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# Caputo-Katugampola type Implicit fractional differential equation with two-point anti-periodic boundary conditions

Saleh S. Redhwan[a], Sadikali L. Shaikh[b], Mohammed S. Abdo[c]

[a] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004 (M.S.), India.*
[b] *Department of Mathematics, Maulana Azad College of Arts, Science and Commerce, RozaBagh, Aurangabad 431004 (M.S.), India.*
[c] *Department of Mathematics, Hodeidah University, Al-Hodeidah-3114, Yemen.*

**Abstract**

The given article describes the implicit fractional differential equation with anti-periodic boundary conditions in the frame of Caputo-Katugampola fractional derivative. We obtain an analogous integral equation of the given problem and prove the existence and uniqueness results of such a problem using the Banach and Krasnoselskii fixed point theorems. Further, by applying generalized Gronwall inequality, the Ulam-Hyers stability results are discussed. To show the effectiveness of the acquired results, convenient examples are presented.

*Keywords:* implicit fractional differential equation fractional derivative and fractional integral anti-periodic conditions fixed point theorem Gronwall inequality.
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## 1. Introduction

Fractional differential equations have lately improved as an interesting area of research. In fact, fractional derivatives types supply a very good appliance for the explanation of store and hereditary properties of various materials and operations. More investigators have demonstrated that fractional differential equations show main parts in many research scopes, such as chemical technology, physics, biotechnology, population

*Email addresses:* saleh.redhwan909@gmail.com (Saleh S. Redhwan), sad.math@gmail.com (Sadikali L. Shaikh), msabdo1977@gmail.com (Mohammed S. Abdo)

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dynamics, and economics. On the forward evolution of the fractional differential equations have been caught much interest lately as study some results on the existence of solutions of fractional integro-differential equations have been studied by many authors by employing fixed point techniques. For more details, see [1, 6, 7, 8, 22, 24, 29, 30, 35] and many other references. Recently, there has been considerable growth in differential equations including the Caputo (Riemann-Liouville, Hilfer, and Hadamard) fractional derivative have been investigated and advanced by employing various tools from the nonlinear analysis. See Podlubny [26], the monographs of Kilbas et al. [21], Malinowska et al. [23], and some articles, for example, [28, 33] and the references cited therein.

In recent years, in [19] the researcher inserted a new fractional integral, which generalizes the Riemann-Liouville and Hadamard integrals into a one form. For more properties such as variational calculus applications, expansion formulas, convexity, control theoretical applications, and integral inequalities and Hermite-Hadamard type inequalities of this new operator and similar operators, can be see in [14, 15, 16]. The identicaling fractional derivatives were established in [18, 20, 23] which named Katugampola fractional operators.

In [17], the existence and uniqueness results of fractional differential equations involving Caputo-Katugampola (CK) derivative are given, the publisher applied the Peano theorem to find the existence and uniqueness of solution for the next Cauchy kind problem

$$^{CK}\mathfrak{D}_{0^+}^{\varsigma;\rho}\upsilon(\vartheta) = \mathfrak{g}(\vartheta, \upsilon(\vartheta)), \quad \vartheta \in [0, T], \tag{1}$$

$$\upsilon^{(k)}(0) = \upsilon_0^{(k)}, \quad k = 0, 1, ..., m-1, \ m = [\varsigma]. \tag{2}$$

In [3], R. Almeida via Gronwall inequality type established the uniqueness of solution of the problem (1)–(2) including $^{CK}D_{a^+}^{\varsigma;\rho}$. Furthermore, de Oliveira and Oliveira inspected the nonlinear fractional differential equations which include Hilfer-Katugampola derivative in [25] of the type

$$^{\rho}\mathfrak{D}_{a^+}^{\varsigma,\beta}\upsilon(\vartheta) = \mathfrak{g}(\vartheta, \upsilon(\vartheta)), \quad \vartheta \in \mathbb{J} = [a, b], \tag{3}$$

$$^{\rho}\mathfrak{J}_{a^+}^{1-\rho}\upsilon(a) = c, \quad \rho = \varsigma + \beta - \varsigma\beta. \tag{4}$$

They examine the existence and uniqueness results by applied the generalized Banach fixed point theorem on the problem (3)–(4).

Nowly, Saleh S. et al in ([27]), investigated the Implicit fractional differential equation (IFDE) with anti-periodic boundary condition involving Caputo-Katugampola of the kind

$$^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta) = \mathfrak{g}(\vartheta, \upsilon(\vartheta), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta)), \ \vartheta \in \mathbb{J} = [a, T].$$

$$\upsilon(a) + \upsilon(T) = 0.$$

Recently some fruitful results have been published in (IFDEs), see the papers [2, 9, 13, 31]. In order to investigate the different types of stability in the Ulam sense for fractional differential equations, we remind the works [10, 32]. Also, discuss some anti-periodic boundary value problems for fractional differential equations in [4, 5].

To the best of our knowledge, to present day not been studied CK-type implicit fractional differential equations with anti-periodic boundary conditions widely. So, in this work, we consider a class of boundary value problems for an implicit fractional differential equation (IFDE), that is

$$^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta) = \mathfrak{g}(\vartheta, \upsilon(\vartheta), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta)), \ \vartheta \in \mathbb{J} = [a, T]. \tag{5}$$

$$\begin{cases} \upsilon(a) + \upsilon(T) = 0 \\ \upsilon'(a) + \upsilon'(T) = 0 \end{cases}, \tag{6}$$

where $1 < \varsigma < 2$, ${}^{CK}D_{a^+}^{\varsigma;\rho}$ is the fractional derivatives of order $\varsigma$ in the CK sense, and $g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a suitable function.

The main aim of the article is to study the existence, uniqueness and Ulam-Hyers stability of solutions of the given problem (5)–(6). Our study is established on the Picard and fixed point techniques [12], and generalized Gronwall inequality [34].

The proposed problem is more general than that found in the literature. Moreover, our dissection can also be used in the corresponding problems by selecting the appropriate parameter of $\rho$ , i.e., The (CK) fractional derivative ${}^{CK}D_{a^+}^{\varsigma;\rho}$ is an interpolator of the following fractional derivatives: standard Caputo ($\rho \to 1$, $a \to 0$) [10] , Caputo-Hadamard ($\rho \to 0$) [11], Liouville ($\rho \to 1$, $a \to 0$) [21], and Weyl ($\rho \to 1$, $a \to -\infty$) [21].

The remainder of the article is displayed as follows: In the Section 2, we recall some essential definitions and properties which will be useful throughout this article and we proving some axiom lemmas which play a key role in the sequel. Section 3 includes sure sufficient conditions to establish the existence and uniqueness results of the problem (5)–(6) via fixed point techniques of Banach and Krasnoselskii. The stability analysis in the concept Ulam–Hyers of the suggested system is inspected in Section 4 by using generalized Gronwall inequality. At the end, some examples are involved to illustrate the applicability of the obtained results in Section 5.

## 2. Preliminaries

In this Section, we will present some preliminaries and essential lemmas, basic definitions, lemmas of fractional calculus theory and nonlinear analysis which are applied in this paper.

