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On the global behavior of the rational difference equation $y_{n+1} = \frac{\alpha_n + y_{n-r}}{\alpha_n + y_{n-k}}$

Sihem Oudina[a], Mohamed Amine Kerker[a], Abdelouahab Salmi[a]

[a]*Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University, Annaba, Algeria.*

Abstract

In this article, we study the global behavior of the following higher-order non-autonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_{n-r}}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ...,$$

where $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence of positive numbers, k, r are nonnegative integers such that $r < k$, and the initial values $y_{-k}, ..., y_0$ are nonnegative real numbers. We show that all positive solutions are oscillatory about the equilibrium point $\bar{y} = 1$. Furthermore, we establish sufficient conditions under which the unique equilibrium attracts all positive solutions.

Keywords: Rational difference equation   Oscillation Local stability   Global attractivity.
2020 MSC: 39A22, 39A30

1. Introduction

In the few last decades, nonlinear difference equations have gained a great interest. They are the discrete analogues of differential equations and arise whenever an independent variable can have only discrete values. Besides their theoretical importance, their applications are wide-spread, ranging from modeling to discretizations (see [3, 5, 7, 13] and the references therein).

Many real-world problems are non-autonomous. Namely, they involve time-dependent parameters, controls and various other effects. In principle, the time dependence can be periodic, almost periodic or arbitrary. Since the

Email addresses: sihem.oudina20@gmail.com (Sihem Oudina), a_kerker@yahoo.com (Mohamed Amine Kerker), asalmia@yahoo.com (Abdelouahab Salmi)

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methods and results for autonomous systems are no longer applicable for non-autonomous equations, their qualitative study is much more complicated, and they require special attention.

In [14], Kocic *et al.* studied the following high order difference equation

$$y_{n+1} = \frac{a + by_n}{A + y_{n-k}}, \quad n \in \mathbb{N}, \tag{1}$$

with a, b, A are nonnegative real numbers and k is a positive integer. They showed, among others, that the positive equilibrium point of the Eq. (1) is globally asymptotically stable. These results were extended in [6] and [12] to the following non-autonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ... \tag{2}$$

Precisely, Dekkar *et al.* [6] considered Eq. (2) in the case where $\{\alpha_n\}_{n\geq 0}$ is a periodic sequence of positive numbers with period T, while Kerker *et al.* [12] studied Eq. (2) when $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence. Recently, Kerker and Bouaziz [11] studied the oscillation and the global attractivity for the more general difference equation

$$y_{n+1} = \frac{\alpha_n + y_{n-r}}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ..., \tag{3}$$

where $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence of positive numbers and $r < k$ are positive integers. For more related works see [1, 2, 4, 9, 10, 16, 17, 18, 19].

In the present work, we consider a generalization of Eq. (3) by taking the sequence $\{\alpha_n\}_{n\geq 0}$ to be bounded. We show that every positive solution of (3) is oscillatory about the equilibrium point $\bar{y} = 1$. Furthermore, we establish sufficient conditions under which the unique equilibrium $\bar{y} = 1$ attracts all positive solutions of Eq. (3). We present some examples to confirm our results.

2. Preliminaries

In this preliminary section, we recall some notions and results about the theory of difference equations. For more details we refer readers to [8, 15].

Let I be an interval of real numbers and let $f : \mathbb{N} \times I^{k+1} \longrightarrow I$ be a continuously differentiable function. Consider the difference equation

$$y_{n+1} = f(n, y_n, y_{n-1}, ..., y_{n-k}), \quad n \geq 0, \tag{4}$$

with $y_0, y_{-1}, ..., y_{-k} \in I$.

Definition 2.1. *A point $\bar{y} \in I$ such that $\bar{y} = f(n, \bar{y}, \bar{y}, ..., \bar{y})$ for all $n \geq 0$, is called an equilibrium point of Eq. (4).*

Definition 2.2. *An equilibrium point $\bar{y}$ of (4) is said to be*

1. *Stable if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that for all $y_0, y_{-1}, ..., y_{-k} \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$ then $|y_n - \bar{y}| < \varepsilon$, for all $n \geq -k$. Otherwise, the equilibrium $\bar{y}$ is called unstable.*
2. *Attractive if there exists $\mu > 0$ such that for all $y_0, y_{-1}, ..., y_{-k} \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \mu$, then*
$$\lim_{n\to\infty} y_n = \bar{y}.$$
If $\mu = \infty$, $\bar{y}$ is called globally attractive.
3. *Asymptotically stable if it is stable and attractive.*
4. *Globally asymptotically stable if it is stable and globally attractive.*

Definition 2.3. *A solution $\{y_n\}_{n\geq -k}$ of Eq. (4) is called nonoscillatory if there exists $p \geq -k$ such that either*

$$y_n > \bar{y}, \quad \forall n \geq p \qquad \text{or} \qquad y_n < \bar{y}, \quad \forall n \geq p,$$

and it is called oscillatory if it is not nonoscillatory.

Finally, we state the comparison principle for non-autonomous difference equations.

Lemma 2.4. *Let $z \geq 0$ be a real number, $g(n,z)$ be a nondecreasing function with respect to z for any fixed natural number $n \geq n_0$, $n_0 \in \mathbb{N}$. Suppose that for $n \geq n_0$, we have*

$$x_{n+1} \leq g(n, x_n),$$
$$y_{n+1} \geq g(n, y_n).$$

Then,

$$x_{n_0} \leq y_{n_0}$$

implies that

$$x_n \leq y_n, \qquad \forall n \geq n_0.$$

3. Oscillation of positive solutions

In this section, we study the oscillatory behavior of positive solutions of Eq. (3).

