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Local Fractional Aboodh Transform and its Applications to Solve Linear Local Fractional Differential Equations

Djeloul Ziane[a], Rachid Belgacem[a], Ahmed Bokhari[a]

[a]*Faculty of Exact and Computer Sciences, Hassiba Benbouali University Chlef, Algeria.*

Abstract

In this work we focus on presenting a method for solving local fractional differential equations. This method based on the combination of the Aboodh transform with the local fractional derivative (we can call it local fractional Aboodh transform), where we have provided some important results and properties. We concluded this work by providing illustrative examples, through which we focused on solving some linear local fractional differential equations in order to obtain nondifferential analytical solutions.

Keywords: local fractional calculus local fractional Laplace transform Aboodh transform method local fractional differential equations .
2010 MSC: 65R10, 26A33.

1. Introduction

The search for analytical solutions to fractional differential equations is often difficult, so in many cases researchers focus on studying the existence, uniqueness and properties of solutions [5, 12, 18, 21, 27, 28], while others tend to employ numerical methods to search for approximate solutions.
Transformations defined by integrals play an important role in the resolution of ordinary differential equations, partial differential equations and in the resolution of integral differential equations with integer order or fractional order. It also intervenes in mathematical physics, probability calculus, automatics, engineering, etc.

Email addresses: djeloulz@yahoo.com (Djeloul Ziane), belgacemrachid02@yahoo.fr (Rachid Belgacem), bokhari.ahmed@ymail.com (Ahmed Bokhari)

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Among the most famous transformations, we find the Laplace transform method [24], the Fourier transform method [6], the Hankel transform method [19], the Mellin transform method [14], and there are other transformations that have appeared in the recent period. We cite for example, the Sumudu transform method [29], the Natural transform method [13], the Ezaki transform method [7], the Aboodh transform method [1], the ZZ-transform method [35], the Shehu transform method [17] and others.

Our work in this paper is based on the Aboodh transform method, which was developed in 2013 by K. S. Aboodh [1], and has been used by many researchers in solving differential equations of integer order [2, 4, 15, 16, 20, 23, 36, 38], and differential equations of fractional order [8, 9, 22, 26] and we will extend it to solve linear differential equations with local fractional derivative. We supported this work with illustrative examples showing how to apply this transform with the use of local fractional derivative.

The present paper has been organized as follows. In Section 2, some basic definitions and properties of the local fractional calculus and local fractional Laplace transform method. In Section 3, we present some important results. In Section 4, we apply the local fractional Aboodh transform method (LFETM) to solve the proposed example. Then we finish with the conclusion.

2. Basic of local fractional calculus

In this section, we present the basic definitions and theorems of local fractional derivative, local fractional integral, local fractional Taylor's series, local fractional Mc-Laurin's series and local fractional Laplace transform method.

Definition 2.1. ([25], [32], p. 14) *If there exists the relation*

$$|\Phi(\upsilon) - \Phi(\upsilon_0)| < \gamma^{\eta}, \tag{1}$$

with $|\upsilon - \upsilon_0| < \delta$, *for* γ, $\delta > 0$, *and* γ, $\delta \in \mathbb{R}$. *Now* $\Phi(\upsilon)$ *is called local fractional continuous at* $\upsilon = \upsilon_0$, *denote by* $\lim_{\upsilon \longrightarrow \upsilon_0} \Phi(\upsilon) = \Phi(\upsilon_0)$. *Then* $\Phi(\upsilon)$ *is called local fractional continuous on the interval* (a, b), *denoted by* $\Phi(\upsilon) \in C_\eta(a, b)$.

Definition 2.2. ([25], [32], p.18, p.34) *Setting* $\Phi(\upsilon) \in C_\eta(a, b)$, *the local fractional derivative of* $\Phi(\upsilon)$ *of order* η *at* $\upsilon = \upsilon_0$ *is defined as*

$$\Phi^{(\eta)}(\upsilon) = \left.\frac{d^\eta \Phi}{d\upsilon^\eta}\right|_{\upsilon=\upsilon_0} = \frac{\Delta^\eta(\Phi(\upsilon) - \Phi(\upsilon_0))}{(\upsilon - \upsilon_0)^\eta}, \tag{2}$$

where

$$\Delta^\eta(\Phi(\upsilon) - \Phi(\upsilon_0)) \cong \Gamma(1+\eta)\left[(\Phi(\upsilon) - \Phi(\upsilon_0^\eta)\right]. \tag{3}$$