**Definition 2.1.** *[19] The Katugampola fractional integral of order $\varsigma > 0$ with $\rho > 0$ is defined by*

$$\mathfrak{I}_{a^+}^{\varsigma;\rho} z(\vartheta) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^{\vartheta} \tau^{\rho-1} \left(\vartheta^{\rho} - \tau^{\rho}\right)^{\varsigma-1} z(\tau) d\tau, \ \vartheta > a, \tag{7}$$

*where, $\Gamma(.)$ is a gamma function.*

**Definition 2.2.** *[18] The Katugampola fractional derivative of order $\varsigma$ $(n-1 < \varsigma < n)$ , $(n = [\varsigma] + 1)$ with $\rho > 0$ is defined as*

$$\begin{aligned} \mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) &= \left(\vartheta^{1-\rho} \frac{d}{d\vartheta}\right)^n \mathfrak{I}_{a^+}^{n-\varsigma;\rho} z(\vartheta) \\ &= \frac{\gamma^n \ \rho^{\varsigma-n+1}}{\Gamma(n-\varsigma)} \int_a^{\vartheta} \tau^{\rho-1} \left(\vartheta^{\rho} - \tau^{\rho}\right)^{n-\varsigma-1} z(\tau) d\tau, \ \vartheta > a, \end{aligned} \tag{8}$$

*where $\gamma = \left(\vartheta^{1-\rho} \frac{d}{d\vartheta}\right)$ . In particular, if $0 < \varsigma < 1$, $\rho > 0$, and $z \in C^1(J, \mathbb{R})$, we have*

$$\mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) = \left(\vartheta^{1-\rho} \frac{d}{d\vartheta}\right) \mathfrak{I}_{a^+}^{1-\varsigma;\rho} z(\vartheta) = \frac{\gamma \rho^{\varsigma}}{\Gamma(1-\varsigma)} \int_a^{\vartheta} \tau^{\rho-1} \left(\vartheta^{\rho} - \tau^{\rho}\right)^{-\varsigma} z(\tau) d\tau, \ \vartheta > a.$$

**Definition 2.3.** *[18] Let $\varsigma \geq 0$, $n = [\varsigma] + 1$. The (CK) fractional derivative of order $\varsigma$ with $\rho > 0$ is defined by*

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) = \mathfrak{D}_{a^+}^{\varsigma;\rho} \left[ z(\vartheta) - \sum_{k=0}^{n-1} \frac{z_{\rho}^{(k)}(a)}{k!} \rho^{-k} \left(\vartheta^{\rho} - a^{\rho}\right)^k \right], \tag{9}$$

*where $z_{\rho}^{(k)}(\vartheta) = \left(\vartheta^{1-\rho} \frac{d}{d\vartheta}\right)^k z(\vartheta)$. In case $0 < \varsigma < 1$, and $z \in C^1(J, \mathbb{R})$, we have*

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) = \mathfrak{D}_{a^+}^{\varsigma;\rho} \left[ z(\vartheta) - z(a) \right]. \tag{10}$$

*From (10) and (8), we obtain*

$$\begin{aligned} {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) &= \frac{\gamma \rho^{\varsigma}}{\Gamma(1-\varsigma)} \int_a^{\vartheta} \tau^{\rho-1} \left(\vartheta^{\rho} - \tau^{\rho}\right)^{-\varsigma} \left[z(\tau) - z(a)\right] d\tau, \\ \text{where } \vartheta &> a, \ \gamma = \left(\vartheta^{1-\rho} \frac{d}{d\vartheta}\right). \end{aligned}$$

*Obviously, if* $\varsigma \notin \mathbb{N}_0$*, and* $z \in C^1(J, \mathbb{R})$*, then the (CK) fractional derivative exists a.e, moreover, we have*

$$\begin{aligned} {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} z(\vartheta) &= \frac{\rho^{\varsigma}}{\Gamma(1-\varsigma)} \int_a^{\vartheta} \tau^{\rho-1} \left(\vartheta^{\rho} - \tau^{\rho}\right)^{-\varsigma} z_{\rho}^{(1)}(\tau) d\tau, \ \vartheta > a, \\ &= \mathfrak{I}_{a^+}^{1-\varsigma;\rho} z_{\rho}^{(1)}(\vartheta). \end{aligned}$$

*Also, if* $\varsigma \in \mathbb{N}$*, then* ${}^{CK}D_{a^+}^{\varsigma;\rho} z(\vartheta) = z_{\rho}^{(n)}(\vartheta)$*. Particularly,* ${}^{CK}D_{a^+}^{0;\rho} z(\vartheta) = z_{\rho}^{(0)}(\vartheta) = z(\vartheta)$*.*

**Lemma 2.4.** *[19]* $I_{a^+}^{\varsigma;\rho}$*,* ${}^{CK}D_{a^+}^{\varsigma;\rho}$ *are bounded operators from* $C[a,T]$ *to* $C[a,T]$*.*

**Lemma 2.5.** *[19] Let* $\varsigma > 0$*,* $\beta > 0$*,* $z \in X_c^p(a,T)$ $(1 \leq p \leq \infty)$*,* $\rho$ *,* $c \in \mathbb{R}$*,* $\rho \geq c$*. Then we have*

$$\mathfrak{I}_{a^+}^{\varsigma;\rho} \mathfrak{I}_{a^+}^{\beta;\rho} z(\vartheta) = \mathfrak{I}_{a^+}^{\varsigma+\beta;\rho} z(\vartheta), \qquad {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} \, {}^{\rho}\mathfrak{I}_{a^+}^{\varsigma} z(\vartheta) = z(\vartheta).$$

**Lemma 2.6.** *[19],[18] Let* $\vartheta > a$*,* $\varsigma, \delta \in (0, \infty)$*, and* $I_{a^+}^{\varsigma;\rho}, D_{a^+}^{\varsigma;\rho}$ *and* ${}^{CK}D_{a^+}^{\varsigma;\rho}$ *are according to (7), (8) and (9) respectively. Then we get*

$$\mathfrak{I}_{a^+}^{\varsigma;\rho} \left(\vartheta^{\rho} - a^{\rho}\right)^{\delta-1} = \frac{\rho^{-\varsigma} \ \Gamma(\delta)}{\Gamma(\delta+\varsigma)} \left(\vartheta^{\rho} - a^{\rho}\right)^{\varsigma+\delta-1},$$

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} \left(\vartheta^{\rho} - a^{\rho}\right)^{\delta-1} = \frac{\rho^{+\varsigma} \ \Gamma(\delta)}{\Gamma(\delta-\varsigma)} \left(\vartheta^{\rho} - a^{\rho}\right)^{\delta-\varsigma-1},$$

*and*

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} \left(\vartheta^{\rho} - a^{\rho}\right)^{k} = 0, \quad \varsigma \geq 0, \ k = 0, 1, ..., n-1.$$

*Particularly,* ${}^{CK}D_{a^+}^{\varsigma;\rho}(1) = 0$*.*

**Lemma 2.7.** *[27] Let* $\varsigma, \rho > 0$ *and* $\upsilon \in C(J, \mathbb{R}) \cap C^1(J, \mathbb{R})$*. Then*

1. *The (CK) fractional differential equation*

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} \upsilon(\vartheta) = 0,$$

*has a solution*

$$\upsilon(\vartheta) = c_0 + c_1 \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right) + c_2 \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right)^2 + .... + c_{n-1} \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right)^{n-1},$$

*where* $c_i \in \mathbb{R}$*,* $i = 0, 1, 2, ..., n-1$ *and* $n = [\varsigma] + 1$*.*