Theorem 3.1. *Every positive solution of (3) oscillates about $\bar{y} = 1$.*

Proof. Assume that Eq. (3) has a nonoscillatory solution. Then, there exists $n_0 \geq -k$ such that

$$y_n > 1, \quad \text{for all } n \geq n_0$$

or

$$y_n < 1, \quad \text{for all } n \geq n_0.$$

Suppose that $y_n > 1,\ \forall\ n \geq n_0$. So, for $n \geq n_0 + k$, we have

$$y_{n+1} = y_{n-r}\frac{(\alpha_n/y_{n-r} + 1)}{\alpha_n + y_{n-k}} < y_{n-r}\frac{\alpha_n + 1}{\alpha_n + y_{n-k}} < y_{n-r}. \tag{5}$$

Let p be the smallest integer in $\{n_0 + k, ..., n_0 + k + r\}$ such that

$$y_p = \max\{y_i,\ i = n_0 + k, ..., n_0 + k + r\}.$$

Therefore, there exists a nonnegative integer m and $j \in \{0, ..., r\}$, such that

$$k - r + p = m(r + 1) + j.$$

Consequently, by using (5) we obtain

$$y_{n_0+2k+p+1} = \frac{\alpha_{n_0+2k+p} + y_{n_0+2k-r+p}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}}$$

$$= \frac{\alpha_{n_0+2k+p} + y_{n_0+k+m(r+1)+j}}{\alpha_{n_0+2k+p} + y_{n_0+k+p}} \leq 1,$$

which is a contradiction, and then the proof is complete. □

To confirm our result on the oscillatory behavior of the positive solutions of Eq. (3), we consider the two following numerical examples.

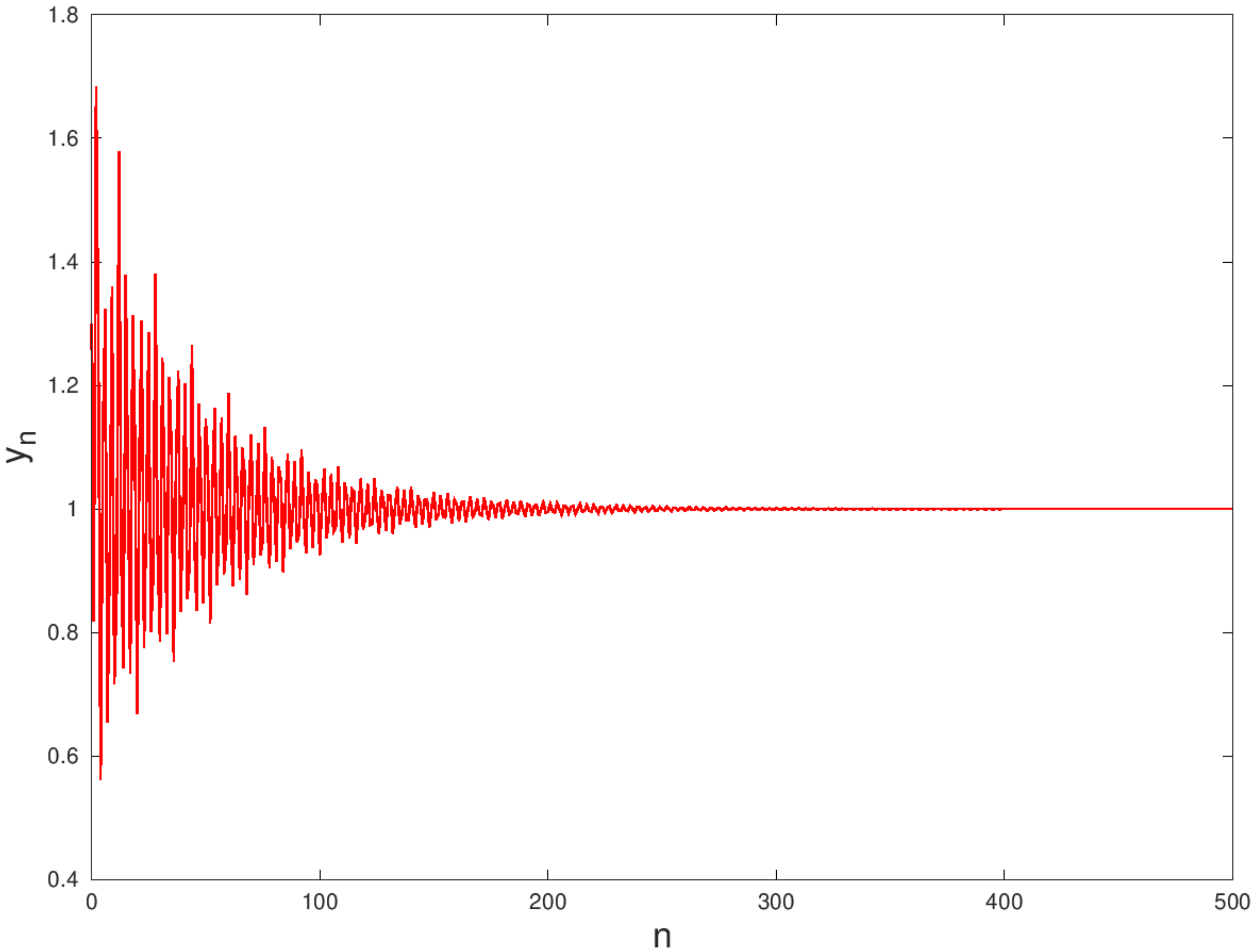

Figure 1: Plot of the solution of equation Eq. (6) with the initial values $y_{-4} = 1.4$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.9$, $y_0 = 1.3$.