The local fractional partial differential operator of order η $(0 < \eta \leqslant 1)$ *was given by*

$$\frac{\partial^\eta \omega(\upsilon_0, \nu)}{\partial \nu^\eta} = \frac{\Delta^\eta(\omega(\upsilon_0, \nu) - \omega(\upsilon_0, \nu_0))}{(\nu - \nu_0)^\eta}, \tag{4}$$

where

$$\Delta^\eta(\omega(\upsilon_0, \nu) - \omega(\upsilon_0, \nu_0)) \cong \Gamma(1+\eta)\left[\omega(\upsilon_0, \nu) - \omega(\upsilon_0, \nu_0)\right]. \tag{5}$$

Definition 2.3. ([25], [32], p. 25) *The local fractional integral of* $\Phi(\upsilon)$ *of order* η *in the interval* $[a, b]$ *is defined as*

$$\begin{aligned} {}_aI_b^{(\eta)}\Phi(\upsilon) &= \frac{1}{\Gamma(1+\eta)} \int_a^b \Phi(\tau)(d\tau)^\eta \\ &= \frac{1}{\Gamma(1+\eta)} \lim_{\Delta\tau \longrightarrow 0} \sum_{i=0}^{N-1} f(\tau_i)(\Delta\tau_i)^\eta, \end{aligned} \tag{6}$$

where $\Delta\tau_i = \tau_{i+1} - \tau_i$, $\Delta\tau = \max\{\Delta\tau_0, \Delta\tau_1, \Delta\tau_2, \cdots\}$ *and* $[\tau_i, \tau_{i+1}]$, $\tau_0 = a$, $\tau_N = b$, *is a partition of the interval* $[a, b]$.

Definition 2.4. ([10], p.113, [25], [37]) *The local fractional Laplace transform of* $\Phi(\upsilon)$ *of order* η *is defined as*

$$L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s) = \frac{1}{\Gamma(1+\eta)}\int_0^\infty E_\eta(-s^\eta\upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta. \tag{7}$$

If $L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s)$*, the inverse formula of* (7) *is defined as*

$$\Phi(\upsilon) = L_\eta^{-1}\left\{F_\eta(s)\right\} = \frac{1}{(2\pi)^\eta}\int_{\beta-i\infty}^{\beta+i\infty} E_\eta(s^\eta\upsilon^\eta)F_\eta(s)(ds)^\eta, \tag{8}$$

where $\Phi(\upsilon)$ *is local fractional continuous,* $s^\eta = \beta^\eta + i^\eta\infty^\eta$*, and* $Re(s) = \beta > 0$.

Theorem 2.5. ([32], p.152) *If* $L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s)$ *and* $\lim_{\upsilon\to\infty}\Phi(\upsilon) = 0$*, then one has*

$$L_\eta\left\{\Phi^{(\eta)}(\upsilon)\right\} = s^\eta L_\eta\left\{\Phi(\upsilon)\right\} - \Phi(0). \tag{9}$$

Proof. (*see*[32], *p*.153) □

Theorem 2.6. ([32], p.153) *If* $L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s)$ *and* $\lim_{\upsilon\to\infty} {}_0I_\upsilon^\eta\Phi(\upsilon) = 0$*,*
then one has

$$L_\eta\left\{{}_0I_\upsilon^\eta\Phi(\upsilon)\right\} = \frac{1}{s^\eta}L_\eta\left\{\Phi(\upsilon)\right\}. \tag{10}$$

Proof. (*see*[32], *p*.153) □

Theorem 2.7. ([32], p.155) *If* $L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s)$ *and* $L_\eta\left\{\Psi(\upsilon)\right\} = \Omega_\eta(s)$*, then one has*

$$L_\eta\left\{\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta\right\} = F_\eta(s)\Omega_\eta(s), \tag{11}$$

where

$$\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta = \frac{1}{\Gamma(1+\eta)}\int_0^\infty \Phi(\varkappa)\Psi(\upsilon-\varkappa)(d\varkappa)^\eta. \tag{12}$$

Proof. (*see*[32], *p*.155) □

Theorem 2.8. ([33]) *Suppose that* $\Phi(\upsilon) \in C_\eta[a,b]$*, then there is a function*

$$\Pi(\upsilon) = {}_aI_\upsilon^{(\eta)}\Phi(\upsilon),$$

the function has its derivative with respect to $(d\upsilon)^\eta$*,*

$$\frac{d^\eta\Pi(\upsilon)}{(d\upsilon)^\eta} = \Phi(\upsilon), \;\; a \leqslant \upsilon \leqslant b.$$

Proof. (*see*[33]) □

3. Main Result

In this section, we derive the local fractional Aboodh transform method ($LFAT$) and some properties are discussed.

