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Fixed points of ρ-nonexpansive mappings using MP iterative process

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Abstract

This research article introduces a new iterative process called MP iteration and proves some convergence and approximation results for the fixed points of ρ-nonexpansive mappings in modular function spaces. To demonstrate that MP iterative process converges faster than some well-known existing iterative processes for ρ-nonexpansive mappings, we construct some numerical examples. In the end, the concept of summably almost T-stability for MP iterative process is discussed.

Keywords: Fixed point ρ-nonexpansive mappings MP iteration summably almost T-stability non-self mappings .

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1. Introduction

In recent years, the convergence results for the fixed points of mappings by using the iterative process have been introduced and studied by many researchers. The authors always focus on three aspects as follows: the

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weak property of the constructed map, the speed of the iteration and the spatial constraints. When someone establishes the existence of fixed point for some mapping then it is not easy to compute the value of that fixed point and due to this reason we take help of an iterative process for finding the value of fixed point. In the past decades, a number of iterative processes have been developed to approximate the fixed points of nonexpansive mappings.

Now, we give a brief introduction of the existing iterative processes. Throughout this section, let $T : E \to E$ be any mapping where E is a nonempty subset of a Banach space X and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $(0, 1)$ satisfying some appropriate conditions where $n \geq 0$.

As the Picard iterative process fails to converge to a fixed point of nonexpansive mapping, in 1953, Mann [11] introduced the following iterative process to approximate the fixed points of nonexpansive mappings:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \end{cases} \tag{1.1}$$

Later, it was discovered that Mann iterative process fails to converge a fixed point of pseudo-contractive mapping. To overcome this drawback of Mann iterative procedure, in 1974, Ishikawa [7] introduced the following two step iterative process to approximate the fixed points of pseudo-contractive mappings:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n \\ v_n = (1-\beta_n)u_n + \beta_n T u_n. \end{cases} \tag{1.2}$$

In 2000, Noor [14] introduced the following three step iterative process for general variational inequalities:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n \\ v_n = (1-\beta_n)u_n + \beta_n T w_n \\ w_n = (1-\gamma_n)u_n + \gamma_n T u_n. \end{cases} \tag{1.3}$$

It is well known that the Picard iterative process converges faster than the Mann iterative process for contraction mappings (see Proposition 3.1 [2]). In 2007, Agarwal et al. [2] introduced the following two step iterative process to approximate the fixed points of nearly asymptotically nonexpansive mappings:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)T u_n + \alpha_n T v_n \\ v_n = (1-\beta_n)u_n + \beta_n T u_n. \end{cases} \tag{1.4}$$

They claimed that the iterative process defined by them converges to a fixed point of contraction mapping with same rate of convergence as the Picard iterative process but converges faster than Mann iterative process.

In 2014, Abbas and Nazir [1] introduced the following three step iterative process to approximate the fixed points of nonexpansive mappings in uniformly convex Banach spaces:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n. \end{cases} \tag{1.5}$$

They also showed that this iterative process converges faster than all of Picard, Mann and Agarwal iterative processes to a fixed point of contraction mapping.

In 2014, Thakur et al. [25] introduced the following iterative process, we call it Thakur iteration, to approximate the fixed points of nonexpansive mappings:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n. \end{cases} \tag{1.6}$$

They also claimed that this iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Agarwal and Abbas and Nazir iterative processes to a fixed point of contraction mapping in sense of Berinde [3].

In 2016, Sahu et al. [21] and Thakur et al. [24] introduced the following iterative process to approximate the fixed points of nonexpansive mappings in uniformly convex Banach spaces:

$$\begin{cases} u_1 \in E \\ u_{n+1} = (1-\alpha_n)Tw_n + \alpha_n Tv_n \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n. \end{cases} \tag{1.7}$$

The authors [21, 24] proved that this iterative process converges faster than all known iterative processes to a fixed point of contraction mapping.