Example 3.2. *We consider the following fifth order difference equation*

$$y_{n+1} = \frac{[\frac{(-1)^n+4}{4}]^2 + y_{n-2}}{[\frac{(-1)^n+4}{4}]^2 + y_{n-4}} \tag{6}$$

with the initial values $y_{-4} = 1.2$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.9$, $y_0 = 1.3$. *The solution of Eq. (6) is oscillatory about* $\bar{y} = 1$, *see Fig. 1.*

Example 3.3. *We consider the following rational difference equation*

$$y_{n+1} = \frac{\frac{n+12}{7n+8} + y_{n-2}}{\frac{n+12}{7n+8} + y_{n-11}} \tag{7}$$

with the initial values $y_{-11} = 0.02$, $y_{-10} = 3.5$, $y_{-9} = 9.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 1.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 11.5$, $y_0 = 0.3$. *The solution of Eq. (7) is oscillatory about* $\bar{y} = 1$, *see Fig. 2.*

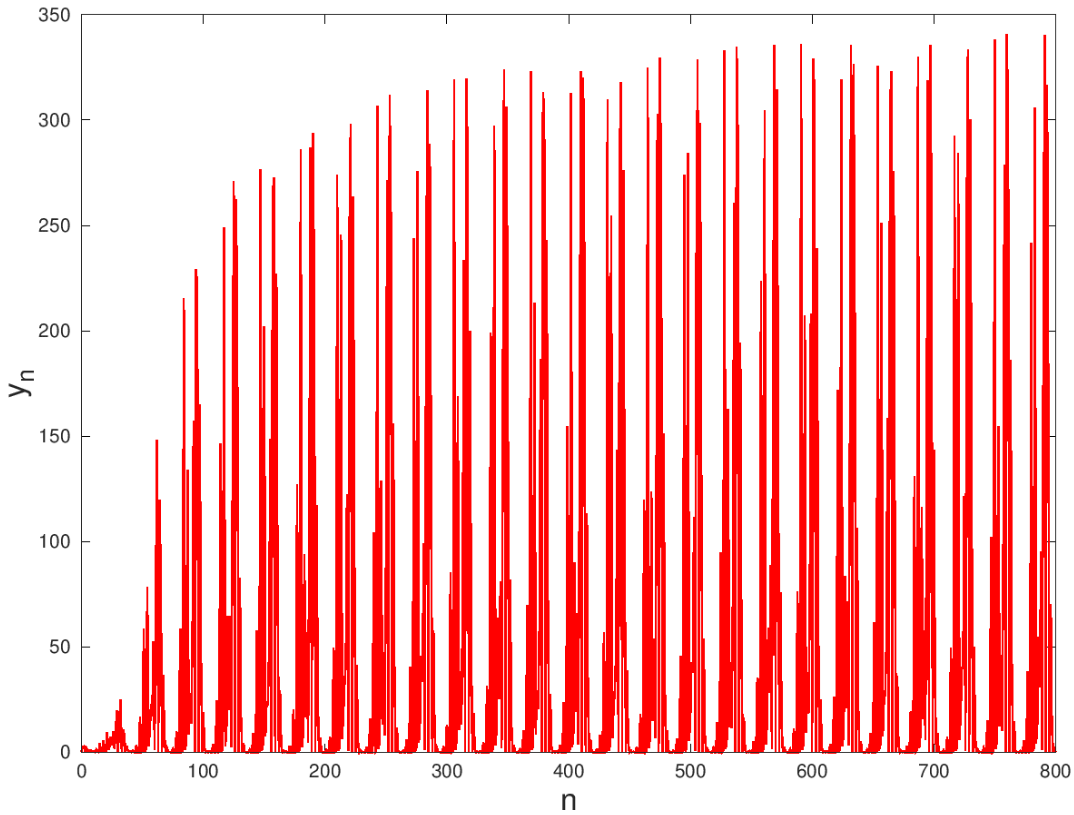

Figure 2: Plot of the solution of Eq. (7) with the initial values $y_{-11} = 0.02$, $y_{-10} = 3.5$, $y_{-9} = 9.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 1.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 11.5$, $y_0 = 0.0.3$

4. Boundedness of positive solutions

Hereafter, we shall use the following notations

$$\alpha = \lim \alpha_n, \qquad a = \inf\{\alpha_n\} \qquad \text{and} \qquad A = \sup\{\alpha_n\}.$$

We have the following result.

Theorem 4.1. *Assume that*

$$a > 1. \tag{8}$$

Then, every positive solution of (3) is bounded.

Proof. We have

$$y_{n+1} = \frac{\alpha_n + y_{n-r}}{\alpha_n + y_{n-k}} \le \frac{A}{a} + \frac{1}{a} y_{n-r}, \tag{9}$$

which gives (see [8, p. 77])

$$y_n \le \frac{A}{a-1} + a^{-\frac{n}{r+1}} \sum_{i=0}^{r} c_i n^i \underset{n\to\infty}{\to} \frac{A}{a-1}.$$

Then, by Lemma 2.4, there exists $M > 0$, such that

$$y_n \le M, \qquad \forall n \ge k.$$

Hence, Eq. (3) yields

$$y_{n+1} \ge \frac{a}{A+M} = m > 0.$$

□

Theorem 4.2. *Assume that $a \le 1$ and that there exists a positive integer m such that*

$$k = r + m(r+1). \tag{10}$$

Then, every positive solution of (3) is bounded.