If there is a new transform operator $LFA_\eta : \Phi(\upsilon) \longrightarrow F_\eta(\nu)$, namely,

$$LFA_\eta \{\Phi(\upsilon)\} = LFA_\eta \left\{ \sum_{k=0}^{\infty} a_k \upsilon^{k\eta} \right\} = \sum_{k=0}^{\infty} a_k \frac{\Gamma(1+k\eta)}{\nu^{k\eta+2}}. \tag{13}$$

For example if $\Phi(\upsilon) = E_\eta(i^\eta \upsilon^\eta)$, we obtain

$$\begin{aligned} LFA_\eta \{E_\eta(i^\eta \upsilon^\eta)\} &= LFA_\eta \left\{ \sum_{k=0}^{\infty} \frac{i^{k\eta} \upsilon^{k\eta}}{\Gamma(1+k\eta)} \right\} \\ &= \sum_{k=0}^{\infty} \frac{i^{k\eta}}{\nu^{k\eta+2}}, \end{aligned} \tag{14}$$

and if $\Phi(\upsilon) = \frac{\upsilon^\eta}{\Gamma(1+\eta)}$, we get

$$LFA_\eta \left\{ \frac{\upsilon^\eta}{\Gamma(1+\eta)} \right\} = \frac{1}{\nu^{\eta+2}}. \tag{15}$$

These results can be generalized by providing the following definition.

Definition 3.1. *The local fractional Elzaki transform of $\Phi(\upsilon)$ of order η is defined as*

$$LFA_\eta \{\Phi(\upsilon)\} = F_\eta(\nu) = \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta) \Phi(\upsilon) (d\upsilon)^\eta, \ 0 < \eta \leqslant 1. \tag{16}$$

The inverse transformation can be obtained as follows

$$LFA_\eta^{-1} \{ F_\eta(\nu)\} = \Phi(\upsilon). \tag{17}$$

Theorem 3.2. (*linearity*). *If $LFA_\eta \{\Phi(\upsilon)\} = F_\eta(\nu)$ and $LFA_\eta \{\Psi(\upsilon)\} = \Omega_\eta(\nu)$, then one has*

$$LFA_\eta \{\lambda \Phi(\upsilon) + \mu \Psi(\upsilon)\} = \lambda F_\eta(\nu) + \mu \Omega_\eta(\nu), \tag{18}$$

where λ and μ are constant.

Proof. Using formula (16), we obtain

$$\begin{aligned} LFA_\eta \{\lambda \Phi(\upsilon) + \mu \Psi(\upsilon)\} &= \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta) \{\lambda \Phi(\upsilon) + \mu \Psi(\upsilon)\} (d\upsilon)^\eta \\ &= \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty [E_\eta(-\nu^\eta \upsilon^\eta) (\lambda \Phi(\upsilon)) + E_\eta(-\nu^\eta \upsilon^\eta) (\mu \Psi(\upsilon))] (d\upsilon)^\eta \\ &= \lambda \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta) \Phi(\upsilon) (d\upsilon)^\eta \ + \\ & \quad \mu \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta) \Psi(\upsilon) (d\upsilon)^\eta \\ &= \lambda LFA_\eta \{\Phi(\upsilon)\} + \ \mu LFA_\eta \{\Psi(\upsilon)\}. \end{aligned}$$

This ends the proof. □

Theorem 3.3. (*local fractional Aboodh-Laplace and Laplace-Aboodh duality*). *If* $L_\eta\left\{\Phi(\upsilon)\right\} = F_\eta(s)$ *and* $LFA_\eta\left\{\Phi\left(\upsilon\right)\right\} = \Omega_\eta(\nu)$, *then one has*

$$L_\eta\left\{\Phi(\upsilon)\right\} = s^\eta \Omega_\eta(s). \tag{19}$$

$$LFA_\eta\left\{\Phi(\upsilon)\right\} = \frac{1}{\nu^\eta} F_\eta(\nu). \tag{20}$$