In 2019, Panwar and Reena [19] proved some approximation results for the fixed points of multivalued ρ-quasi-nonexpansive mappings for a newly defined hybrid iterative process in modular function spaces as follows:

$$\begin{cases} f_1 \in E \\ f_{n+1} \in P_\rho^T(h_n) \\ e_n = \gamma_n u_n + (1-\gamma_n) f_n \\ g_n = \beta_n v_n + (1-\beta_n) f_n \\ h_n = \alpha_n w_n + (1-\alpha_n) f_n \end{cases} \tag{1.8}$$

where the sequences $u_n \in P_\rho^T(f_n), v_n \in P_\rho^T(e_n), w_n \in P_\rho^T(g_n)$.

In 2019, Ritika et al. [22] introduced RK-iteration for generalized nonexpansive mappings in CAT(0) spaces and claimed that the iterative process introduced by them converges faster than some well known iterative processes.

$$\begin{cases} x_0 \in E \\ x_{n+1} \in T v_n \\ v_n = T((1-\alpha_n) y_n + \alpha_n T y_n) \\ y_n = T((1-\beta_n) z_n + \beta_n T z_n) \\ z_n = T((1-\gamma_n) x_n + \gamma_n T x_n). \end{cases} \tag{1.9}$$

Pant et al. established some fixed point results for generalized nonexpansive type mappings in Banach spaces (see [16, 17, 18]).

In [20], Panwar and Reena introduced a new iterative scheme named as AR-iteration for nonexpansive mappings and by providing numerical examples proved that the AR-iteration converges faster than that of Sahu-Thakur iteration and Thakur iteration in modular function spaces.

$$\begin{cases} f_1 \in E \\ f_{n+1} = T g_n \\ g_n = T((1-\alpha_n) f_n + \alpha_n T h_n) \\ h_n = T((1-\beta_n) f_n + \beta_n T e_n) \\ e_n = (1-\gamma_n) f_n + \gamma_n T f_n. \end{cases} \tag{1.10}$$

Motivated by the work done in the area of fixed point theory, we define a new iterative process named as MP iteration to approximate the fixed points of ρ-nonexpansive mappings in modular function spaces. Also, we establish that MP iterative process converges faster than the existing well-known iterative process.

$$\begin{cases} u_1 \in B_\rho \\ u_{n+1} = T^2 v_n \\ v_n = T((1-\alpha_n)Ts_n + \alpha_n T w_n) \\ w_n = T((1-\beta_n)s_n + \beta_n T s_n) \\ s_n = T((1-\gamma_n)u_n + \gamma_n T u_n). \end{cases} \tag{1.11}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in (0,1) are bounded away from both 0 and 1 and B_ρ is a nonempty ρ-bounded, ρ-closed, ρ-convex subset of modular function spaces.

2. Preliminaries

In this section, we recall some basic definitions and needed results to prove our results.

Let Ω be a nonempty set and $\sum$ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a nontrivial δ-ring of subsets of Ω which means that $\mathcal{P}$ is closed under countable intersection, finite union and differences. Suppose that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \sum$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. By ε we denote the linear space of all simple functions with support from $\mathcal{P}$. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence

$$\{g_n\} \subset \varepsilon,\ |g_n| \le |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(w)| < \infty \; \rho - a.e.\}.$$

The definition of modular was given by Musielak and Orlicz in [13].

Definition 2.1. [13] Let X (over field $\mathbb{R}$ or $\mathbb{C}$) be a vector space. A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary elements $f, g \in X$, the following hold:

(i) $\rho(f) = 0 \iff f = 0$
(ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$
(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ whenever $\alpha, \beta \ge 0,\ \ \alpha + \beta = 1$.

If we replace (iii) by

(iv) $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ whenever $\alpha, \beta \ge 0,\ \alpha + \beta = 1$.

Then modular ρ is called convex.

Definition 2.2. [13] If ρ is convex modular in X, then the set defined by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \; as \; \lambda \to 0\}$$

is called modular function space.

