain

$$\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}k\left(\Psi_{\xi}^{\psi}(t,a)\right)^{-1} = \frac{k\left(\Psi_{\xi+X}^{\psi}(t,a)\right)^{-1}}{\Gamma_k(kX)}\left[\frac{1}{k}\int_{0}^{1}(1-\mu)^{X-1}\mu^{\xi-1}d\mu\right].$$

Then, by the definition of the k-beta function

$$B_k(\alpha,\beta) = \frac{1}{k}\int_{0}^{1} t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)},$$

and we have

$$\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}k\left(\Psi_{\xi}^{\psi}(t,a)\right)^{-1} = \frac{k\Gamma_k(k\xi)}{\Gamma_k(k(X+\xi))} = k\Gamma_k(k\xi).$$

Then,

$$\delta_{\psi}^{1}\left(\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}k\left(\Psi_{\xi}^{\psi}(t,a)\right)^{-1}\right) = 0.$$

□

Theorem 2.10. *If* $f \in C_{\xi,k;\psi}^{n}[a,b], n-1 < \frac{\vartheta}{k} < n$, $0 \le r \le 1$, *where* $n \in \mathbb{N}$ *and* $k > 0$, *then*

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\ &= f(t) - \sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))}\left\{\delta_{\psi}^{n-i}\left(\mathcal{J}_{a+}^{k(n-\xi),k;\psi}f(a)\right)\right\},\end{aligned}$$

where

$$\xi = \frac{1}{k}(r(kn-\vartheta)+\vartheta).$$

In particular, if $n = 1$, *we have*

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t) = f(t) - \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(r(k-\vartheta)+\vartheta)}\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}f(a).$$

Proof. From Definition 2.8 and Lemma 2.5, we have

$$\begin{aligned}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t) &= \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ \mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f\right)\right)(t)\\ &= \left(\mathcal{J}_{a+}^{r(kn-\vartheta)+\vartheta,k;\psi}\delta_{\psi}^{n}\left(k^{n}I_{a+}^{(1-r)(kn-\vartheta),k;\psi}f\right)\right)(t)\\ &= \int_{a}^{t}\bar{\Psi}_{k\xi}^{k,\psi}(t,s)\psi'(s)\left\{\delta_{\psi}^{n}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\ &= \frac{-\left(\psi(t)-\psi(a)\right)^{\xi-1}}{k\Gamma_{k}(k\xi)}\left\{\delta_{\psi}^{n-1}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &+ \frac{\xi-1}{k\Gamma_{k}(k\xi)}\int_{a}^{t}\frac{\psi'(s)}{\left(\psi(t)-\psi(s)\right)^{2-\xi}}\left\{\delta_{\psi}^{n-1}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds.\end{aligned}$$

Using the proprieties of the gamma and k-gamma functions, we get

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\ &= \frac{-\left(\psi(t)-\psi(a)\right)^{\xi-1}}{k^{\xi}\Gamma(\xi)}\left\{\delta_{\psi}^{n-1}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &+ \frac{1}{k^{\xi}\Gamma(\xi-1)}\int_{a}^{t}\frac{\psi'(s)}{\left(\psi(t)-\psi(s)\right)^{2-\xi}}\left\{\delta_{\psi}^{n-1}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds.\end{aligned}$$

So, with integration by parts n times, we obtain

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\ &= -\sum_{i=1}^{n}\frac{\left(\psi(t)-\psi(a)\right)^{\xi-i}}{k^{\xi}\Gamma(\xi-i+1)}\left\{\delta_{\psi}^{n-i}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &\quad+ \frac{1}{k^{\xi-n}\Gamma(\xi-n)}\int_{a}^{t}\frac{\psi'(s)}{\left(\psi(t)-\psi(s)\right)^{n+1-\xi}}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)ds,\\ &= -\sum_{i=1}^{n}\frac{\left(\psi(t)-\psi(a)\right)^{\xi-i}}{k^{i}\Gamma_{k}(k(\xi-i+1))}\left\{\delta_{\psi}^{n-i}\left(k^{n}\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &\quad+ \frac{1}{k\Gamma_{k}(k(\xi-n))}\int_{a}^{t}\frac{\psi'(s)}{\left(\psi(t)-\psi(s)\right)^{n+1-\xi}}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)ds,\\ &= -\sum_{i=1}^{n}\frac{\left(\psi(t)-\psi(a)\right)^{\xi-i}}{k^{i-n}\Gamma_{k}(k(\xi-i+1))}\left\{\delta_{\psi}^{n-i}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &\quad+ \mathcal{J}_{a+}^{k(\xi-n),k;\psi}I_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(t).\end{aligned}$$