Proof. We show formula (19). Using the formula (7) gives

$$\begin{aligned} L_\eta\left\{\Phi(\upsilon)\right\} &= \frac{1}{\Gamma(1+\eta)} \int_0^\infty E_\eta(-s^\eta \upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta \\ &= s^\eta \left(\frac{1}{\Gamma(1+\eta)} \frac{1}{s^\eta} \int_0^\infty E_\eta(-s^\eta \upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta \right) \\ &= s^\eta \Omega_\eta(s). \end{aligned}$$

Proof of the formula (20). We have

$$LFA_\eta\left\{\Phi(\upsilon)\right\} = \frac{1}{\Gamma(1+\eta)} \frac{1}{\nu^\eta} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta.$$

then

$$LFA_\eta\left\{\Phi(\upsilon)\right\} = \frac{1}{\nu^\eta} \left(\frac{1}{\Gamma(1+\eta)} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta \right),$$

therefore, we get

$$LFA_\eta\left\{\Phi(\upsilon)\right\} = \frac{1}{\nu^\eta} F_\eta(\nu).$$

This and the proof. □

Theorem 3.4. (*local fractional Aboodh transform of local fractional derivative*). *If* $LFA_\eta\left\{\Phi(\upsilon)\right\} = \Omega_\eta(\nu)$, *then one has*

$$LFA_\eta\left\{D_{0+}^{\sigma}\Phi(\upsilon)\right\} = \nu^\eta \Omega_\eta(\nu) - \frac{\Phi(0)}{\nu^\eta},\ 0 < \eta \leqslant 1, \tag{21}$$

and

$$LFA_\eta\left\{D_{0+}^{n\sigma}\Phi(\upsilon)\right\} = \nu^{n\eta} \Omega_\eta(\nu) - \sum_{k=0}^{n-1} \frac{\Phi^{(k\eta)}(0)}{\nu^{(2-n+k)\eta}},\ 0 < \eta \leqslant 1. \tag{22}$$

Proof. We proof the formula (21). Using the formula (16) and the integral by parts [11], we get the following

$$\begin{aligned}
LFA_\eta\left\{\Phi^{(\eta)}(\upsilon)\right\} &= \frac{1}{\Gamma(1+\eta)}\frac{1}{\nu^\eta}\int\limits_0^\infty E_\eta(-\nu^\eta\upsilon^\eta)\Phi^{(\eta)}(\upsilon)(d\upsilon)^\eta \\
&= \frac{1}{\Gamma(1+\eta)}\frac{1}{\nu^\eta}\left([-\Gamma(1+\sigma)\Phi(0)]+\nu^\eta\lim_{t\longrightarrow\infty}\int\limits_0^t E_\eta(-\nu^\eta\upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta\right) \\
&= -\frac{1}{\nu^\eta}\Phi(0)+\nu^\eta\left(\frac{1}{\Gamma(1+\eta)}\frac{1}{\nu^\eta}\int\limits_0^\infty E_\eta(-\nu^\eta\upsilon^\eta)\Phi(\upsilon)(d\upsilon)^\eta\right) \\
&= \nu^\eta\Omega_\eta(\nu)-\frac{\Phi(0)}{\nu^\eta}.
\end{aligned}$$

To demonstrate the validity of the formula (22), we use mathematical induction.
If $n=1$ and according to formula (22), we obtain

$$LFA_\eta\left\{\Phi^{(\eta)}(\upsilon)\right\}=\nu^\eta\Omega_\eta(\nu)-\frac{\Phi(0)}{\nu^\eta},$$

so, according to the (21), we note that the formula holds when $n=1$.
Assume inductively that the formula holds for n, so that

$$LFA_\eta\left\{D_{0+}^{n\sigma}\Phi(\upsilon)\right\}=\nu^{n\eta}\Omega_\eta(\nu)-\sum_{k=0}^{n-1}\frac{\Phi^{(k\eta)}(0)}{\nu^{(2-n+k)\eta}}. \tag{23}$$