Definition 2.3. [9] Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudo modular if

(1.) $\rho(0) = 0$;
(2.) ρ is monotone, i.e., $|f(w)| \leq |g(w)|$ for any $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(3.) ρ is orthogonally sub-additive, i.e., $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A)+\rho(f\chi_B)$ for any $A, B \in \sum$ such that $A\cap B \neq \phi$, $f \in \mathcal{M}_\infty$;
(4.) ρ has Fatou property, i.e., $|f_n(w)| \uparrow |f(w)|$ for $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(5.) ρ is order continuous in ε, i.e., $g_n \in \varepsilon$ and $|g_n(w)| \downarrow 0$, then $\rho(g_n) \downarrow 0$.

Let ρ be a regular convex pseudo modular then ρ is regular convex function modular if $\rho(f) = 0$ implies $f = 0$ a.e. The class of all nonzero regular convex function modular on Ω is denoted by $\mathfrak{R}$.

In 1988, Kozlowski introduced the following condition in modular function spaces:

Definition 2.4. [10] $\rho \in \mathfrak{R}$ is said to satisfy Δ_2-condition if $\sup_{n\geq1} \rho(2u_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to ϕ and $\sup_{n\geq1} \rho(u_n, D_k) \to 0$ as $k \to \infty$.

In 2012, the following lemma was proved by Dehaish and Kozlowski [5] to establish that generalized Mann and Ishikawa iterative processes converge almost everywhere to a fixed point of asymptotic pointwise nonexpansive mapping T in modular function spaces.

Lemma 2.5. *[5] Let $\rho \in \mathfrak{R}$ satisfying (UUC1) and $\{t_n\} \subset (0, 1)$ be bounded away from both 0 and 1. If there exists $R > 0$ such that*

$$\lim_{n\to\infty} \sup \rho(u_n) \leq R,\ \lim_{n\to\infty} \sup \rho(v_n) \leq R \text{ and}$$
$$\lim_{n\to\infty} \rho(t_n u_n + (1 - t_n)v_n) = R, \text{ then}$$
$$\lim_{n\to\infty} \rho(u_n - v_n) = 0.$$

The sequence $\{t_n\} \subset (0, 1)$ is said to be bounded away from 0 if there exists $a > 0$ such that $t_n \geq a$ for all $n \in \mathbb{N}$. Similarly the sequence $\{t_n\} \subset (0, 1)$ is said to be bounded away from 1 if there exists $b < 1$ such that $t_n \leq b$ for all $n \in \mathbb{N}$.

In 1974, Senter and Dotson [23] gave a condition for nonexpansive mapping T that assures convergence of certain iterates to fixed point T as follows:

Definition 2.6. [23] Let D be a subset of L_ρ. Then $T : D \to D$ is said to satisfy condition (I) if there exists a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0, \psi(r) > 0$ for all $r \in (0, \infty)$ such that
$$\rho(f - Tf) \geq \psi(d_\rho(f, F_\rho(T))) \text{ for all } f \in D.$$

The following uniform convexity type properties of the function modular ρ is defined by Khamsi and Kozlowski [8] in 2011:

Definition 2.7. [8] Let ρ be a nonzero regular convex function modular defined on Ω.

(i) Let $r > 0, \epsilon > 0$. Define $D_1(r, \epsilon) = \{(f, g) : f, g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho(f - g) \geq \epsilon r\}$.
Let $\delta_1(r, \epsilon) = \inf \left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f, g) \in D_1(r, \epsilon)\right\}$ if $D_1(r, \epsilon) \neq \phi$ and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \phi$. One says that ρ satisfies (UC1) if for every $r > 0, \epsilon > 0, \delta_1(r, \epsilon) > 0$.
Observe that for every $D_1(r, \epsilon) \neq \phi$, $\epsilon > 0$ small enough.
(ii) One says that ρ satisfies (UUC1) if for every $s \geq 0, \epsilon > 0$, there exists $\eta_1(r, \epsilon) > 0$ depending only on s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(r, \epsilon) > 0$ for any $r > s$.

The following definition was given by Musielak in [12]:

Definition 2.8. [12] Let $\rho \in \mathfrak{R}$.

(a) A sequence $\{f_n\}$ is ρ-convergent to f, that is, $f_n \to f$ if and only if $\rho(f_n - f) \to 0$ as $n \to \infty$.
(b) A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $m, n \to \infty$.