Proof. Let $\{y_n\}_{n\ge -k}$ be a positive solution of (3). Assume, for the sake of contradiction, that the solutions are unbounded. Then, there exists a subsequence $\{y_{n_i+1}\}$ such that

$$\lim_{i\to\infty} y_{n_i+1} = +\infty, \qquad y_{n_i+1} = \max\{y_n : n \le n_i + 1\}.$$

From (9) we have

$$\lim_{i\to\infty} y_{n_i-r} = +\infty. \tag{11}$$

Furthermore,

$$0 \le y_{n_i+1} - y_{n_i-r} = \frac{\alpha_{n_i}(1 - y_{n_i-r}) + (1 - y_{n_i-k}) y_{n_i-r}}{\alpha_{n_i} + y_{n_i-k}}$$

which implies that

$$y_{n_i-k} \le \frac{\alpha_n}{y_{n_i-r}} + 1 - \alpha_n. \tag{12}$$

In view of (11) and (12), we see that $\{y_{n_i-k}\}$ is bounded. Hence, by applying (9) repeatedly we obtain

$$y_{n_i-r} \le \begin{cases} m + y_{n_i-k}, & \text{if } a = 1, \\ \frac{A}{1-a}(a^{-m} - 1) + a^{-m} y_{n_i-k}, & \text{if } a < 1, \end{cases}$$

which implies that $\{y_{n_i-r}\}$ is also bounded. This is a contradiction. □

5. Global asymptotic stability

In this section, we investigate the global asymptotic stability of the equilibrium point. First, we have the following local stability result.

Theorem 5.1. *Assume that (8) holds. Then, $\bar{y} = 1$ is stable.*

Proof. Choose $M > A/(a-1)$ such that

$$y_{-k}, ..., y_0 \in \left(\frac{1}{A+M}, M\right).$$

Therefore, it is easy to check that

$$y_n \in \left(\frac{1}{A+M}, M\right), \qquad \forall n \geq -k. \tag{13}$$

Next, setting

$$M(\varepsilon) = \min\left\{1+\varepsilon, \frac{1}{1-\varepsilon} - A\right\}$$

and

$$\delta(\varepsilon) = \min\left\{M(\varepsilon) - 1, 1 - \frac{1}{A + M(\varepsilon)}\right\},$$

for $\varepsilon \in (0,1)$, we obtain

$$(1-\delta, 1+\delta) \subseteq \left(\frac{1}{A+M}, M\right) \subseteq (1-\varepsilon, 1+\varepsilon). \tag{14}$$

Now, if we take $y_{-k}, ..., y_0 \in \mathbb{R}_+$ with $|y_{-k} - 1| + |y_{-k+1} - 1| + ... + |y_0 - 1| < \varepsilon$, then (13), combined with (14), yields

$$|y_n - 1| < \varepsilon, \qquad \forall n \geq -k,$$

and so $\bar{y}$ is stable. □

In the next theorem, we establish the global attractivity of the equilibrium point.

Theorem 5.2. *Assume that (8) holds. Then, $\bar{y} = 1$ is the global attractor of all positive solutions of Eq. (3).*

Proof. Let $\{y_n\}_{n \geq -k}$ be an arbitrary positive solution of (3). Set

$$I = \liminf_{n\to\infty} y_n \qquad \text{and} \qquad S = \limsup_{n\to\infty} y_n$$

which by Theorem 4.1 exist. Let $\{n_p\}$ and $\{n_q\}$ be an infinite increasing sequences of positive integers such that

$$\lim_{q\to\infty} y_{n_q+1} = I \quad \text{and} \quad \lim_{p\to\infty} y_{n_p+1} = S.$$

By taking subsequences, if necessary, we assume that $\{\alpha_{n_p}\}_p$, $\{\alpha_{n_q}\}_q$, $\{y_{n_p-r}\}_p$, $\{y_{n_q-r}\}_q$, $\{y_{n_p-k}\}_p$ and $\{y_{n_q-k}\}_q$ converge to A_0, a_0, L_r, l_r, L_k and l_k respectively. Clearly

$$l_r, L_r, l_k, L_k \in [I, S] \quad \text{and} \quad a_0, A_0 \in [a, A].$$

Then, the Eq. (3) yields

$$I = \frac{a_0 + l_r}{a_0 + l_k} \geq \frac{a_0 + I}{a_0 + S}$$

and

$$S = \frac{A_0 + L_r}{A_0 + L_k} \leq \frac{A_0 + S}{A_0 + I}.$$

Since the function $(x+I)/(x+S)$ is non-decreasing, we have

$$I \geq \frac{a+I}{a+S}. \tag{15}$$

Similarly, since $(x+S)/(x+I)$ is non-increasing, we obtain

$$S \leq \frac{a+S}{a+I}. \tag{16}$$

Combining (15) with (16) gives

$$a+(1-a)I \leq IS \leq a+(1-a)S.$$

Consequently, since $a>1$ we obtain $I \geq S$, and so the sequence $\{y_n\}$ is convergent to the unique limit $l=1$. □

From Theorems 5.1 and 5.2 we obtain the following result.