It remains to show that (22) is true for $n+1$. Let $D_{0+}^{n\sigma}\Phi(\upsilon)=\phi(\upsilon)$, (where $LFA_\eta\{\phi(\upsilon)\}=\psi_\eta(\nu)$) and according to (21) and (23), we have

$$\begin{aligned}
LFA_\eta\left[D_{0+}^{(n+1)\sigma}\Phi(\upsilon)\right] &= LFA_\eta\left[D_{0+}^{\sigma}\phi(\upsilon)\right]=\nu^\eta\psi_\eta(\nu)-\frac{\phi(0)}{\nu^\eta} \\
&= \nu^\eta\left[\nu^{n\eta}\Omega_\eta(\nu)-\sum_{k=0}^{n-1}\frac{\Phi^{(k\eta)}(0)}{\nu^{(2-n+k)\eta}}\right]-\frac{\phi(0)}{\nu^\eta} \\
&= \nu^{(n+1)\eta}\Omega_\eta(\nu)-\sum_{k=0}^{n-1}\frac{\Phi^{(k\eta)}(0)}{\nu^{(1-n+k)\eta}}-\frac{D_{0+}^{n\sigma}\Phi^{(n\eta)}(0)}{\nu^\eta} \\
&= \nu^{(n+1)\eta}\Omega_\eta(\nu)-\sum_{k=0}^{n}\frac{\Phi^{(k\eta)}(0)}{\nu^{(1-n+k)\eta}}.
\end{aligned}$$

Therefore the formula (22) is true for $n+1$.
Thus by the principle of mathematical induction, for all $n\geqslant 1$ the formula (22) holds.

□

Theorem 3.5. (*Local fractional Aboodh transform of local fractional integral*). *If* $LFA_\eta\{\Phi(\upsilon)\}=\Omega_\eta(\nu)$, *then one has*

$$LFA_\eta\left\{{}_0I_\upsilon^{(\eta)}\Phi(\upsilon)\right\}=\frac{1}{\nu^\eta}\Omega_\eta(\nu). \tag{24}$$

Proof. Let $P(\upsilon)={}_0I_\upsilon^{(\eta)}\Phi(\upsilon)$. According to the (theorem 3.2.9, [33]), we get

$$D_{0+}^{\eta}P(\upsilon)=\Phi(\upsilon), \tag{25}$$

and $P(0)=0$.

Taking the local fractional Aboodh transform on both sides of Equ.(25), we have

$$LFA_\eta \left\{D_{0+}^{\eta} P(\upsilon)\right\} = LFA_\eta \left\{\Phi(\upsilon)\right\}.$$

Which give

$$\nu^\eta LFA_\eta \left\{P(\upsilon)\right\} = \Omega_\eta(\nu),$$

because $P(0) = 0$, and $LFA_\eta \left\{\Phi(\upsilon)\right\} = \Omega_\eta(\nu)$.
Thus we get

$$LFA_\eta \left\{{}_0I_\upsilon^{(\eta)} \Phi(\upsilon)\right\} = \frac{1}{\nu^\eta} \Omega_\eta(\nu).$$

□

Theorem 3.6. (*local fractional convolution*). *If* $LFA_\eta \left\{\Phi(\upsilon)\right\} = F_\eta(\nu)$ *and* $LFA_\eta \left\{\Psi(\upsilon)\right\} = \Omega_\eta(\nu)$, *then one has*

$$LFA_\eta \left\{\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta\right\} = \nu^\eta F_\eta(\nu)\Omega_\eta(\nu),$$

where

$$\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta = \frac{1}{\Gamma(1+\eta)} \int_0^\infty \Phi(\varkappa)\Psi(\upsilon - \varkappa)(d\varkappa)^\eta.$$

Proof. The Laplace transform of fractional order of the function $\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta$, is given by

$$L_\eta \left\{\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta\right\} = L_\eta \left\{\Phi(\upsilon)\right\} L_\eta \{\Psi(\upsilon)\}.$$

Using the formula (19), gives

$$\begin{aligned} LFA_\eta \left\{\left(\Phi(\upsilon) * \Psi(\upsilon)\right)_\eta\right\} &= \frac{1}{\nu^\eta} L_\eta\{\Phi(\upsilon) * \Psi(\upsilon)\} \\ &= \nu^\eta \left(\frac{1}{\nu^\eta} L_\eta\{\Phi(\upsilon)\} \frac{1}{\nu^\eta} L_\eta\{\Psi(\upsilon)\}\right) \\ &= \nu^\eta F_\eta(\nu)\Omega_\eta(\nu). \end{aligned}$$