2. *If* $\upsilon, {}^{CK}D_{a^+}^{\varsigma;\rho} \upsilon \in C(J, \mathbb{R}) \cap C^1(J, \mathbb{R})$*. Then*

$$\begin{aligned} \mathfrak{I}_{a^+}^{\varsigma;\rho} \, {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho} \upsilon(\vartheta) &= \upsilon(\vartheta) + c_0 + c_1 \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right) + c_2 \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right)^2 \\ &\quad + .... + c_{n-1} \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right)^{n-1}, \end{aligned} \tag{11}$$

*where* $c_i \in \mathbb{R}$*,* $i = 0, 1, 2, ..., n-1$ *and* $n = [\varsigma] + 1$*.*

**Lemma 2.8.** *[34] Let $\varsigma > 0$, $\upsilon, w$ be two integrable functions and $z$ a continuous function, with domain $[a,T]$. Assume that $\upsilon$ and $w$ are nonnegative; and let $z$ is nonnegative and nondecreasing. If*

$$\upsilon(\vartheta) \leq w(\vartheta) + z(\vartheta)\rho^{1-\varsigma} \int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1}\upsilon(\tau)d\tau, \quad \vartheta \in [a,T],$$

*then*

$$\upsilon(\vartheta) \leq w(\vartheta) + \int_a^\vartheta \left[\sum_{k=1}^{\infty} \frac{\rho^{1-k\varsigma}(z(\vartheta)\Gamma(\varsigma))^k}{\Gamma(k\varsigma)}\tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{k\varsigma-1}w(\tau)\right] d\tau, \qquad \vartheta \in [a,T].$$

**Remark 2.9.** *In particular, if $w(\vartheta)$ be a nondecreasing function on $J$. Then we have*

$$\upsilon(\vartheta) \leq w(\vartheta) E_\varsigma\left[\mathfrak{g}(\vartheta)\Gamma(\varsigma)\left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)^\varsigma\right], \ \vartheta \in [a,T],$$

*where $E_\varsigma(\cdot)$ is the Mittag-Leffler function defined by*

$$E_\varsigma(\upsilon) = \sum_{k=0}^{\infty} \frac{\upsilon^k}{\Gamma(\varsigma k + 1)}, \quad \upsilon \in \mathbb{C}, \ Re(\varsigma) > 0.$$

*Here we can suffice to refer to Banach's fixed point theorem [12] and Krasnoselskii's fixed point theorem [12].*

## 3. Existence and uniqueness results:

In this Section, we will discuss the existence and uniqueness of solution to the problem (5)–(6) via the fixed point technique. The next lemma plays a necessary role in the following analysis.

**Lemma 3.1.** *Let $1 < \varsigma < 2$, $\rho > 0$ and $w \in C(J,\mathbb{R})$. Then a function $\upsilon$ is a solution of the fractional boundary value problem*

$$^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta) = w(\vartheta), \quad \vartheta \in \mathbb{J}, \tag{12}$$

$$\begin{cases} \upsilon(a) + \upsilon(T) = 0 \\ \upsilon'(a) + \upsilon'(T) = 0 \end{cases}, \tag{13}$$

*if $\upsilon(\vartheta)$ satisfies the following fractional integral equation*

$$\begin{aligned} \upsilon(\vartheta) \ = \ & \left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left[\left(\frac{T^\rho - a^\rho}{2\rho}\right) - \left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)\right] \\ & \times \frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-2}w(\tau)d\tau \\ & - \frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-1}w(\tau)d\tau \\ & + \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1}w(\tau)d\tau. \end{aligned} \tag{14}$$

*Proof.* Applying $I_{a^+}^{\varsigma;\rho}$ on both sides of (12), and employing Lemma (2.7), we get

$$\upsilon(\vartheta) = c_0 + c_1\left(\frac{\vartheta^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1}w(\tau)d\tau, \tag{15}$$

and

$$\upsilon'(\vartheta) = c_1\vartheta^{\rho-1} + \frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-2}w(\tau)d\tau, \tag{16}$$

where $c_0, c_1 \in \mathbb{R}$. Take the limit of the equations (15) and (16) as $\vartheta \to T$, and using boundary conditions (13), we attain

$$c_1 = -\frac{1}{T^{\rho-1}+a^{\rho-1}} \frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2} w(\tau)d\tau, \tag{17}$$

and

$$\begin{aligned} c_0 &= \frac{1}{2}\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho-a^\rho}{\rho}\right)\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}w(\tau)d\tau \\ &\quad -\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}w(\tau)d\tau. \end{aligned} \tag{18}$$

Substitute (17) and (18) into (15), we deduce

$$\begin{aligned} \upsilon(\vartheta) &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho-a^\rho}{2\rho}\right)-\left(\frac{\vartheta^\rho-a^\rho}{\rho}\right)\right] \\ &\quad \times\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}w(\tau)d\tau \\ &\quad -\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}w(\tau)d\tau \\ &\quad +\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-1}w(\tau)d\tau. \end{aligned}$$

□

As a result of Lemma 3.1, we have the following lemma:

**Lemma 3.2.** *Assume that $g : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with the help of Lemma 3.1. Then, the solution of the problem (5)–(6) is given by*

$$\begin{aligned} \upsilon(\vartheta) &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho-a^\rho}{2\rho}\right)-\left(\frac{\vartheta^\rho-a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\ &\quad \times\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau \\ &\quad -\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau \\ &\quad +\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau. \end{aligned}$$

In view of Lemma 3.2, we define the operator $\Pi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\begin{aligned} \Pi\upsilon(\vartheta) &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho-a^\rho}{2\rho}\right)-\left(\frac{\vartheta^\rho-a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\ &\quad \times\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau \\ &\quad -\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau \\ &\quad +\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau. \end{aligned} \tag{19}$$

Observe that, the problem (5)–(6) has a solution if and only if the operator $\Pi$ has fixed point.

Now, we present results on the existence, uniqueness of the solution of problem (5)–(6) via Banach's fixed point theorem.

**Theorem 3.3.** *Let $g : J \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfies the following condition:*

**(H$_1$)** *There exists a constant $0 < \lambda < 1$ such that:*

$$|\mathfrak{g}(\vartheta, \upsilon_1, \upsilon_2) - \mathfrak{g}(\vartheta, y_1, y_2)| \leq \lambda \left[|\upsilon_1 - y_1| + |\upsilon_2 - y_2|\right], \quad \forall \vartheta \in \mathbb{J}, \ \upsilon_i, y_i \in \mathbb{R}, (i = 1, 2).$$

*If*

$$\begin{aligned} \Upsilon &= \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\frac{\lambda\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho - a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)} + \frac{\lambda\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho - a^\rho)^\varsigma}{\Gamma(\varsigma+1)}\right] \\ &< 1, \end{aligned} \tag{20}$$