Then by using Lemma 2.5, we obtain

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_{k}^{H}\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\ &= f(t)-\sum_{i=1}^{n}\frac{\left(\psi(t)-\psi(a)\right)^{\xi-i}}{k^{i-n}\Gamma_{k}(k(\xi-i+1))}\left\{\delta_{\psi}^{n-i}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}.\end{aligned}$$

□

Lemma 2.11. *Let $\vartheta > 0, 0 \leq r \leq 1$, and $x \in C^1_{\xi,k;\psi}(J)$, where $k > 0$. Then for $t \in (a,b]$,*

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t) = x(t).$$

Proof. We have from Definition 2.8 and Lemma 2.5 that, for $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$,

$$\begin{aligned}\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t) &= \left(\mathcal{J}^{r(k-\vartheta),k;\psi}_{a+}\delta^1_\psi\left(k\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+}\mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)\right)(t)\\ &= \left(\mathcal{J}^{k\xi-\vartheta,k;\psi}_{a+}\delta^1_\psi\left(k\mathcal{J}^{(1-r)(k-\vartheta)+\vartheta,k;\psi}_{a+}x\right)\right)(t)\\ &= \left(\mathcal{J}^{k\xi-\vartheta,k;\psi}_{a+}\delta^1_\psi\left(k\mathcal{J}^{k-k\xi+\vartheta,k;\psi}_{a+}x\right)\right)(t),\end{aligned}$$

and then, we obtain

$$\begin{aligned}\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t) = &\int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}}\delta^1_\psi\left[\int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}\right]ds\\ &\times \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}.\end{aligned} \tag{6}$$

On other hand, by integrating by parts, we have

$$\begin{aligned}\int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = \frac{1}{1-\xi+\frac{\vartheta}{k}}\Bigg[&x(a)\left(\psi(s)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}\\ &+\int_a^s \frac{x'(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-1-\frac{\vartheta}{k}}}\Bigg],\end{aligned}$$

and then, by applying δ^1_ψ, we get

$$\delta^1_\psi\int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = x(a)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}} + \int_a^s \frac{x'(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}. \tag{7}$$

Now, substituting (7) into Equation (6), and by Dirichlet's formula and the properties of the k-gamma function, we get

$$\begin{aligned}&\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t)\\ &= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\int_a^t \frac{x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}}ds}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}}\right.\\ &\left.+\int_a^t x'(t)dt\int_s^t \frac{\psi'(s)d\tau}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}\left(\psi(s)-\psi(\tau)\right)^{\xi-\frac{\vartheta}{k}}}\right].\end{aligned}$$

Making the following change of variables $\mu = \dfrac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ in the integral from a to t and similarly changing the variable in the integral from s to t, then we have

$$\begin{aligned}&\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t)\\ &= \left[\int_a^t x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}}\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1}ds\right.\\ &\quad\left.+\int_a^t x'(t)dt\int_s^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1}\left(\psi(s)-\psi(\tau)\right)^{-\xi+\frac{\vartheta}{k}}d\tau\right]\end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\\
&= \left[\frac{1}{k}\int_0^1 \mu^{-\xi+\frac{\vartheta}{k}}\,(1-\mu)^{\xi-\frac{\vartheta}{k}-1}\,d\mu\right]\left(x(a)+\int_a^t x'(t)dt\right)\\
&\times \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\\
&= \left[\frac{1}{k}\int_0^1 \mu^{(1-(\xi-\frac{\vartheta}{k}))-1}\,(1-\mu)^{\xi-\frac{\vartheta}{k}-1}\,d\mu\right]\left(x(a)+\int_a^t x'(t)dt\right)\\
&\times \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}.
\end{aligned}
$$