This completes the proof. □

Aboodh transform of somes special functions

In all of the following results, we relied on the formula (16), and some of the results found in references [3], [34]

1) If $\Phi(\upsilon) = 1$, we get

$$\begin{aligned} LFA_\eta \{1\} &= \frac{1}{\nu^\eta} \frac{1}{\Gamma(1+\eta)} \int_0^\infty E_\eta(-\nu^\eta \upsilon^\eta)(d\upsilon)^\eta \\ &= \frac{1}{\nu^\eta} \lim_{\varkappa \longrightarrow \infty} \left[\frac{-1}{\nu^\eta} E_\eta(-\nu^\eta \upsilon^\eta)\right]_0^\varkappa \\ &= \frac{1}{\nu^{2\eta}} \end{aligned}$$

2) If $\Phi(\upsilon)=\frac{\upsilon^\eta}{\Gamma(1+\eta)}$ $(0<\eta\leqslant 1)$, using the integral by parts [11], we get the following

$$\begin{aligned}LFA_\eta\{\upsilon^\eta\} &= \frac{1}{\nu^\eta}\frac{1}{\Gamma(1+\eta)}\int_0^\infty E_\eta(-\nu^\eta\upsilon^\eta)\upsilon^\eta(d\upsilon)^\eta\\ &= \frac{1}{\nu^\eta}\frac{1}{\Gamma(1+\eta)}\lim_{\varkappa\longrightarrow\infty}\left(\int_0^\varkappa\left(\frac{-1}{\nu^\eta}E_\eta(-\nu^\eta\upsilon^\eta)\right)^{(\eta)}\frac{\upsilon^\eta}{\Gamma(1+\eta)}(d\upsilon)^\eta\right)\\ &= \frac{1}{\nu^{2\eta}}\frac{1}{\Gamma(1+\eta)}\lim_{\varkappa\longrightarrow\infty}\left(\int_0^\varkappa E_\eta(-\nu^\eta\upsilon^\eta)(d\upsilon)^\eta\right)\end{aligned}$$

Because $\lim_{\varkappa\longrightarrow\infty}\left[\frac{-1}{\nu^\eta}E_\eta(-\nu^\eta\upsilon^\eta)\frac{\upsilon^\eta}{\Gamma(1+\eta)}\right]_0^\varkappa=0$.
Therefore

$$\begin{aligned}LFA_\eta\{\upsilon^\eta\} &= \frac{1}{\nu^{2\eta}}\lim_{\varkappa\longrightarrow\infty}\left[\frac{-1}{\nu^\eta}E_\eta(-\nu^\eta\upsilon^\eta)\right]_0^\varkappa\\ &= \frac{1}{\nu^{3\eta}}.\end{aligned}$$

3) If $\Phi(\upsilon)=E_\eta(a\upsilon^\eta)$, using the formula (16), we get

$$\begin{aligned}LFA_\eta\{E_\eta(a\upsilon^\eta)\} &= \frac{1}{\nu^\eta}\frac{1}{\Gamma(1+\eta)}\int_0^\infty E_\eta(-\nu^\eta\upsilon^\eta)E_\eta(a\upsilon^\eta)(d\upsilon)^\eta\\ &= \frac{1}{\nu^\eta}\frac{1}{\Gamma(1+\eta)}\int_0^\infty E_\eta((a-\nu^\eta)\upsilon^\eta)(d\upsilon)^\eta\\ &= \frac{1}{\nu^\eta}\lim_{\varkappa\longrightarrow\infty}\left[\frac{1}{a-\nu^\eta}E_\eta(-\nu^\eta\upsilon^\eta)\right]_0^\varkappa\\ &= \frac{1}{\nu^{2\eta}-a\nu^\eta}\end{aligned}$$