Theorem 5.3. *Assume that (8) holds. Then, the equilibrium point $\bar{y}=1$ of Eq. (3) is globally asymptotically stable.*

Next, when Condition (8) does not hold, we have

Theorem 5.4. *Assume that $a>0$ and $k=2r+1$. Then $\bar{y}$ is the global attractor of all positive solutions of Eq. (3).*

Proof. Let $\{y_n\}_{n\geq -k}$ be an arbitrary positive solution of (3). In view of Theorem **??**, it suffices to show that all positive solutions of Eq. (3) which are oscillatory about $\bar{y}$ are attracted to it. Setting

$$z_n^{(i)} = y_{n(r+1)+i}, \qquad \text{for all } n \geq n_0+k \text{ and } i=0,...,r,$$

it follows from Eq. (3) that

$$z_{n+1}^{(i)} = \frac{\beta_n^{(i)} + z_n^{(i)}}{\beta_n^{(i)} + z_{n-1}^{(i)}}, \qquad \text{for all } n \geq 0 \text{ and } i=0,...,r,$$

where $\beta_n^{(i)} = \alpha_{n(r+1)+i+r}$. Hence, all subsequences $\left\{z_n^{(i)}\right\}_n$, $i=0,...,r$, satisfy the same second order difference equation

$$z_{n+1} = \frac{\beta_n + z_n}{\beta_n + z_{n-1}}. \tag{17}$$

In the sequel, we will show that $\bar{y}=1$ is the global attractor of all positive solutions of Eq. (17). Let $\{z_n\}_{n\geq -k}$ be an oscillatory positive solution of (17), and let $\{p_i\}$ and $\{p_i\}$ be sequences of integers such that $y_{p_0}<1$ and for $i=0,1,2,...$

$$\{z_{p_i+1},...,z_{q_i}\} \quad \text{is a positive semicycle,} \tag{18}$$

and

$$\{z_{q_i+1},...,z_{p_{i+1}}\} \quad \text{is a negative semicycle.} \tag{19}$$

For each $i=0,1,2,...$, let P_i and Q_i be the smallest integers in $\{p_i+1,...,q_i\}$ and $\{q_i+1,...,p_{i+1}\}$, respectively, such that

$$z_{P_i} = \max\{z_{p_i+1},...,z_{q_i}\} \quad \text{and} \quad z_{Q_i} = \min\{z_{q_i+1},...,z_{p_{i+1}}\}.$$

From (17) it follows that the extreme point in any semicycle occurs in one of the first two terms of the semicycle. Consequently, we have $\forall i \geq 0$

$$p_i+1 \leq P_i \leq p_i+2 \quad \text{and} \quad q_i+1 \leq Q_i \leq q_i+2. \tag{20}$$

Let

$$I = \liminf_{n\to\infty} z_n = \liminf_{i\to\infty} z_{Q_i} \quad \text{and} \quad S = \limsup_{n\to\infty} z_n = \limsup_{i\to\infty} z_{P_i}. \tag{21}$$

Next, by Theorems 4.1 and 4.2, I and S exist, and they satisfy

$$0 < I \le 1 \le S < \infty.$$

By definition of I and S, $\forall\varepsilon,\ 0 < \varepsilon < I$ and $\delta > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$I - \varepsilon \le z_n \le S + \delta, \quad \forall n \ge n_0.$$

Now, we consider the positive semicycle (18) with $p_i \ge n_0 + 1$. Then, we have

$$\begin{aligned} z_{n-1} &\ge I - \varepsilon, \quad \text{for } n = p_i, ..., P_i - 1, \\ z_n &\ge 1, \quad \text{for } n = p_i + 1, P_i - 1. \end{aligned}$$

From (20) we distinguish two cases:

Case 1: $P_i = p_i + 1$. In this case we have

$$\begin{aligned} z_{P_i} = z_{p_i+1} &= \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \\ &\le \frac{\beta_{p_i} + 1}{\beta_{p_i} + I - \varepsilon}. \end{aligned}$$

Since the function $(x + 1)/(x + I - \varepsilon)$ is non-increasing, we obtain

$$z_{P_i} \le \frac{a+1}{a+I-\varepsilon}, \quad \forall\varepsilon > 0,$$

and so

$$S \le \frac{a+1}{a+I}. \tag{22}$$

Case 2: $P_i = p_i + 2$. In this case, we have

$$\begin{aligned} z_{P_i} = z_{p_i+2} &= \frac{z_{p_i+2}}{z_{p_i+1}} \times z_{p_i+1} \\ &= \frac{\beta_{p_i+1}/z_{p_i+1} + 1}{\beta_{p_i+1} + z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \\ &\le \frac{\beta_{p_i+1} + 1}{\beta_{p_i+1} + z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \end{aligned}$$

and since the function $(x + 1)/(x + z_{p_i})$ is non-increasing, we have

$$z_{P_i} \le \frac{a+1}{a+z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}}.$$

To estimate the last term in the right hand side of this inequality, we have:

Case 2-a: If $z_{p_i-1} \le z_{p_i}$, then the function $(x + z_{p_i})/(x + z_{p_i-1})$ is non-increasing, and then

$$\begin{aligned} z_{P_i} &\le \frac{a+1}{a+z_{p_i}} \times \frac{a+z_{p_i}}{a+z_{p_i-1}} \\ &\le \frac{a+1}{a+I-\varepsilon}, \quad \forall\varepsilon > 0. \end{aligned}$$

Case 2-b: If $z_{p_i-1} \ge z_{p_i}$, then

$$\begin{aligned} z_{P_i} &\le \frac{a+1}{a+z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i}} \\ &\le \frac{a+1}{a+I-\varepsilon}, \quad \forall\varepsilon > 0. \end{aligned}$$

Therefore, in the two sub-cases we obtain the inequality (22).