(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence $\{f_n\} \subset B$, the convergence $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(d) A set $B \subset L_\rho$ is called ρ-bounded if ρ-diameter is finite; the ρ-diameter of B is defined as
$$\delta_\rho(B) = \sup\{\rho(f-g) : f, g \in B\}.$$

(e) A set $B \subset L_\rho$ is called ρ-compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.

Proposition 2.9. *[9] Let* $\rho \in \Re$.

(i) L_ρ *is* ρ*-complete.*

(ii) ρ*-balls* $B_\rho(f, r) = \{g \in L_\rho : \rho(f-g) \leq r\}$ *are* ρ*-closed.*

(iii) If $\rho(\lambda f_n) \to 0$ *for* $\lambda > 0$ *then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n \to 0\ \rho - a.e.$ *as* $n \to \infty$.

(iv) If $\{f_n\}$ *is* ρ*-a.e. convergent to* f *then* $\rho(f) \leq \liminf_{n\to\infty} \rho(f_n)$. *(This property is equivalent to Fatou property.)*

(v) If $L^0_\rho = \{f \in L_\rho : \rho(f, .)$ *is order continuous\} and* $S_\rho = \{f \in L_\rho : \lambda f \in L^0_\rho$ *for any* $\lambda > 0\}$, *then* $S_\rho \subseteq L^0_\rho \subseteq L_\rho$.

3. Convergence and Approximation Results

In the following section, some convergence and approximation results are proved for the fixed point of ρ-nonexpansive mappings using MP iteration (1.11).

Lemma 3.1. *If* $B_\rho \neq \phi$ *is a* ρ*-bounded,* ρ*-closed,* ρ*-convex subset of* L_ρ *and* $T : B_\rho \to B_\rho$ *is a* ρ*-nonexpansive mapping with* $F_\rho(T) \neq \phi$. *For arbitrarily chosen* $u_1 \in B_\rho$, *suppose* $\{u_n\}$ *is the sequence, generated by MP iterative process* (1.11), *then* $\lim_{n\to\infty} \rho(u_n - p^*)$ *exists for any* $p^* \in F_\rho(T)$.

Proof. Let $p^* \in F_\rho(T)$, where T is a ρ-nonexpansive mapping, then using Proposition 2.9 and convexity of ρ, we have
$$\rho(u_{n+1} - p^*) = \rho(T^2 v_n - p^*) \leq \rho(Tv_n - p^*) \leq \rho(v_n - p^*). \tag{3.1}$$

$$\begin{aligned}\rho(v_n - p^*) &= \rho(T((1-\alpha_n)Ts_n + \alpha_n Tw_n) - p^*)\\ &\leq \rho((1-\alpha_n)Ts_n + \alpha_n Tw_n - p^*)\\ &\leq (1-\alpha_n)\rho(Ts_n - p^*) + \alpha_n \rho(Tw_n - p^*)\\ &\leq (1-\alpha_n)\rho(s_n - p^*) + \alpha_n \rho(w_n - p^*).\end{aligned}$$

$$\rho(v_n - p^*) \leq (1-\alpha_n)\rho(s_n - p^*) + \alpha_n \rho(w_n - p^*). \tag{3.2}$$

$$\begin{aligned}\rho(w_n - p^*) &= \rho(T((1-\beta_n)s_n + \beta_n Ts_n) - p^*)\\ &\leq \rho((1-\beta_n)s_n + \beta_n Ts_n - p^*)\\ &\leq (1-\beta_n)\rho(s_n - p^*) + \beta_n \rho(Ts_n - p^*) \leq \rho(s_n - p^*).\end{aligned}$$

$$\rho(w_n - p^*) \leq \rho(s_n - p^*). \tag{3.3}$$

$$\begin{aligned}\rho(s_n - p^*) &= \rho(T((1-\gamma_n)u_n + \gamma_n Tu_n) - p^*)\\ &\leq \rho((1-\gamma_n)u_n + \gamma_n Tu_n - p^*)\\ &\leq (1-\gamma_n)\rho(u_n - p^*) + \gamma_n \rho(Tu_n - p^*)\\ &\leq (1-\gamma_n)\rho(u_n - p^*) + \gamma_n \rho(u_n - p^*).\end{aligned}$$