4) If $\Phi(\upsilon)=\sin_\eta(a\upsilon^\eta)$ $(0<\eta\leqslant 1)$, using the formula (16), we get

$$\begin{aligned}LFA_\eta\{\sin_\eta(a\upsilon^\eta)\} &= \frac{1}{\nu^\eta}\frac{1}{\Gamma(1+\eta)}\int_0^\infty E_\eta(-\nu^\eta\upsilon^\eta)\frac{E_\eta(ai^\eta\upsilon^\eta)-E_\eta(-ai^\eta\upsilon^\eta)}{2i^\eta}(d\upsilon)^\eta\\ &= \frac{1}{2i^\eta\nu^\eta}\frac{1}{\Gamma(1+\eta)}\int_0^\infty\left[E_\eta((ai^\eta-\nu^\eta)\upsilon^\eta)-E_\eta((-ai^\eta-\nu^\eta)\upsilon^\eta)\right](d\upsilon)^\eta\\ &= \frac{1}{2i^\eta\nu^\eta}\lim_{\varkappa\longrightarrow\infty}\left[\left(\frac{E_\eta((ai^\eta-\nu^\eta)\upsilon^\eta)}{ai^\eta-\nu^\eta}-\frac{E_\eta((ai^\eta-\nu^\eta)\upsilon^\eta)}{-ai^\eta-\nu^\eta}\right)\right]_0^\varkappa\end{aligned}$$

After the calculations we find

$$LFA_\eta\{\sin_\eta(a\upsilon^\eta)\}=\frac{a}{\nu^\eta(\nu^{2\eta}+a^2)}.$$

5) If $\Phi(\upsilon)=\cos_\eta(a\upsilon^\eta)$ $(0<\eta\leqslant 1)$, knowing that $\cos_\eta(a\upsilon^\eta)=\frac{E_\eta(ai^\eta\upsilon^\eta)+E_\eta(-ai^\eta\upsilon^\eta)}{2}$, and by following the same previous steps, we get

$$LFA_{\eta}\{\cos_{\eta}(a\upsilon^{\eta})\} = \frac{1}{\nu^{2\eta}+a^{2}}.$$

6) If $\Phi(\upsilon) = \sinh_{\eta}(a\upsilon^{\eta})$ $(0 < \eta \leqslant 1)$, we obtain

$$\begin{aligned} LFA_{\eta}\left\{\sinh_{\eta}(a\upsilon^{\eta})\right\} &= \frac{1}{\nu^{\eta}}\frac{1}{\Gamma(1+\eta)}\int_{0}^{\infty} E_{\eta}(-\nu^{\eta}\upsilon^{\eta})\frac{E_{\eta}\left(a\upsilon^{\eta}\right)-E_{\eta}\left(-a\upsilon^{\eta}\right)}{2}(d\upsilon)^{\eta} \\ &= \frac{1}{2\nu^{\eta}}\frac{1}{\Gamma(1+\eta)}\int_{0}^{\infty}\left[E_{\eta}((a-\nu^{\eta})\upsilon^{\eta}) - E_{\eta}((-a-\nu^{\eta})\upsilon^{\eta})\right](d\upsilon)^{\eta} \\ &= \frac{1}{2\nu^{\eta}}\lim_{\varkappa\longrightarrow\infty}\left[\left(\frac{E_{\eta}((a-\nu^{\eta})\upsilon^{\eta})}{a-\nu^{\eta}} + \frac{E_{\eta}((-a-\nu^{\eta})\upsilon^{\eta})}{a+\nu^{\eta}}\right)\right]_{0}^{\varkappa} \end{aligned}$$

By performing simple operations, we find

$$LFA_{\eta}\left\{\sinh_{\eta}(a\upsilon^{\eta})\right\} = \frac{a}{\nu^{\eta}\left(\nu^{2\eta}-a^{2}\right)}.$$

5) If $\Phi(\upsilon) = \cosh_{\eta}(a\upsilon^{\eta})$ $(0 < \eta \leqslant 1)$, knowing that $\cosh_{\eta}(a\upsilon^{\eta}) = \frac{E_{\eta}(a\upsilon^{\eta})+E_{\eta}(-a\upsilon^{\eta})}{2}$, and by following the same previous steps, we get

$$LFA_{\eta}\left\{\cosh_{\eta}(a\upsilon^{\eta})\right\} = \frac{1}{\nu^{2\eta}-a^{2}}.$$

4. Ilustrative Examples

In this section, we will apply the local fractional Aboodh transform $(LFAT)$ to some suggested local fractional differential equations.