Then by the definition of the k-beta function, we obtain

$$
\begin{aligned}
\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t) &= \frac{[\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)]}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left(x(a)+\int_a^t x'(t)dt\right)\\
&= x(a)+\int_a^t x'(t)dt\\
&= x(t).
\end{aligned}
$$

□

3. Main Results

We consider the following fractional differential equation

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t)=\varphi(t),\quad t\in(a,b], \tag{8}$$

where $0<\vartheta<k, 0\le r\le 1$, with the conditions

$$\alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+)+\alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(b)=\alpha_3, \tag{9}$$

$$x(t)=\varpi(t),\quad t\in[a-\lambda,a],\ \lambda>0, \tag{10}$$

$$x(t)=\tilde{\varpi}(t),\quad t\in\left[b,b+\tilde{\lambda}\right],\ \tilde{\lambda}>0, \tag{11}$$

where $\xi=\dfrac{r(k-\vartheta)+\vartheta}{k}$, $k>0$, $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ such that $\alpha_1+\alpha_2\neq 0$ and where $\varphi(\cdot)\in C(J,\mathbb{R})$, $\varpi(\cdot)\in\mathcal{C}$ and $\tilde{\varpi}(\cdot)\in\tilde{\mathcal{C}}$.

The following theorem will be used in a result for the existence of a unique solution for the problem (8)-(11).

Theorem 3.1. *x satisfies* (8)-(11) *if and only if it satisfies*

$$
x(t)=\begin{cases}
\dfrac{\alpha_3-\alpha_2\left(\mathcal{J}^{k(1-\xi)+\vartheta,k;\psi}_{a+}\varphi\right)(b)}{(\alpha_1+\alpha_2)\Gamma_k(k\xi)\Psi^{\psi}_{\xi}(t,a)}+\left(\mathcal{J}^{\vartheta,k;\psi}_{a+}\varphi\right)(t), & t\in(a,b],\\
\varpi(t),\quad t\in[a-\lambda,a], &\\
\tilde{\varpi}(t),\quad t\in\left[b,b+\tilde{\lambda}\right]. &
\end{cases} \tag{12}
$$

Proof. For both directions of the proof, (10) and (11) are trivially satisfied.

Assume that x satisfies the equations (8)-(11). By applying the fractional integral operator $\mathcal{J}_{a+}^{\vartheta,k;\psi}(\cdot)$ on both sides of the fractional equation (8), we have

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)=\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t),$$

and using Theorem 2.10, we get

$$x(t)=\frac{\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x(a)}{\Psi_\xi^\psi(t,a)\Gamma_k(k\xi)}+\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t). \tag{13}$$

Applying $\mathcal{J}_{a+}^{k(1-\xi),k;\psi}(\cdot)$ on both sides of (13), using Lemma 2.5, Lemma 2.6 and taking $t=b$, we have

$$\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(b)=\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x(a)+\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b). \tag{14}$$

Multiplying both sides of (14) by α_2, we get

$$\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(b)=\alpha_2\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x(a)+\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b).$$

Using condition (9), we obtain

$$\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(b)=\alpha_3-\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+).$$

Thus

$$\alpha_3-\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+)=\alpha_2\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x(a)+\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b).$$

Then

$$\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+)=\frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b). \tag{15}$$

Substituting (15) into (13), we obtain (12).

For the converse, let us now prove that if x satisfies equation (12), then it satisfies (8)-(11). Applying the fractional derivative operator ${}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}(\cdot)$ on both sides of the fractional equation (12), then we get

$$\begin{aligned}\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}x\right)(t)=\ &{}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\left(\frac{\alpha_3-\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b)}{(\alpha_1+\alpha_2)\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)}\right)\\&+\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t).\end{aligned}$$

Using Lemma 2.9 and Lemma 2.11, we obtain equation (8). Now we apply the operator $\mathcal{J}_{a+}^{k(1-\xi),k;\psi}(\cdot)$ to equation (12), to obtain

$$\begin{aligned}\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(t)=\mathcal{J}_{a+}^{k(1-\xi),k;\psi}&\left(\frac{\alpha_3-\alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b)}{(\alpha_1+\alpha_2)\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)}\right)\\&+\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t).\end{aligned}$$

Now, using Lemma 2.5 and 2.6, we get

$$\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(t)=\frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b)$$

$$+\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t). \tag{16}$$

Using Theorem 2.7 with $t \to a$, we obtain

$$\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+) = \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b). \tag{17}$$

Next, taking $t = b$ in (16), we have

$$\begin{aligned}\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(b) &= \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b)\\ &\quad+\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b).\end{aligned} \tag{18}$$

From (17) and (18), we obtain (9). This completes the proof. □

As a consequence of Theorem 3.1, we have the following result.