*then the problem (5)–(6) has a unique solution on $J$.*

*Proof.* First, we show that $\Pi B_\varepsilon \subseteq B_\varepsilon$, such that be $\Pi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by (19) and

$$\mathbb{B}_\varepsilon = \{\upsilon \in C(\mathbb{J}, \mathbb{R}), \|\upsilon\| \leq \varepsilon\}, \tag{21}$$

with select $\varepsilon \geq \frac{\Theta}{1-\Upsilon}$, where $\Upsilon < 1$ and

$$\Theta = \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\frac{\mu\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho - a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)} + \frac{\mu\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho - a^\rho)^\varsigma}{\Gamma(\varsigma+1)}\right],$$

and $\sup_{\vartheta\in\mathbb{J}} |\mathfrak{g}(\vartheta, 0, 0)| := \mu < \infty$. Put $G_\upsilon(\vartheta) := g(\vartheta, \upsilon(\vartheta), {}^{CK}D_{a^+}^{\varsigma;\rho}\upsilon(\vartheta))$. For any $\upsilon \in B_\varepsilon$, we get by our hypotheses that

$$\begin{aligned} |\Pi\upsilon(\vartheta)| &\leq \sup_{\vartheta\in\mathbb{J}} |\Pi\upsilon(\vartheta)| \\ &\leq \sup_{\vartheta\in\mathbb{J}} \left\{\left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left[\left(\frac{T^\rho - a^\rho}{2\rho}\right) + \left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\right. \\ &\quad \times \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-2} |\mathbb{G}_\upsilon(\tau)| \, d\tau + \\ &\quad \frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-1} |\mathbb{G}_\upsilon(\tau)| \, d\tau \\ &\quad \left. + \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1} |\mathbb{G}_\upsilon(\tau)| \, d\tau\right\}. \end{aligned}$$

From (H$_1$), we get

$$\begin{aligned} |\mathbb{G}_\upsilon(\tau)| &= \left|\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))\right| \\ &\leq \left|\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau)) - \mathfrak{g}(\tau, 0, 0)\right| + |\mathfrak{g}(\tau, 0, 0)| \\ &\leq \lambda |\upsilon(\tau)| + \lambda \left|{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau)\right| + \mu \\ &= \lambda\varepsilon + \lambda |\mathbb{G}_\upsilon(\tau)| + \mu, \end{aligned}$$

which implies

$$|\mathbb{G}_\upsilon(\tau)| \leq \frac{(\lambda\varepsilon + \mu)}{1-\lambda}. \tag{22}$$

Consequently,

$$\begin{aligned}
&|\Pi\upsilon(\vartheta)| \\
\leq\ & \sup_{\vartheta\in\mathbb{J}}\left\{\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^{\rho}-a^{\rho}}{2\rho}\right)+\left(\frac{\vartheta^{\rho}-a^{\rho}}{\rho}\right)\right]\frac{(\lambda\varepsilon+\mu)}{1-\lambda}\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\right. \\
&\times\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}d\tau+\frac{1}{2}\frac{(\lambda\varepsilon+\mu)}{1-\lambda}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-1}d\tau \\
&\left.+\frac{(\lambda\varepsilon+\mu)}{1-\lambda}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}d\tau\right\}. \\
\leq\ & \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^{\rho}-a^{\rho}}{2\rho}\right)+\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\right] \\
&\times\left[\frac{(\lambda\varepsilon+\mu)}{1-\lambda}\frac{\rho^{1-\varsigma}}{}\frac{(T^{\rho}-a^{\rho})^{\varsigma-1}}{\Gamma(\varsigma)}\right]+\frac{3}{2}\frac{(\lambda\varepsilon+\mu)}{1-\lambda}\frac{\rho^{-\varsigma}}{}\frac{(T^{\rho}-a^{\rho})^{\varsigma}}{\Gamma(\varsigma+1)} \\
\leq\ & \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\lambda\rho^{1-\varsigma}}{1-\lambda}\frac{(T^{\rho}-a^{\rho})^{\varsigma-1}}{\Gamma(\varsigma)}\right. \\
&\left.+\frac{\lambda\rho^{-\varsigma}}{1-\lambda}\frac{(T^{\rho}-a^{\rho})^{\varsigma}}{\Gamma(\varsigma+1)}\right]\varepsilon \\
&+\frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\mu\rho^{1-\varsigma}}{1-\lambda}\frac{(T^{\rho}-a^{\rho})^{\varsigma-1}}{\Gamma(\varsigma)}\right. \\
&\left.+\frac{\mu\rho^{-\varsigma}}{1-\lambda}\frac{(T^{\rho}-a^{\rho})^{\varsigma}}{\Gamma(\varsigma+1)}\right] \\
=\ & \Upsilon\varepsilon+\Theta<\varepsilon.
\end{aligned}$$

Therefore,

$$\|\Pi\upsilon\|<\varepsilon,$$

which gives that $\Pi\upsilon\in B_{\varepsilon}$. Moreover, by (19), and lammas 2.5, 2.6, we obtain

$${}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\Pi\upsilon(\vartheta)={}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\,\mathfrak{I}_{a^+}^{\varsigma;\rho}\mathbb{G}_{\upsilon}(\vartheta)=\mathbb{G}_{\upsilon}(\vartheta).$$

Since $G_{\upsilon}(\cdot)$ is continuous on $J$, the operator ${}^{CK}D_{a^+}^{\varsigma;\rho}\Pi\upsilon(\vartheta)$ is continuous on $J$, that is $\Pi\ B_{\varepsilon}\subseteq B_{\varepsilon}$.

Second, we apply the Banach fixed point theorem to show that $\Pi$ has a fixed point. In fact, it sufficient to prove that $\Pi$ is contraction map. Set $\upsilon_1,\upsilon_2\in C(J,\mathbb{R})$ and for $\vartheta\in J$. Then, we get

$$\begin{aligned}
&|\Pi\upsilon_1(\vartheta)-\Pi\upsilon_2(\vartheta)| \\
\leq\ & \frac{3}{2}\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\
&\times\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}\left|\mathbb{G}_{\upsilon_1}(\tau)-\mathbb{G}_{\upsilon_2}(\tau)\right|d\tau \\
&+\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-1}\left|\mathbb{G}_{\upsilon_1}(\tau)-\mathbb{G}_{\upsilon_2}(\tau)\right|d\tau \\
&+\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\left|\mathbb{G}_{\upsilon_1}(\tau)-\mathbb{G}_{\upsilon_2}(\tau)\right|d\tau.
\end{aligned}$$

By $(\mathrm{H}_1)$, we have

$$\begin{aligned}
&|\mathbb{G}_{\upsilon_1}(\tau)-\mathbb{G}_{\upsilon_2}(\tau)| \\
=\ & \left|\mathfrak{g}(\tau,\upsilon_1(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon_1(\tau))-\mathfrak{g}(\tau,\upsilon_1(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon_2(\tau))\right| \\
\leq\ & \lambda\left|\upsilon_1-\upsilon_2\right|+\lambda\left|{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon_1(\tau)-{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon_2(\tau)\right| \\
=\ & \lambda\left|\upsilon_1-\upsilon_2\right|+\lambda\left|\mathbb{G}_{\upsilon_1}(\tau)-\mathbb{G}_{\upsilon_2}(\tau)\right|,
\end{aligned}$$

which implies

$$|\mathbb{G}_{\upsilon_1}(\tau) - \mathbb{G}_{\upsilon_2}(\tau)| \leq \frac{\lambda}{1-\lambda} |\upsilon_1 - \upsilon_2| . \tag{23}$$

Thus

$$\begin{aligned} & \|\Pi\upsilon_1 - \Pi\upsilon_2\| \\ \leq & \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\frac{\lambda\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)} \right. \\ & \left. + \frac{\lambda\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma}}{\Gamma(\varsigma+1)}\right] \|\upsilon_1-\upsilon_2\| \\ \leq & \Upsilon \|\upsilon_1-\upsilon_2\| \end{aligned}$$

Since $\Upsilon < 1$ shows that the operator $\Pi$ is contraction mapping. Then, a unique solution exists on $J$ to the problem (5)–(6) by virtue of the Banach's fixed point theorem [12], and this finishes the proof. □

Our second existence result for the problem (5)–(6) is based on the Krasnoselskii$'$s fixed point theorem [12].