$$\rho(s_n - p^*) \leq \rho(u_n - p^*). \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we get $\rho(v_n - p^*) \leq \rho(u_n - p^*)$. Then from (3.1), we have

$$\rho(u_{n+1} - p^*) \leq \rho(u_n - p^*). \tag{3.5}$$

From (3.5), we conclude that $\{\rho(u_n - p^*)\}$ is a bounded and nonincreasing sequence $\forall\ p^* \in F_\rho(T)$ which implies that $\lim_{n\to\infty} \rho(u_n - p^*)$ exists. □

Theorem 3.2. *Let $\rho \in \Re$ satisfy (UUC1), Δ_2-conditions and B_ρ be a nonempty ρ-closed, ρ-convex subset of L_ρ and $T : B_\rho \to B_\rho$ be a ρ-nonexpansive mapping with $F_\rho(T) \neq \phi$. Suppose $\{u_n\}$ is the sequence in B_ρ defined by MP iterative process* (1.11)*. Then $\lim_{n\to\infty} \rho(u_n - Tu_n) = 0$.*

Proof. Let $p^* \in F_\rho(T)$. Using Lemma 3.1, $\lim_{n\to\infty} \rho(u_n - p^*)$ exists and let

$$\lim_{n\to\infty} \rho(u_n - p^*) = K. \tag{3.6}$$

From (3.2), (3.3) and (3.4), we obtain

$$\lim_{n\to\infty} \sup \rho(v_n - p^*) \leq \lim_{n\to\infty} \sup \rho(u_n - p^*) = K. \tag{3.7}$$

$$\lim_{n\to\infty} \sup \rho(w_n - p^*) \leq \lim_{n\to\infty} \sup \rho(u_n - p^*) = K. \tag{3.8}$$

$$\lim_{n\to\infty} \sup \rho(s_n - p^*) \leq \lim_{n\to\infty} \sup \rho(u_n - p^*) = K. \tag{3.9}$$

Also, $\rho(Tu_n - p^*) \leq \rho(u_n - p^*)$, $\rho(Tw_n - p^*) \leq \rho(w_n - p^*)$, $\rho(Tv_n - p^*) \leq \rho(v_n - p^*)$ and $\rho(Ts_n - p^*) \leq \rho(s_n - p^*)$, therefore

$$\lim_{n\to\infty} \sup \rho(Tv_n - p^*) \leq K. \tag{3.10}$$

$$\lim_{n\to\infty} \sup \rho(Tw_n - p^*) \leq K. \tag{3.11}$$

$$\lim_{n\to\infty} \sup \rho(Ts_n - p^*) \leq K. \tag{3.12}$$

Using (3.1), (3.2), (3.3) and (3.4), $\rho(u_{n+1} - p^*) \leq \rho(v_n - p^*) \leq \rho(w_n - p^*) \leq \rho(s_n - p^*) \leq \rho(u_n - p^*)$ which implies $\rho(u_{n+1} - p^*) \leq \rho(s_n - p^*) \leq \rho(u_n - p^*)$ and hence

$$\lim_{n\to\infty} \rho(s_n - p^*) = K. \tag{3.13}$$

$$\begin{aligned}
\rho(s_n - p^*) &= \rho(T((1-\gamma_n)u_n + \gamma_n Tu_n) - p^*) \\
&\leq \rho((1-\gamma_n)u_n + \gamma_n Tu_n - p^*) \\
\rho(s_n - p^*) &\leq \rho((1-\gamma_n)(u_n - p^*) + \gamma_n(Tu_n - p^*)) \leq \rho(u_n - p^*).
\end{aligned}$$

Applying limit and using (3.6) and (3.13), we get

$$\lim_{n\to\infty} \rho((1-\gamma_n)(u_n - p^*) + \gamma_n(Tu_n - p^*)) = K. \tag{3.14}$$