Lemma 3.2. *Let* $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$ *where* $0<\vartheta<k$ *and* $0 \le r \le 1$, *let* $f : J \times C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right) \times \mathbb{R} \to \mathbb{R}$ *be a continuous function, and let* $\varpi(\cdot) \in \mathcal{C}$ *and* $\tilde{\varpi}(\cdot) \in \tilde{\mathcal{C}}$. *Then* $x \in \mathbb{F}$ *satisfies the problem* (1)-(4) *if and only if* x *is the fixed point of the operator* $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ *defined by*

$$(\mathcal{T}x)(t) = \begin{cases} \dfrac{\alpha_3 - \alpha_2\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b)}{(\alpha_1+\alpha_2)\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)} + \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t), & t \in (a,b],\\ \varpi(t), \quad t \in [a-\lambda,a], & \\ \tilde{\varpi}(t), \quad t \in \left[b, b+\tilde{\lambda}\right], & \end{cases} \tag{19}$$

where φ *is a function satisfying the functional equation*

$$\varphi(t) = f(t, x_t(\cdot), \varphi(t)).$$

We may employ Theorem 2.4 to easily demonstrate that for $x \in \mathbb{F}$, we have $\mathcal{T}x \in \mathbb{F}$, where $\mathcal{T}$ is the operator defined in (19).

The following hypotheses will be used in the sequel :

(*Ax1*) The function $f : J \times C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(*Ax2*) There exist constants $\zeta_1 > 0$ and $0 < \zeta_2 < 1$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \zeta_1\|x_1-x_2\|_{[-\lambda,\tilde{\lambda}]} + \zeta_2|y_1-y_2|$$

for any $x_1, x_2 \in C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right)$, $y_1, y_2 \in \mathbb{R}$ and $t \in (a,b]$.

(*Ax3*) There exist functions $q_1, q_2, q_3 \in C(J,\mathbb{R}_+)$ with

$$q_1^* = \sup_{t\in J} q_1(t),\ q_2^* = \sup_{t\in J} q_2(t),\ q_3^* = \sup_{t\in J} q_3(t) < 1,$$

such that

$$|f(t,x,y)| \le q_1(t) + q_2(t)\|x\|_{[-\lambda,\tilde{\lambda}]} + q_3(t)|y|$$

for any $x \in C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right)$, $y \in \mathbb{R}$ and $t \in (a,b]$.

We are now in a position to state and prove our existence result for the problem (1)-(4) based on Banach's fixed point theorem [11].

Theorem 3.3. *Assume* (Ax1) *and* (Ax2) *hold. If*

$$\mathcal{L} = \frac{\zeta_1 \left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{1-\zeta_2}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)}+\frac{1}{\Gamma_k(\vartheta+k)}\right] < 1, \tag{20}$$

then the problem (1)-(4) *has a unique solution in* $\mathbb{F}$.

Proof. We show that the operator $\mathcal{T}$ defined in (19) has a unique fixed point in $\mathbb{F}$.
Let $x,y\in\mathbb{F}$. Then for any $t\in[a-\lambda,a]\cup\left[b,b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x(t)-\mathcal{T}y(t)|=0.$$

Thus

$$\|\mathcal{T}x-\mathcal{T}y\|_{\mathcal{C}}=\|\mathcal{T}x-\mathcal{T}y\|_{\tilde{\mathcal{C}}}=0. \tag{21}$$

Further, for $t\in(a,b]$ we have

$$|\mathcal{T}x(t)-\mathcal{T}y(t)| \le \frac{|\alpha_2|\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi_1(s)-\varphi_2(s)|\right)(b)}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)} + \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}|\varphi_1(s)-\varphi_2(s)|\right)(t),$$

where φ_1 and φ_1 be functions satisfying the functional equations

$$\varphi_1(t)=f(t,x_t(\cdot),\varphi_1(t)),$$
$$\varphi_2(t)=f(t,y_t(\cdot),\varphi_2(t)).$$