**Theorem 3.4.** *Assume that* $(H_1)$ *holds. If*

$$\begin{aligned} & \Lambda \\ : & = \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\frac{\lambda\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)} + \frac{\lambda\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma}}{\Gamma(\varsigma+1)}\right] < \frac{1}{2}, \end{aligned}$$

*then there exist at least one solution of the problem (5)–(6) on* $J$.

*Proof.* Consider the operator $\Pi$ defined by (19). Set the ball $B_{\varepsilon_0} := \{\upsilon \in C(J, \mathbb{R}) : \|\upsilon\| \leq \varepsilon_0\}$, with $\varepsilon_0 \geq 2\mu\Lambda$, where $\mu$ is defined as in Theorem 3.3. Moreover, we define two operators $\Pi_1$ and $\Pi_2$ on $B_{\varepsilon_0}$ by

$$\begin{aligned} \Pi_1\upsilon(\vartheta) &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho - a^\rho}{2\rho}\right) - \left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\ & \times \int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}\mathbb{G}_\upsilon(\tau)d\tau - \frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}\mathbb{G}_\upsilon(\tau)d\tau, \end{aligned}$$

and

$$\Pi_2\upsilon(\vartheta) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-1}\mathbb{G}_\upsilon(\tau)d\tau.$$

Taking into account that $\Pi_1$ and $\Pi_2$ are defined on $B_{\varepsilon_0}$, and for any $\upsilon \in C(J, \mathbb{R})$,

$$\Pi\upsilon(\vartheta) = \Pi_1\upsilon(\vartheta) + \Pi_2\upsilon(\vartheta), \quad \vartheta \in \mathbb{J}.$$

The proof will be divided into several stages as follows:

Stage 1: $\Pi_1\upsilon_1 + \Pi_2\upsilon_2 \in B_{\varepsilon_0}$ for every $\upsilon_1, \upsilon_2 \in B_{\varepsilon_0}$.

For $\upsilon_1 \in B_{\varepsilon_0}$ and applying the same arguments in (22), we have

$$|\mathbb{G}_{\upsilon_1}(\tau)| \leq \frac{(\lambda\varepsilon_0 + \mu)}{1-\lambda}.$$

Similarly, for $\upsilon_2 \in B_{\varepsilon_0}$, we obain

$$|\mathbb{G}_{\upsilon_2}(\tau)| \leq \frac{(\lambda\varepsilon_0 + \mu)}{1-\lambda}.$$

Now, for $\upsilon_1, \upsilon_2 \in B_{\varepsilon_0}$ and $\vartheta \in J$, we get

$$\begin{aligned}
& |\Pi_1\upsilon_1(\vartheta) + \Pi_2\upsilon_2(\vartheta)| \\
\leq\ & |\Pi_1\upsilon_1(\vartheta)| + |\Pi_2\upsilon_2(\vartheta)| \\
\leq\ & \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{3}{2}\frac{T^\rho - a^\rho}{\rho}\right)\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \times \int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-2}\left|\mathbb{G}_{\upsilon_1}(\tau)\right| d\tau \\
& + \frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\varsigma-1}\left|\mathbb{G}_{\upsilon_1}(\tau)\right| d\tau \\
& + \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-1}\left|\mathbb{G}_{\upsilon_2}(\tau)\right| d\tau \\
\leq\ & \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho-a^\rho}{2\rho}\right)+\left(\frac{T^\rho-a^\rho}{\rho}\right)\right]\frac{(\lambda\varepsilon_0+\mu)\,\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)} \\
& + \frac{3}{2}\frac{(\lambda\varepsilon_0+\mu)\,\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma}}{\Gamma(\varsigma+1)} \\
\leq\ & \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho-a^\rho}{\rho}\right)\frac{\lambda\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)}\right. \\
& \left. + \frac{\lambda\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma}}{\Gamma(\varsigma+1)}\right]\varepsilon_0 \\
& + \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho-a^\rho}{\rho}\right)\frac{\mu\rho^{1-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma-1}}{\Gamma(\varsigma)}\right. \\
& \left. + \frac{\mu\rho^{-\varsigma}}{1-\lambda}\frac{(T^\rho-a^\rho)^{\varsigma}}{\Gamma(\varsigma+1)}\right],
\end{aligned}$$

which implies

$$\|\Pi_1\upsilon_1 + \Pi_2\upsilon_2\| \leq \varepsilon_0. \tag{24}$$

This proves that $\Pi_1\upsilon_1 + \Pi_2\upsilon_2 \in B_{\varepsilon_0}$ for every $\upsilon_1, \upsilon_2 \in B_{\varepsilon_0}$.

Stage 2 $\Pi_1$ is a contration mapping on $B_{\varepsilon_0}$.

Since $\Pi$ is contraction mapping as in Theorem 3.3, then $\Pi_1$ is a contraction mapping too.

Stage 3. We will prove in three steps that the operator $\Pi_2$ on $B_{\varepsilon_0}$ is completely continuous.

Step 1, the operator $\Pi_2$ is continuous because of the continuity of $G_\upsilon(\cdot)$.

Step 2, It is easy to prove that

$$\|\Pi_2\upsilon\| \leq \frac{(\lambda\varepsilon_0+\mu)}{1-\lambda}\frac{\rho^{-\varsigma}}{\Gamma(\varsigma+1)}(T^\rho-a^\rho)^\varsigma < \varepsilon_0,$$

due to definitions of $\Lambda$ and $\varepsilon_0$. This verifies that $\Pi_2$ is uniformly bounded on $B_{\varepsilon_0}$

Step 3, we show that $\Pi_2$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$, i.e., $(\Pi_2\mathbb{B}_{\varepsilon_0})$ is equicontinuous. We estimate the derivative of $\Pi_2\upsilon(\vartheta)$

$$\begin{aligned}
\left|(\Pi_2\upsilon)'(\vartheta)\right| &= \left|\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma-1)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-2}\mathbb{G}_\upsilon(\tau)d\tau\right| \\
&\leq \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma-1)}\int_a^\vartheta \tau^{\rho-1}(\vartheta^\rho-\tau^\rho)^{\varsigma-2}\left|\mathbb{G}_\upsilon(\tau)\right| d\tau \\
&\leq \frac{(\lambda\varepsilon_0+\mu)}{1-\lambda}\frac{\rho^{-\varsigma}}{\Gamma(\varsigma)}(T^\rho-a^\rho)^{\varsigma-1} := K
\end{aligned}$$