Example 4.1. *First, we consider the following local fractional differential equation of order η, $(0 < \eta \leqslant 1)$*

$$\frac{d^{\eta}\psi\left(\upsilon\right)}{d\upsilon^{\eta}} + \psi\left(\upsilon\right) = -1, \tag{26}$$

with the initial condition $\psi(0) = 0$.
Taking local fractional Aboodh transform on both sides of given equation,we have

$$\nu^{\eta}\ LFA_{\eta}\left\{\psi\left(\upsilon\right)\right\} - \frac{\psi\left(0\right)}{\nu^{\eta}} + LFA_{\eta}\left\{\psi\left(\upsilon\right)\right\} = -LFA_{\eta}\left\{1\right\}. \tag{27}$$

Then

$$\left(\nu^{\eta}\ +1\right)\ LFA_{\eta}\left\{\psi\left(\upsilon\right)\right\} = -\frac{1}{\nu^{2\eta}}. \tag{28}$$

Which give

$$\begin{aligned} LFA_{\eta}\left\{\psi\left(\upsilon\right)\right\} &= -\frac{1}{\nu^{2\eta}\left(\nu^{\eta}\ +1\right)} \\ &= \frac{1}{\nu^{2\eta}\ +\nu^{\eta}} - \frac{1}{\nu^{2\eta}}. \end{aligned} \tag{29}$$

By applying the inverse transformation on both sides of equation (29)*, we get*

$$\psi\left(\upsilon\right) = E_{\eta}(-\upsilon^{\eta}) - 1, \tag{30}$$

Example 4.2. *Next, we consider the following local fractional differential equation of order* η, $(0 < \eta \leqslant 1)$

$$\frac{d^{\eta}\psi(\upsilon)}{d\upsilon^{\eta}} - 2\psi(\upsilon) = 2, \tag{31}$$

with the initial condition

$$\psi(0) = 1. \tag{32}$$

Taking local fractional Aboodh transform on both sides of equation (31), *we have*

$$\nu^{\eta}\ LFA_{\eta}\{\psi(\upsilon)\} - 2LFA_{\eta}\{\psi(\upsilon)\} = \frac{2}{\nu^{2\eta}}. \tag{33}$$

By following the same steps as the previous example, we obtain

$$LFA_{\eta}\{\psi(\upsilon)\} = \frac{2}{\nu^{\eta}(\nu^{\eta}-2)} - \frac{1}{\nu^{2\eta}}. \tag{34}$$

Take the inverse transformation on both sides of equation (34),*we get*

$$\psi(\upsilon) = 2E(2\nu^{\eta}) - 1. \tag{35}$$

Result (35) *represents the exact solution to the equation* (31).

Example 4.3. *Finally, we consider the following local fractional differential equation of order* 2η, $(0 < \eta \leqslant 1)$

$$\frac{d^{2\eta}\psi(\upsilon)}{d\upsilon^{2\eta}} + \psi(\upsilon) = -\frac{\upsilon^{\eta}}{\Gamma(1+\eta)}, \tag{36}$$

subject to the initial conditions

$$\psi(0) = 0,\ \frac{d^{\eta}\psi(0)}{d\upsilon^{\eta}} = 0. \tag{37}$$

Taking local fractional Aboodh transform on both sides of equation (36), *we have*

$$\nu^{2\eta}LFA_{\eta}\{\psi(\upsilon)\} + LFA_{\eta}\{\psi(\upsilon)\} = -\frac{1}{\nu^{3\eta}}. \tag{38}$$

By following the same steps as the previous example, we obtain

$$LFA_{\eta}\{\psi(\upsilon)\} = \frac{1}{\nu^{\eta}(\nu^{\eta}+1)} - \frac{1}{\nu^{3\eta}}. \tag{39}$$

Take the inverse transformation on both sides of equation (39), *yields*

$$\psi(\upsilon) = \sin_{\eta}(\upsilon^{\eta}) - \frac{\upsilon^{\eta}}{\Gamma(1+\eta)}. \tag{40}$$

Result (40) *represents the exact solution to the equation* (36).

5. Conclusion

The basic idea that we presented in this work is based on combining the Aboodh transform with the local fractional derivative, where we presented some important results and properties of this combination. And in order to prove the effectiveness of this method, we applied it to solving some linear local fractional differential equations, as we saw that the solutions are accurate and of the type of nondifferentiable functions. Based on the results of the proposed examples, we can say that this method is practical and effective in solving other forms of linear local fractional differential equations.

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