By $(Ax2)$, we have

$$\begin{aligned}|\varphi_1(t)-\varphi_2(t)| &= |f(t,x_t,\varphi_1(t))-f(t,y_t,\varphi_2(t))| \\ &\le \zeta_1\|x_t-y_t\|_{[-\lambda,\tilde{\lambda}]}+\zeta_2|\varphi_1(t)-\varphi_2(t)|.\end{aligned}$$

Then,

$$|\varphi_1(t)-\varphi_2(t)|\le\frac{\zeta_1}{1-\zeta_2}\|x_t-y_t\|_{[-\lambda,\tilde{\lambda}]}.$$

Therefore, for each $t\in(a,b]$

$$\begin{aligned}&|\mathcal{T}x(t)-\mathcal{T}y(t)| \\ &\le \frac{\zeta_1|\alpha_2|\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\|x_s-y_s\|_{[-\lambda,\tilde{\lambda}]}\right)(b)}{(1-\zeta_2)|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)} \\ &\quad+\frac{\zeta_1}{(1-\zeta_2)}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\|x_s-y_s\|_{[-\lambda,\tilde{\lambda}]}\right)(t) \\ &\le \frac{\zeta_1\|x-y\|_{\mathbb{F}}}{1-\zeta_2}\left[\frac{|\alpha_2|\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(b)}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)}+\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}(1)\right)(t)\right].\end{aligned}$$

By Lemma 2.6, we have

$$|\mathcal{T}x(t)-\mathcal{T}y(t)| \leq \left[\frac{\zeta_1|\alpha_2|\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Gamma_k(k(1-\xi)+\vartheta+k)\Psi_\xi^\psi(t,a)} + \frac{\zeta_1\left(\psi(t)-\psi(a)\right)^{\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(\vartheta+k)}\right]\|x-y\|_{\mathbb{F}}.$$

Hence

$$\left|\Psi_\xi^\psi(t,a)\left(\mathcal{T}x(t)-\mathcal{T}y(t)\right)\right| \leq \left[\frac{\zeta_1|\alpha_2|\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)} + \frac{\zeta_1\left(\psi(t)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(\vartheta+k)}\right]\|x-y\|_{\mathbb{F}},$$

which implies that

$$\|\mathcal{T}x-\mathcal{T}y\|_{C_{\xi,k;\psi}} \leq \frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{1-\zeta_2}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)} + \frac{1}{\Gamma_k(\vartheta+k)}\right]\|x-y\|_{\mathbb{F}}.$$

Thus

$$\|\mathcal{T}x-\mathcal{T}y\|_{C_{\xi,k;\psi}} \leq \mathcal{L}\|x-y\|_{\mathbb{F}}. \tag{22}$$

By (21) and (22), we obtain

$$\begin{aligned}\|\mathcal{T}x-\mathcal{T}y\|_{\mathbb{F}} &= \max\left\{\|\mathcal{T}x-\mathcal{T}y\|_{\mathcal{C}}, \|\mathcal{T}x-\mathcal{T}y\|_{\tilde{\mathcal{C}}}, \|\mathcal{T}x-\mathcal{T}y\|_{C_{\xi,k;\psi}}\right\}\\ &\leq \mathcal{L}\|x-y\|_{\mathbb{F}}.\end{aligned}$$

By (20), the operator $\mathcal{T}$ is a contraction on $\mathbb{F}$. Hence, by Banach's contraction principle, $\mathcal{T}$ has a unique fixed point $x \in \mathbb{F}$, which is a solution to our problem (1)-(4). □

Our next existence result for the problem (1)-(4) is based on based on Schauder's fixed point theorem [11].

Theorem 3.4. *Assume* (Ax1)-(Ax3) *hold. If*

$$\begin{aligned}\ell = \frac{q_2^*}{1-q_3^*}\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}\left[\frac{1}{\Gamma_k(\vartheta+k)} + \frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)}\right]\\ < 1,\end{aligned} \tag{23}$$

then the problem (1)-(4) *has at least one solution in* $\mathbb{F}$.