Now, Let $\vartheta_1, \vartheta_2 \in J$, with $\vartheta_1 < \vartheta_2$ and for any $\upsilon \in B_{\varepsilon_0}$. Then we have

$$|\Pi_2\upsilon(\vartheta_1) - \Pi_2\upsilon(\vartheta_2)| = \int_{\vartheta_1}^{\vartheta_2}\left|(\Pi_2\upsilon)'(\tau)\right| d\tau \leq K(\vartheta_2-\vartheta_1).$$

As $\vartheta_1 \longrightarrow \vartheta_2$ the right-hand side of the above inequality is not dependent on $\upsilon$ and tends to zero. Therefore,

$$|\Pi_2\upsilon(\vartheta_1) - \Pi_2\upsilon(\vartheta_2)| \to 0, \ \forall \ |\vartheta_2 - \vartheta_1| \to 0, \ \upsilon \in \mathbb{B}_{\varepsilon_0}.$$

This proves that $\Pi_2$ is equicontinuous on $B_{\varepsilon_0}$. An application of Arzela-Ascoli Theorem shows that $\Pi_2$ is relatively compact on $B_{\varepsilon_0}$. Hence all the assumptions of Krasnoselskii's fixed point theorem are satisfied. Thus, we deduce that the problem (5)–(6) has at least one solution on $J$. □

## 4. Ulam-Hyers stability

This part is devoted to discussing the Ulam-Hyers and generalized Ulam-Hyers stability of solution of (CK)-type for the problem (5)–(6).

**Definition 4.1.** *[10, 32], The problem (5)–(6) is Ulam-Hyers stable, if there exists a real number $K_f > 0$, such that for each $\varepsilon > 0$ and for each solution $\widetilde{\upsilon} \in C(J, \mathbb{R})$ of the inequality*

$$\left|{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\vartheta) - \mathfrak{g}(\vartheta, \widetilde{\upsilon}(\vartheta), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\vartheta))\right| \le \varepsilon, \quad \vartheta \in \mathbb{J}, \tag{25}$$

*there exists a solution $\upsilon \in C(J, \mathbb{R})$ for the problem (5)-(6) such that*

$$|\widetilde{\upsilon}(\vartheta) - \upsilon(\vartheta)| \le K_f\varepsilon, \quad \vartheta \in \mathbb{J}.$$

**Definition 4.2.** *[10, 32] The problem (5)-(6) is generalized Ulam-Hyers stable if there exists $\Psi \in C([0, \infty), [0, \infty))$ with $\Psi(0) = 0$, such that for each solution $\widetilde{\upsilon} \in C(J, R)$ of the inequality*

$$\left|{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\ \widetilde{\upsilon}(\vartheta) - \mathfrak{g}(\vartheta, \widetilde{\upsilon}(\vartheta), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\vartheta))\right| \le \varepsilon, \quad \vartheta \in \mathbb{J}, \tag{26}$$

*there exists a solution $\upsilon \in C(J, \mathbb{R})$ for the problem (5)-(6) such that*

$$|\widetilde{\upsilon}(\vartheta) - \upsilon(\vartheta)| \le \Psi(\varepsilon), \ \vartheta \in \mathbb{J}.$$

**Remark 4.3.** *[10] Let $\varsigma, \rho > 0$. A function $\widetilde{\upsilon} \in C(J, \mathbb{R})$ is a solution of the inequality (25) if and only if there exist a function $h_{\widetilde{\upsilon}} \in C(J, \mathbb{R})$ such that*

1. *$|h_{\widetilde{\upsilon}}(\vartheta)| \le \varepsilon$ for all $\vartheta \in J$,*
2. *$D_{a^+}^{\varsigma;\rho}\ \widetilde{\upsilon}(\vartheta) = g(\vartheta, \widetilde{\upsilon}(\vartheta), {}^{CK}D_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\vartheta)) + h_{\widetilde{\upsilon}}(\vartheta)$, $\vartheta \in J$.*

**Lemma 4.4.** *Let $\widetilde{\upsilon} \in C(J, \mathbb{R})$ is a solution of the inequality (25). Then $\widetilde{\upsilon}$ is a solution of the following integral inequality:*

$$\left|\widetilde{\upsilon}(\vartheta) - Z_{\widetilde{\upsilon}} - \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta} \tau^{\rho-1}\left(\vartheta^{\rho} - a^{\rho}\right)^{\varsigma-1} \mathfrak{g}(\tau, \widetilde{\upsilon}(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau\right| \le \zeta\varepsilon,$$

*where*

$$\zeta := \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left(\frac{T^{\rho} - a^{\rho}}{\rho}\right)\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\left(T^{\rho} - a^{\rho}\right)^{\varsigma-1} + \frac{\rho^{-\varsigma}}{\Gamma(\varsigma+1)}\left(T^{\rho} - a^{\rho}\right)^{\varsigma}\right],$$

*and*

$$\begin{aligned} Z_{\widetilde{\upsilon}} &= \left(\frac{1}{T^{\rho-1} + a^{\rho-1}}\right)\left[\left(\frac{T^{\rho} - a^{\rho}}{2\rho}\right) - \left(\frac{\vartheta^{\rho} - a^{\rho}}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\ &\times \int_a^T \tau^{\rho-1}(T^{\rho} - \tau^{\rho})^{\varsigma-2}\mathfrak{g}(\tau, \widetilde{\upsilon}(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau \\ &- \frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^{\rho} - \tau^{\rho})^{\varsigma-1}\mathfrak{g}(\tau, \widetilde{\upsilon}(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau. \end{aligned} \tag{27}$$

*Proof.* Thanks to Remark 4.3, and Theorem 3.3, we get

$$\begin{aligned}\widetilde{\upsilon}(\vartheta) &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^{\rho}-a^{\rho}}{2\rho}\right)-\left(\frac{\vartheta^{\rho}-a^{\rho}}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\\ &\times\int_a^T \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}\left[\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))+h_{\widetilde{\upsilon}}(\tau))\right]d\tau\\ &-\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-1}\left[\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))+h_{\widetilde{\upsilon}}(\tau))\right]d\tau\\ &+\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta} \tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\left[\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))+h_{\widetilde{\upsilon}}(\tau))\right]d\tau. \end{aligned}\tag{28}$$

It follows that

$$\begin{aligned}&\left|\widetilde{\upsilon}(\vartheta)-Z_{\widetilde{\upsilon}}-\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}\left(\vartheta^{\rho}-a^{\rho}\right)^{\varsigma-1}\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau\right|\\ \leq\ &\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{3}{2}\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}\left|h_{\widetilde{\upsilon}}(\tau)\right|d\tau\\ &\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T\tau^{\rho-1}\left(T^{\rho}-a^{\rho}\right)^{\varsigma-1}\left|h_{\widetilde{\upsilon}}(\vartheta)\right|d\tau+\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}\left(\vartheta^{\rho}-a^{\rho}\right)^{\varsigma-1}\left|h_{\widetilde{\upsilon}}(\vartheta)\right|d\tau\\ \leq\ &\frac{3\varepsilon}{2}\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}d\tau\\ &+\frac{\varepsilon}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T\tau^{\rho-1}\left(T^{\rho}-a^{\rho}\right)^{\varsigma-1}d\tau+\varepsilon\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}\left(\vartheta^{\rho}-a^{\rho}\right)^{\varsigma-1}d\tau\\ \leq\ &\zeta\varepsilon.\end{aligned}$$

□

**Theorem 4.5.** *Suppose that the hypotheses of Theorem 3.3 are satisfied. Then the problem (5)-(6) is Ulam-Hyers stable.*