Similarly, by considering negative semicycles, in the two cases $Q_i = q_i + 1$ and $Q_i = q_i + 2$ we obtain

$$I \geq \frac{a+1}{a+S}. \tag{23}$$

Combining (22) with (23) gives

$$a + 1 - aI \leq IS \leq a + 1 - aS.$$

Consequently, since $a > 0$ we obtain $I \geq S$, and so the sequence $\{z_n\}$ is convergent to the unique limit $l = 1$. □

Now, we give four illustrative examples:

Example 5.5. *We consider the following fifth order difference equation*

$$y_{n+1} = \frac{5 + \cos(n\pi) + \frac{2}{n+1} + y_{n-2}}{5 + \cos(n\pi) + \frac{2}{n+1} + y_{n-4}}, \tag{24}$$

with the initial values $y_{-4} = 0.3$, $y_{-3} = 5.5$, $y_{-2} = 0.9$, $y_{-1} = 4.2$ and $y_0 = 1.2$. From Theorem 5.2, the equilibrium point $\bar{y} = 1$ is the global attractor of all positive solution of Eq. (24), see Fig. 3.

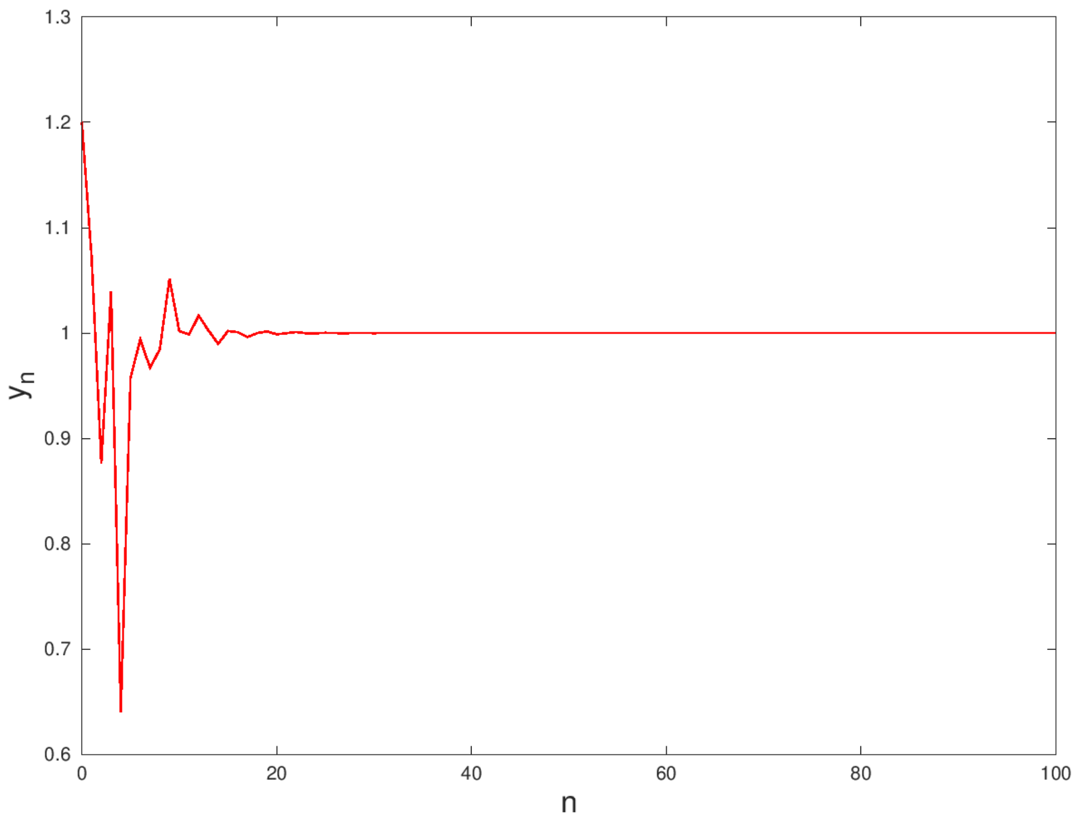

Figure 3: Plot of the solution of Eq. (24) with the initial values $y_{-4} = 0.3$, $y_{-3} = 5.5$, $y_{-2} = 0.9$, $y_{-1} = 4.2$ and $y_0 = 1.2$.