Proof. In several steps, we will use Schauder's fixed point theorem to prove that the operator $\mathcal{T}$ defined in (19) has a fixed point.

Step 1: The operator $\mathcal{T}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathbb{F}$. For each $t \in [a-\lambda, a] \cup \left[b, b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x(t)| = 0.$$

And for $t \in (a,b]$, we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x(t)| \leq \frac{|\alpha_2|(\psi(t)-\psi(a))^{\xi-1}}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)} \left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi} |\varphi_n(s)-\varphi(s)|\right)(b) + \left(\mathcal{J}_{a+}^{\vartheta,k;\psi} |\varphi_n(s)-\varphi(s)|\right)(t),$$

where φ_1 and φ_1 be functions satisfying the functional equations

$$\varphi(t) = f(t, x_t(\cdot), \varphi(t)),$$
$$\varphi_n(t) = f(t, x_{nt}(\cdot), \varphi_n(t)).$$

Since $x_n \to x$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in (a,b]$, and since f is continuous, then we have

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathcal{T}(B_M) \subset B_M$.
Let M a positive constant such that

$$M \geq \max\left\{\frac{|\alpha_3|}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)(1-\ell)} + \frac{q_1^*\ell}{q_2^*(1-\ell)}, \|\varpi\|_{\mathcal{C}}, \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}\right\}.$$

We define the following bounded closed set

$$B_M = \{x \in \mathbb{F} : \|x\|_{\mathbb{F}} \leq M\}.$$

For each $t \in [a-\lambda, a]$, we have

$$|\mathcal{T}x(t)| \leq \|\varpi\|_{\mathcal{C}},$$

and for each $t \in \left[b, b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x(t)| \leq \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}.$$

Further, for each $t \in (a,b]$, (19) implies that

$$|\mathcal{T}x(t)| \leq \frac{(\psi(t)-\psi(a))^{\xi-1}}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)} \left[|\alpha_3| + |\alpha_2| \left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi} |f(s,x_s,\varphi(s))|\right)(b)\right] + \left(\mathcal{J}_{a+}^{\vartheta,k;\psi} |f(s,x_s,\varphi(s))|\right)(t). \tag{24}$$

By the hypothesis $(Ax3)$, for $t \in (a,b]$, we have

$$|\varphi(t)| = |f(t,x_t,\varphi(t))| \leq q_1(t) + q_2(t)\|x_t\|_{[-\lambda,\tilde{\lambda}]} + q_3(t)|\varphi(t)|,$$

which implies that

$$|\varphi(t)| \leq q_1^* + q_2^*M + q_3^*|\varphi(t)|,$$

then

$$|\varphi(t)| \leq \frac{q_1^* + q_2^*M}{1-q_3^*} := \Delta.$$

Thus for $t \in (a, b]$, from (24) we get

$$|\Psi_\xi^\psi(t,a)\mathcal{T}x(t)| \leq \frac{|\alpha_3|}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)} + \frac{|\alpha_2|\Delta}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)} \left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(b) + \Delta\Psi_\xi^\psi(t,a)\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}(1)\right)(t).$$

By Lemma 2.6, we have

$$|\Psi_\xi^\psi(t,a)\mathcal{T}x(t)| \leq \frac{|\alpha_3|}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)} + \Delta\left[\frac{|\alpha_2|\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)} + \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}\right].$$

Thus

$$|\Psi_\xi^\psi(t,a)\mathcal{T}x(t)| \leq \frac{|\alpha_3|}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)} + \Delta\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}\left[\frac{1}{\Gamma_k(\vartheta+k)} + \frac{|\alpha_2|}{|\alpha_1 + \alpha_2|\Gamma_k(k\xi)\Gamma_k(2k-k\xi+\vartheta)}\right] \leq M.$$