*Proof.* Let $\widetilde{\upsilon}\in C(J,\mathbb{R})$ be a function which satisfies the inequality (25) and $\varepsilon>0$, and let $\upsilon\in C(J,\mathbb{R})$ be the unique solution of the following (CK) fractional differential equation

$$^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\,\upsilon(\vartheta)=\mathfrak{g}(\vartheta,\upsilon(\vartheta),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\vartheta)),\quad \vartheta\in\mathbb{J},\tag{29}$$

with the conditions

$$\begin{cases}\upsilon(a)=\widetilde{\upsilon}(a),\ \upsilon(T)=\widetilde{\upsilon}(T),\\ \upsilon'(a)=\widetilde{\upsilon}'(a),\ \upsilon'(T)=\widetilde{\upsilon}'(T),\end{cases}\tag{30}$$

where $1<\varsigma<2$. By using Lemma 3.1, we can easily see that $\upsilon(\cdot)$ satisfies the integral equation

$$\upsilon(\vartheta)=Z_{\upsilon}+\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau,$$

where

$$\begin{aligned}Z_{\upsilon} &= \left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^{\rho}-a^{\rho}}{2\rho}\right)-\left(\frac{\vartheta^{\rho}-a^{\rho}}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)}\\ &\times\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-2}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau\\ &-\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T\tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\varsigma-1}\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau.\end{aligned}$$

Using Lemma 4.4, we get

$$\left|\widetilde{\upsilon}(\vartheta) - Z_{\widetilde{\upsilon}} - \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta} \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau\right| \leq \zeta\varepsilon. \tag{31}$$

Due (30), it is easily seen that $|Z_{\widetilde{\upsilon}} - Z_{\upsilon}| \to 0$. Indeed, from ($\mathrm{H}_1$) and (30), we deduce that

$$\begin{aligned}
|Z_{\widetilde{\upsilon}} - Z_{\upsilon}| &= \Bigg|\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho - a^\rho}{2\rho}\right) - \left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\
&\quad \times \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-2}\mathfrak{g}(\tau, \widetilde{\upsilon}(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau \\
&\quad -\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau, \widetilde{\upsilon}(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau \\
&\quad -\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left[\left(\frac{T^\rho - a^\rho}{2\rho}\right) - \left(\frac{\vartheta^\rho - a^\rho}{\rho}\right)\right]\frac{\rho^{2-\varsigma}}{\Gamma(\varsigma-1)} \\
&\quad \times \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-2}\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau \\
&\quad +\frac{1}{2}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau\Bigg| \\
&\leq \frac{3}{2}\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\mathfrak{I}_{a^+}^{\varsigma-1;\rho} \\
&\quad \times \left|\mathfrak{g}(T, \widetilde{\upsilon}(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(T)) - \mathfrak{g}(T, \upsilon(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(T))\right| \\
&\quad +\frac{1}{2}\,\mathfrak{I}_{a^+}^{\varsigma;\rho}\left|\mathfrak{g}(T, \widetilde{\upsilon}(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(T)) - \mathfrak{g}(T, \upsilon(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(T))\right|.
\end{aligned}$$

Since,

$$\begin{aligned}
&\left|\mathfrak{g}(T, \widetilde{\upsilon}(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(T)) - \mathfrak{g}(T, \upsilon(T), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(T))\right| \\
\leq\ & \lambda\left|\widetilde{\upsilon}(T) - \upsilon(T)\right| + \lambda\left|{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(T) - {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(T)\right| \\
\leq\ & \frac{\lambda}{1-\lambda}\left|\widetilde{\upsilon}(T) - \upsilon(T)\right|.
\end{aligned} \tag{32}$$

From (30), we have $\widetilde{\upsilon}(T) - \upsilon(T) = 0$. Hence,

$$\begin{aligned}
&|Z_{\widetilde{\upsilon}} - Z_{\upsilon}| \\
\leq\ & \frac{3}{2}\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^\rho - a^\rho}{\rho}\right)\frac{\lambda}{(1-\lambda)}\mathfrak{I}_{a^+}^{\varsigma-1;\rho}\ \left|\widetilde{\upsilon}(T) - \upsilon(T)\right| \\
&+\frac{\lambda}{2(1-\lambda)}\mathfrak{I}_{a^+}^{\varsigma;\rho}\ \left|\widetilde{\upsilon}(T) - \upsilon(T)\right| \\
\to\ & 0.
\end{aligned}$$

Hence,

$$\upsilon(\vartheta) = Z_{\widetilde{\upsilon}} + \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta} \tau^{\rho-1}(\vartheta^\rho - \tau^\rho)^{\varsigma-1}\mathfrak{g}(\tau, \upsilon(\tau), {}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))d\tau.$$

Through (31), $(H_1)$ and (32), we arrive at

$$\begin{aligned}
&\left|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)\right| \\
\leq\;& \left|\widetilde{\upsilon}(\vartheta)-Z_{\widetilde{\upsilon}}-\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))d\tau\right| \\
&+\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\left|\mathfrak{g}(\tau,\widetilde{\upsilon}(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\widetilde{\upsilon}(\tau))-\mathfrak{g}(\tau,\upsilon(\tau),{}^{CK}\mathfrak{D}_{a^+}^{\varsigma;\rho}\upsilon(\tau))\right|d\tau \\
\leq\;& \zeta\varepsilon+\frac{\lambda}{1-\lambda}\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_a^{\vartheta}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{\varsigma-1}\left|\widetilde{\upsilon}(\tau)-\upsilon(\tau)\right|d\tau.
\end{aligned}$$

By applying Lemma 2.8, and Remark 2.9, it get that

$$\begin{aligned}
&\left|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)\right| \\
\leq\;& \zeta\varepsilon+\int_a^{\vartheta}\left[\sum_{k=1}^{\infty}\frac{\rho^{1-k\varsigma}\left(\frac{\lambda}{1-\lambda}\rho^{1-\varsigma}\right)^k}{\Gamma(k\varsigma)}\tau^{\rho-1}(\vartheta^{\rho}-\tau^{\rho})^{k\varsigma-1}\zeta\varepsilon\right]d\tau \\
\leq\;& \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}(T^{\rho}-a^{\rho})^{\varsigma-1}+\frac{\rho^{-\varsigma}}{\Gamma(\varsigma+1)}(T^{\rho}-a^{\rho})^{\varsigma}\right]\varepsilon \\
&\times E_{\varsigma}\left(\frac{\lambda}{1-\lambda}\left(\frac{\vartheta^{\rho}-a^{\rho}}{\rho}\right)^{\varsigma}\right).
\end{aligned}$$

For

$$\begin{aligned}
K_f \;=\;& \frac{3}{2}\left[\left(\frac{1}{T^{\rho-1}+a^{\rho-1}}\right)\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}(T^{\rho}-a^{\rho})^{\varsigma-1}+\frac{\rho^{-\varsigma}}{\Gamma(\varsigma+1)}(T^{\rho}-a^{\rho})^{\varsigma}\right] \\
& E_{\varsigma}\left(\frac{\lambda}{1-\lambda}\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)^{\varsigma}\right),
\end{aligned}$$

we get

$$\left|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)\right|\leq K_f\varepsilon. \tag{33}$$

This shows that the problem (5)–(6) is Ulam-Hyers stable. □

**Corollary 4.6.** *Let the hypotheses of Theorem 4.5 hold. Suppose that $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\Psi(0)=0$. Then tne problem (12)–(13) is generalized Ulam-Hyers stable.*

*Proof.* One can repeat the manner similar to Theorem 4.5 with setting $K_f\varepsilon=\Psi(\varepsilon)$, and $\Psi(0)=0$, we deduce that

$$\left|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)\right|\leq\Psi(\varepsilon).$$

□

## 5. Examples

Here, we stipulate some interpretative examples to support the obtained results.