Equations (3.6), (3.14) and Lemma 2.5 establish that

$$\lim_{n\to\infty} \rho(u_n - Tu_n) = 0.$$
□

Theorem 3.3. *Let $\rho \in \Re$ satisfy (UUC1), Δ_2-conditions. If $B_\rho \neq \phi$ is a ρ-compact, ρ-convex subset of L_ρ and $T : B_\rho \to B_\rho$ is ρ-nonexpansive mapping. Suppose that $\{u_n\}$ is a sequence in B_ρ defined by MP iterative process* (1.11)*, then $\{u_n\}$ is ρ-convergent to a fixed point of T.*

Proof. Due to ρ-compactness B_ρ, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty}\rho(u_{n_k}-w)=0$ for $w\in B_\rho$. Using continuity ρ-nonexpansive mapping and convexity of ρ, we have

$$\begin{aligned}\rho\left(\frac{w-Tw}{3}\right)&=\rho\left(\frac{w-u_{n_k}}{3}+\frac{u_{n_k}-Tu_{n_k}}{3}+\frac{Tu_{n_k}-Tw}{3}\right)\\&\le\frac{1}{3}\rho(w-u_{n_k})+\frac{1}{3}\rho(u_{n_k}-Tu_{n_k})+\frac{1}{3}\rho(Tu_{n_k}-Tw)\\&\le\rho(w-u_{n_k})+\rho(u_{n_k}-Tu_{n_k})+\rho(Tu_{n_k}-Tw).\end{aligned}$$

Using Theorem 3.2 and continuity of T, $\rho\left(\frac{w-Tw}{3}\right)\to 0$ as $k\to\infty$, that is, $\rho\left(\frac{w-Tw}{3}\right)=0$ which shows that w is a fixed point of T. This proves that $\{u_n\}$ ρ-converges to a fixed point of T. □

Theorem 3.4. *Let $\rho\in\mathfrak{R}$ satisfy (UUC1), Δ_2-condition and B_ρ be a nonempty ρ-compact, ρ-convex subset of L_ρ and $T:B_\rho\to B_\rho$ be a ρ-nonexpansive mapping satisfying condition (I). If $\{u_n\}$ is a sequence in B_ρ defined by MP iterative process* (1.11)*, then $\{u_n\}$ is ρ-convergent to a fixed point of T.*

Proof. Using Lemma 3.1, $\lim_{n\to\infty}\rho(u_n-p^*)$ exists $\forall\ p^*\in F_\rho(T)$.
If $\lim_{n\to\infty}\rho(u_n-p^*)=0$, then nothing to do. Assume that $\lim_{n\to\infty}\rho(u_n-p^*)=K>0$. By the same lemma,

$$\rho(u_{n+1}-p^*)\le\rho(u_n-p^*)\text{ for all }p^*\in F_\rho(T).$$
$$\implies d_\rho(u_{n+1},F_\rho(T))\le d_\rho(u_n,F_\rho(T))$$

so, $\lim_{n\to\infty}d_\rho(u_n,F_\rho(T))$ exists. By Theorem 3.2 and condition (I),

$$0=\lim_{n\to\infty}d_\rho(u_n,Tu_n)\ge\lim_{n\to\infty}\psi(d_\rho(u_n,F_\rho(T))).$$

Since ϕ is increasing function and $\phi(0)=0$, therefore, $\lim_{n\to\infty}d_\rho(u_n,F_\rho(T))=0$. We now prove that $\{u_n\}$ is ρ-Cauchy sequence in B_ρ.
Let $\epsilon>0$ be arbitrary. Then there exists an integer $m_0\in\mathbb{N}$ such that $d_\rho(u_n,F_\rho(T))<\frac{\epsilon}{2}$, for all $n\ge m_0$. Particularly, $\inf\{\rho(f_{m_0}-p^*):p^*\in F_\rho(T)\}<\frac{\epsilon}{2}$. Thus, there exists a $p_0\in F_\rho(T)$ such that $\rho(u_{m_0}-p_0)<\epsilon$. Now, for all $m,n\ge m_0$, we have

$$\begin{aligned}\rho\left(\frac{u_m-u_n}{2}\right)&\le\frac{1}{2}\rho(u_m-p_0)+\frac{1}{2}\rho(u_n-p_0)\\&\le\frac{1}{2}\rho(u_{m_0}-p_0)+\frac{1}{2}\rho(u_{m_0}-p_0)\le\epsilon.\end{aligned}$$