Then, for each $t \in \left[a-\lambda, b+\tilde{\lambda}\right]$ we obtain

$$\|\mathcal{T}x\|_{\mathbb{F}} \leq M.$$

Step 3: $\mathcal{T}(B_M)$ is relatively compact.
Let $\tau_1, \tau_2 \in (a, b]$, $\tau_1 < \tau_2$ and let $x \in B_M$. Then

$$\begin{aligned}
&\left|\Psi_\xi^\psi(\tau_1,a)\mathcal{T}x(\tau_1) - \Psi_\xi^\psi(\tau_2,a)\mathcal{T}x(\tau_2)\right| \\
&\leq \left|\Psi_\xi^\psi(\tau_1,a)\left(\mathcal{J}_{a^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_1) - \Psi_\xi^\psi(\tau_2,a)\left(\mathcal{J}_{a^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_2)\right| \\
&\leq \int_a^{\tau_1}\left|\Psi_\xi^\psi(\tau_1,a)\bar{\Psi}_\vartheta^{k,\psi}(\tau_1,s) - \Psi_\xi^\psi(\tau_2,a)\bar{\Psi}_\vartheta^{k,\psi}(\tau_2,s)\right||\psi'(s)\varphi(s)|ds \\
&+ \left|\Psi_\xi^\psi(\tau_2,a)\left(\mathcal{J}_{\tau_1^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_2)\right|.
\end{aligned}$$

By Lemma 2.6, we get

$$\begin{aligned}
&\left|\Psi_\xi^\psi(\tau_1,a)\mathcal{T}x(\tau_1) - \Psi_\xi^\psi(\tau_2,a)\mathcal{T}x(\tau_2)\right| \\
&\leq \Delta\int_a^{\tau_1}\left|\Psi_\xi^\psi(\tau_1,a)\bar{\Psi}_\vartheta^{k,\psi}(\tau_1,s) - \Psi_\xi^\psi(\tau_2,a)\bar{\Psi}_\vartheta^{k,\psi}(\tau_2,s)\right||\psi'(s)|ds \\
&+ \frac{\Delta\Psi_\xi^\psi(\tau_2,a)\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}.
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the other cases is obvious, thus we omit the details. From Step 1 to Step 3, along with the Arzela-Ascoli theorem, we conclude that $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ continuous and compact. As a consequence of Schauder's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point which is a solution of the problem (1)-(4). □

4. Examples

In this section, we investigate specific cases of our problem (1)-(4), with $J = [1,3]$, $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$ and

$$f(t,x_1,x_2) = \frac{1}{33+31e^{3-t}}\left[1 - \frac{x_1}{2+x_1} + \frac{x_2}{1+|x_2|}\right],$$

where $t \in J$, $x_1 \in C\left(\left[-\lambda,\tilde{\lambda}\right],\mathbb{R}\right)$ and $x_2 \in \mathbb{R}$.

Example 4.1. *Taking $r \to 0$, $\vartheta = \frac{1}{2}$, $k = 1$, $\psi(t) = \ln(t)$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\lambda = \tilde{\lambda} = \frac{1}{2}$ and $\xi = \frac{1}{2}$, we obtain a boundary value problem which is a particular case of problem (1)-(4) with Hadamard fractional derivative, given by*

$$\left({}^H_1\mathcal{D}^{\frac{1}{2},0;\psi}_{1^+}x\right)(t) = \left({}^H\mathbb{D}^{\frac{1}{2}}_{1^+}x\right)(t) = f\left(t, x_t(\cdot), \left({}^H\mathbb{D}^{\frac{1}{2}}_{1^+}x\right)(t)\right), \quad t \in (1,3], \tag{25}$$

$$\left(\mathcal{J}^{\frac{1}{2},1;\psi}_{1^+}x\right)(1) + 2\left(\mathcal{J}^{\frac{1}{2},1;\psi}_{1^+}x\right)(3) = 3. \tag{26}$$

$$x(t) = \varpi(t), \quad t \in \left[\frac{1}{2},1\right], \tag{27}$$

$$x(t) = \tilde{\varpi}(t), \quad t \in \left[3,\frac{7}{2}\right]. \tag{28}$$

We have

$$C_{\xi,k;\psi}(J) = C_{\frac{1}{2},1;\psi}(J) = \left\{x : (1,3] \to \mathbb{R} : \sqrt{\ln(t)}x \in C(J,\mathbb{R})\right\}.$$

Then

$$\mathbb{F} = \left\{x : \left[\frac{1}{2},\frac{7}{2}\right] \to \mathbb{R} : x|_{\left[\frac{1}{2},1\right]} \in \mathcal{C},\ x|_{\left[3,\frac{7}{2}\right]} \in \tilde{\mathcal{C}} \text{ and } x|_{(1,3]} \in C_{\frac{1}{2},1;\psi}(J)\right\}.$$