**Example 5.1.** *Consider the following problem:*

$${}^{CK}\mathfrak{D}_{0^+}^{\frac{3}{2};\frac{1}{2}}\upsilon(\vartheta)=\begin{cases}\left[\frac{1}{9}e^{\sqrt{\vartheta}+1}+\dfrac{5+|\upsilon(\vartheta)|+\left|{}^{CK}\mathfrak{D}_{0^+}^{\frac{3}{2};\frac{1}{2}}\upsilon(\vartheta)\right|}{12e^{2-\vartheta}\left(1+|\upsilon(\vartheta)|+\left|{}^{CK}\mathfrak{D}_{0^+}^{\frac{3}{2};\frac{1}{2}}\upsilon(\vartheta)\right|\right)}\right], & \vartheta\in[0,1], \\ \upsilon(0)+\upsilon(1)=0, & \\ \upsilon'(0)+\upsilon'(1)=0 & \end{cases}. \tag{34}$$

*Take:*

$$\mathfrak{g}(\vartheta,u,v)=\left[\frac{1}{9}e^{\sqrt{\vartheta}+1}+\frac{5+u+v}{12e^{2-\vartheta}\left(1+u+v\right)}\right],\quad \vartheta\in[0,1],u,v\in\mathbb{R}^{+},$$

*with* $\varsigma=\frac{3}{2}$ *and* $\rho=\frac{1}{2}$. *Obviously, the function* $g\in C([0,1])$. *For each* $u,v,u^{*},v^{*}\in\mathbb{R}^{+}$ *and* $\vartheta\in[0,1]$

$$\begin{aligned}|\mathfrak{g}(\vartheta,u,v)-\mathfrak{g}(\vartheta,u^{*},v^{*})| &= \left|\frac{5+u+v}{12e^{2-\vartheta}\left(1+u+v\right)}-\frac{5+u^{*}+v^{*}}{12e^{2-\vartheta}\left(1+u^{*}+v^{*}\right)}\right| \\ &\le \frac{1}{12e^{2-\vartheta}}\left(|u-u^{*}|+|v-v^{*}|\right) \\ &\le \frac{1}{12e}\left(|u-u^{*}|+|v-v^{*}|\right).\end{aligned}$$

*Hence, the assumption* $(H_1)$ *is satisfied with* $\lambda=\frac{1}{12e}$. *We can easily check that* $\Upsilon=-\frac{5}{6}\frac{\sqrt{2}}{\sqrt{\pi}e\left(\frac{1}{12e}-1\right)}=0.252\,34<1$. *Since all the assumptions of Theorem (3.3) are obtained, therefore problem (34) has a unique solution.*

**Example 5.2.** *Consider the following problem:*

$$\begin{cases} {}^{CK}\mathfrak{D}_{0^{+}}^{\frac{4}{3};\frac{3}{2}}\upsilon(\vartheta)=\frac{|\upsilon(\vartheta)|+\cos\left|{}^{CK}\mathfrak{D}_{0^{+}}^{\frac{4}{3};\frac{3}{2}}\upsilon(\vartheta)\right|}{20(\vartheta+2)(1+|\upsilon(\vartheta)|)},\ \vartheta\in[0,1] \\ \upsilon(0)=-\upsilon(1), \\ \upsilon'(0)=-\upsilon'(1)\end{cases}. \tag{35}$$

*Put:*

$$\mathfrak{g}(\vartheta,u,v)=\frac{u+\cos v}{20(\vartheta+2)(1+u)},\quad \vartheta\in[0,1],u,v\in\mathbb{R}^{+},$$

*with* $\varsigma=\frac{4}{3},\rho=\frac{3}{2}$ *and* $T=1$. *Now, for each* $u,v,u^{*},v^{*}\in\mathbb{R}^{+}$ *and* $\vartheta\in[0,1]$

$$\begin{aligned}|\mathfrak{g}(\vartheta,u,v)-\mathfrak{g}(\vartheta,u^{*},v^{*})| &= \left|\frac{u+\cos v}{20(\vartheta+2)(1+u)}-\frac{u^{*}+\cos v^{*}}{20(\vartheta+2)(1+u^{*})}\right| \\ &\le \frac{1}{20}\left(|u-u^{*}|+|v-v^{*}|\right).\end{aligned}$$

*Hence, the assumption* $(H_1)$ *is satisfied with* $\lambda=\frac{1}{20}$. *We can easily check that* $\Upsilon\approx .009<1$. *By Theorem 3.3 the problem (35) has a unique solution.*

*We can see that all the required assumptions of Theorem 4.5 are satisfied. In consequence, the suggested problem (34) is Ulam-Hyers, generalized Ulam-Hyers stable.*

*According to Theorem 4.5, for* $\varepsilon>0$, *any* $\widetilde{\upsilon}\in C([0,1],\mathbb{R})$ *satisfies the following inequality*

$$\left|{}^{CK}\mathfrak{D}_{0^{+}}^{\frac{3}{2};\frac{1}{2}}\widetilde{\upsilon}(\vartheta)-\left[\frac{1}{3}e^{\sqrt{\vartheta}+1}+\frac{4+|\widetilde{\upsilon}(\vartheta)|+\left|\mathfrak{D}_{0^{+}}^{\frac{3}{2};\frac{1}{2}}\widetilde{\upsilon}(\vartheta)\right|}{16e^{2-\vartheta}\left(1+|\widetilde{\upsilon}(\vartheta)|+\left|\mathfrak{D}_{0^{+}}^{\frac{3}{2};\frac{1}{2}}\widetilde{\upsilon}(\vartheta)\right|\right)}\right]\right|\le\varepsilon,\quad \vartheta\in[0,1],$$

*there exists a solution* $\upsilon\in C([0,1],\mathbb{R})$ *for the problem (34) such that*

$$|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)|\le K_{f}\varepsilon,\quad \vartheta\in[0,1],$$

*where* $K_f=10\sqrt{\frac{2}{\pi}}E_{\frac{1}{2}}\left(2\frac{\sqrt{2}}{16e-1}\right)$. *furthermore, if we put* $K_f\varepsilon=\Psi(\varepsilon)$, *and* $\Psi(0)=0$, *then*

$$|\widetilde{\upsilon}(\vartheta)-\upsilon(\vartheta)|\le\Psi(\varepsilon),\quad \vartheta\in[0,1].$$

## 6. Conclusions

In this article we have studied a type of a nonlinear IFDE with the anti-periodic boundary condition involving a Caputo-Katugampola fractional derivative. We have also established sufficient conditions ensuring existence, uniqueness and Ulam-Hyers stability of solutions for a proposed proplem by applying some fixed point theorems and generalized Gronwall inequality. We confident the obtained results here will havea favorable impact on the evolution of more applications in applied sciences and engineering.

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