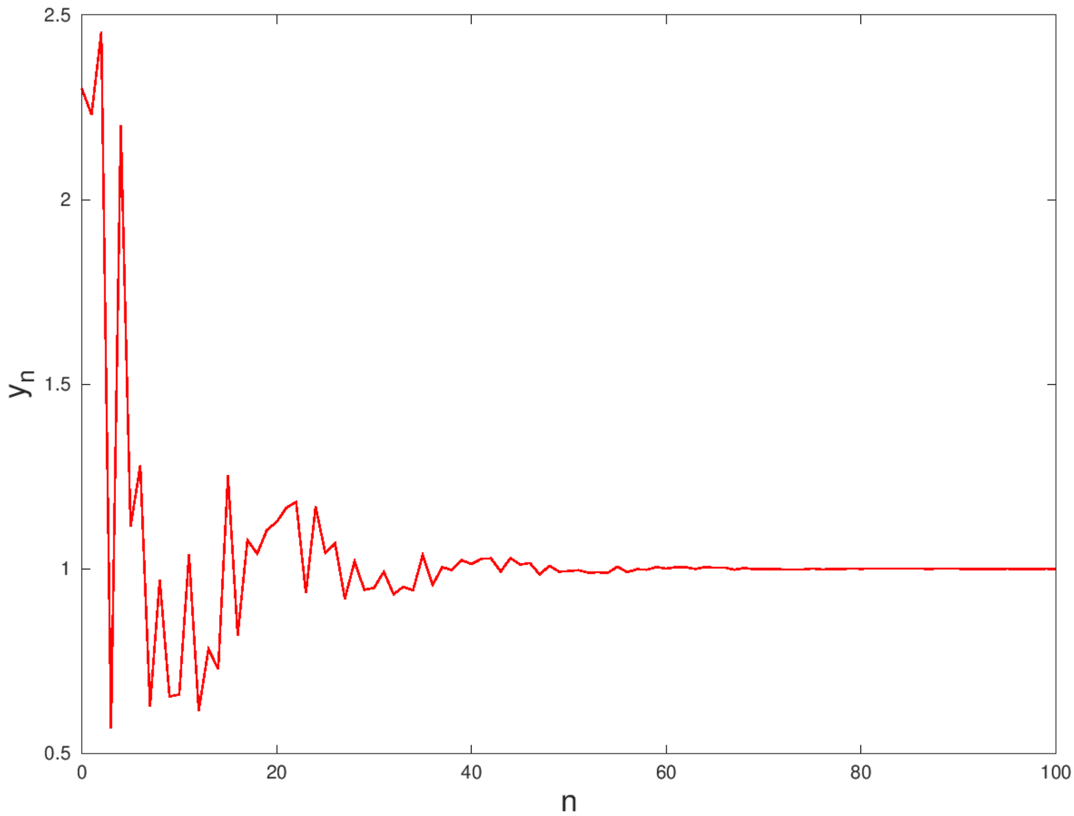

Figure 4: Plot of the solution of Eq. (25) with the initial values $y_{-7} = 0.3$, $y_{-6} = 0.1$, $y_{-5} = 3.4$, $y_{-4} = 0.5$, $y_{-3} = 1.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ and $y_0 = 2.3$.

Example 5.6. *We consider the following eighth order rational difference equation*

$$y_{n+1} = \frac{\alpha_n + y_{n-3}}{\alpha_n + y_{n-7}}, \tag{25}$$

where $\alpha_n = 1 - 2^{-(n+1)}$*. Here, we take the initial values as follows:* $y_{-7} = 0.3$*,* $y_{-6} = 0.1$*,* $y_{-5} = 3.4$*,* $y_{-4} = 0.5$*,* $y_{-3} = 1.9$*,* $y_{-2} = 1.7$*,* $y_{-1} = 1.5$ *and* $y_0 = 2.3$*. From Theorem 5.4, the point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (25), see Fig. 4.*

Example 5.7. *We consider the following sixth order rational difference equation*

$$y_{n+1} = \frac{\alpha_n + y_{n-2}}{\alpha_n + y_{n-5}}, \tag{26}$$

where

$$\alpha_n = \begin{cases} 1/5, & \text{if } n = 3j, \\ 3/10, & \text{if } n = 3j+1, \\ 2/5, & \text{if } n = 3j+2. \end{cases}$$

Here, we take the initial values as follows: $y_{-5} = 0.8$*,* $y_{-4} = 0.3$*,* $y_{-3} = 2.9$*,* $y_{-2} = 1.7$*,* $y_{-1} = 1.5$ *and* $y_0 = 1.3$*. From Theorem 5.4, the point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (26), see Fig. 5.*

Example 5.8. *We consider the following rational difference equation*

$$y_{n+1} = \frac{2 + (n+1)^{-2} + y_{n-1}}{2 + (n+1)^{-2} + y_{n-10}}, \tag{27}$$

with the initial values $y_{-10} = 2.7$*,* $y_{-9} = 1.1$*,* $y_{-8} = 0.3$*,* $y_{-7} = 0.3$*,* $y_{-6} = 0.3$*,* $y_{-5} = 3.4$*,* $y_{-4} = 2.5$*,* $y_{-3} = 1.1$*,* $y_{-2} = 0.5$*,* $y_{-1} = 0.5$*,* $y_0 = 1.8$*. From Theorem 5.4, the point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (27), see Fig. 6.*