Since the function f is continuous, then the condition (Ax1) *is satisfied.*
For each $x_1 \in C\left(\left[-\frac{1}{2},\frac{1}{2}\right],\mathbb{R}\right)$, $x_2 \in \mathbb{R}$ and $t \in J$, we have

$$|f(t,x_1,x_2)| \le \frac{1}{33+31e^{3-t}}\left(2 + \|x_1\|_{[-\lambda,\tilde{\lambda}]} + |x_2|\right).$$

Then, the condition (Ax3) *is satisfied*

$$q_1(t) = \frac{2}{33+31e^{3-t}},\ q_2(t) = q_3(t) = \frac{1}{33+31e^{3-t}},$$

and

$$q_1^* = \frac{2}{64},\ q_2^* = q_3^* = \frac{1}{64}.$$

We have

$$\ell = \frac{\ln(3)}{63}\left[\frac{2}{\sqrt{\pi}} + \frac{2}{3\sqrt{\pi}}\right] \approx 0.0262360046401739 < 1.$$

Then, by Theorem 3.4, we deduce that the problem (25)-(28) *has at least one solution in $\mathbb{F}$.*
Further, for each $x_1, y_1 \in C\left(\left[-\frac{1}{2},\frac{1}{2}\right],\mathbb{R}\right)$, $x_2, y_2 \in \mathbb{R}$ and $t \in J$, we have

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \frac{1}{33+31e^{3-t}}\left(\|x_1-y_1\|_{[-\lambda,\tilde{\lambda}]} + |x_2-y_2|\right),$$

and then, the condition (Ax2) *is satisfied with $\zeta_1 = \zeta_2 = \dfrac{1}{64}$. And since $\mathcal{L} = \ell$, then all the assumptions of Theorem 3.3 are satisfied. Consequently, the problem* (25)-(28) *has a unique solution in $\mathbb{F}$.*

Example 4.2. *Taking* $r \to \frac{1}{2}$, $\vartheta = \frac{1}{2}$, $k = 1$, $\psi(t) = t$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\lambda = \tilde{\lambda} = \frac{1}{5}$ *and* $\xi = \frac{3}{4}$, *we obtain an initial value problem which is a particular case of problem* (1)-(4) *with Hilfer fractional derivative, given by*

$$\left({}_1^H\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{2};\psi} x\right)(t) = \left({}^H\mathbb{D}_{1^+}^{\frac{1}{2},\frac{1}{2}} x\right)(t) = f\left(t, x_t(\cdot), \left({}^H\mathbb{D}_{1^+}^{\frac{1}{2},\frac{1}{2}} x\right)(t)\right), \quad t \in (1,3], \tag{29}$$

$$\left(\mathcal{J}_{1^+}^{\frac{1}{4},1;\psi} x\right)(1) = 0. \tag{30}$$

$$x(t) = 0, \quad t \in \left[\frac{4}{5}, 1\right], \tag{31}$$

$$x(t) = 0, \quad t \in \left[3, \frac{16}{5}\right]. \tag{32}$$

We have

$$C_{\xi,k;\psi}(J) = C_{\frac{3}{4},1;\psi}(J) = \left\{x : (1,3] \to \mathbb{R} : (t-1)^{\frac{1}{4}} x \in C(J,\mathbb{R})\right\},$$

and then

$$\mathbb{F} = \left\{x : \left[\frac{4}{5}, \frac{16}{5}\right] \to \mathbb{R} : x|_{\left[\frac{4}{5},1\right]} \in \mathcal{C}, \;\; x|_{\left[3,\frac{16}{5}\right]} \in \tilde{\mathcal{C}} \text{ and } x|_{(1,3]} \in C_{\frac{3}{4},1;\psi}(J)\right\}.$$

Also

$$\mathcal{L} = \frac{2^{\frac{7}{4}}}{63\sqrt{\pi}} \approx 0.0301222221161139 < 1.$$

As all the conditions of Theorem 3.3 are satisfied, then the problem (29)-(32) *has a unique solution in* $\mathbb{F}$.

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