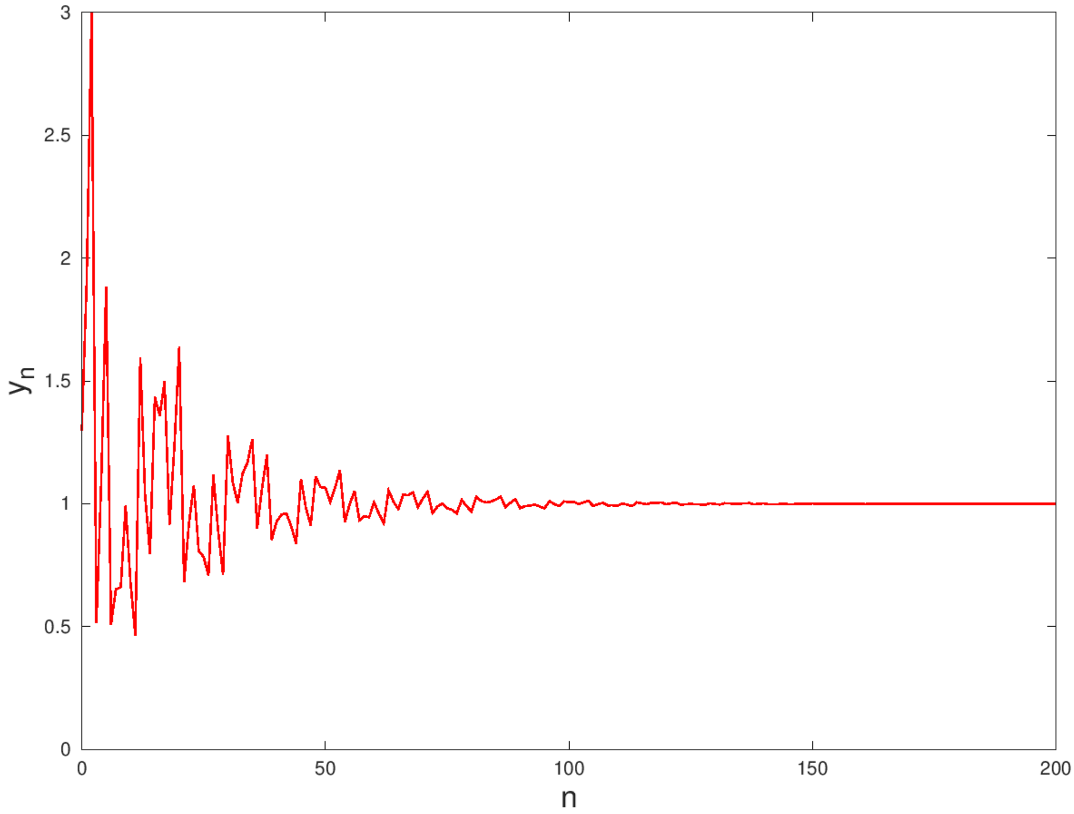

Figure 5: Plot of the solution of Eq. (26) with the initial values $y_{-5} = 0.8$, $y_{-4} = 0.3$, $y_{-3} = 2.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ and $y_0 = 1.3$.

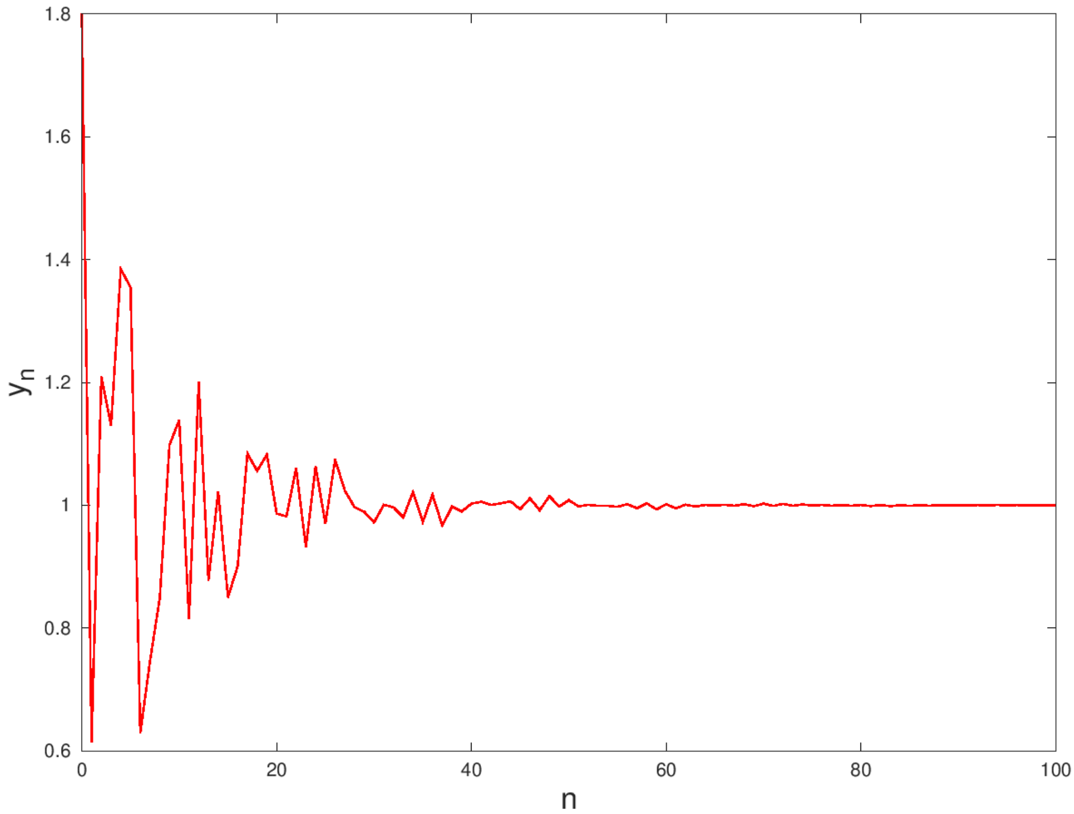

Figure 6: Plot of the solution of Eq. (27) with the initial values $y_{-10} = 2.7$, $y_{-9} = 1.1$, $y_{-8} = 0.3$, $y_{-7} = 0.3$, $y_{-6} = 0.3$, $y_{-5} = 3.4$, $y_{-4} = 2.5$, $y_{-3} = 1.1$, $y_{-2} = 0.5$, $y_{-1} = 0.5$, $y_0 = 1.8$.

6. Conclusion

In this paper, some properties of the higher order rational difference equation (3) were studied. Namely, we proved that every positive solution of (3) is oscillatory about the equilibrium point $\bar{y} = 1$. Also, we investigated the boundedness of solutions. Furthermore, we studied the global attractivity of equilibrium $\bar{y}$ in both cases $\inf\{\alpha_n\} > 1$ and $0 < \inf\{\alpha_n\} \leq 1$. In this last case, our result is valid only if $k = 2r + 1$. Finally, in order to illustrate obtained results some numerical examples are presented.

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