Since Δ_2-condition is satisfied by ρ, therefore by Proposition 2.9, $\{u_n\}$ is a ρ-Cauchy sequence in B_ρ. Due to completeness of L_ρ and ρ-closedness of B_ρ, then there must exists an $u\in B_\rho$ such that $\rho(u_n-u)\to 0\ \ as\ \ n\to\infty$. Hence, $\{u_n\}$ ρ-converges to a fixed point of T. □

4. Numerical Examples

In this section, we provide numerical examples to establish the fact that the rate of convergence of MP iteration (1.11) is faster than that of Sahu-Thakur iteration (1.7), Thakur iteration (1.6), RK iteration (1.9) and AR iteration (1.10). To prove our claim, we consider the following examples:

Example 4.1. *Let the real number system $\mathbb{R}$ be the space modulared as $\rho(u)=|u|$. It follows that ρ satisfies (UUC1) and Δ_2-condition. Define $B_\rho=\{u\in L_\rho:0\le u<\infty\}$ and a mapping $T:B_\rho\to B_\rho$ as $T=\frac{u+2}{2}$. Clearly, B_ρ is a nonempty ρ-compact, ρ-bounded and ρ-convex subset of $L_\rho=\mathbb{R}$ and T is ρ-nonexpansive mapping with $F_\rho(T)\neq\phi$. Let $\psi:[0,\infty)\to[0,\infty)$ be a nondecreasing continuous function by $\psi(r)=\frac{r}{4}$.*

Now, we show that T satisfies condition (I), that is, $\rho(u - Tu) \geq \psi(d_\rho(u, F_\rho(T)))$, $\forall\ u \in B_\rho$.
If $u \in (0, 2)$ then

$$\begin{aligned}\rho(u - Tu) &= \rho\left(u - \frac{u+2}{2}\right)\\ &= \left|u - \frac{u+2}{2}\right| = \left|\frac{u-2}{2}\right|.\\ \psi(d_\rho(u, F_\rho(T))) &= \psi(d_\rho(u, \{2\}))\\ &= \psi(|u-2|) = \left|\frac{u-2}{4}\right|.\end{aligned}$$

If $u \in F_\rho(T) = \{2\}$ then clearly $\rho(u - Tu) = 0 = \psi(d_\rho(u, F_\rho(T)))$. When $u \in (2, \infty)$ then

$$\begin{aligned}\rho(u - Tu) &= \rho\left(u - \frac{u+2}{2}\right)\\ &= \left|u - \frac{u+2}{2}\right| = \frac{u-2}{2}.\\ \psi(d_\rho(u, F_\rho(T))) &= \psi(d_\rho(u, \{2\}))\\ &= \psi(|u-2|) = \frac{u-2}{4}.\end{aligned}$$

From above calculations, for all $u \in B_\rho$, we have $\rho(u - Tu) \geq \psi(d_\rho(u, F_\rho(T)))$. We now prove that T is a ρ-nonexpansive mapping.

$$\begin{aligned}\rho(Tu - Tv) = |Tu - Tv| &= \left|\frac{u+2}{2} - \frac{v+2}{2}\right|\\ &= \left|\frac{u-v}{2}\right| \leq \rho(u - v).\end{aligned}$$

Therefore, T is ρ-nonexpansive mapping. In last, we show that $\{u_n\}$ is ρ-convergent to the fixed point of T by constructing a table. In table 1, the rate of convergence of MP iteration (1.11) *with RK iteration* (1.9)*, AR-iteration* (1.10)*, Sahu-Thakur iteration* (1.7) *and Thakur iteration* (1.6) *is compared and figure 4.1 shows the graphical representation of comparison of rate of convergence of iterative processes* (1.6)*,* (1.7)*,* (1.9) *and* (1.11)*.*

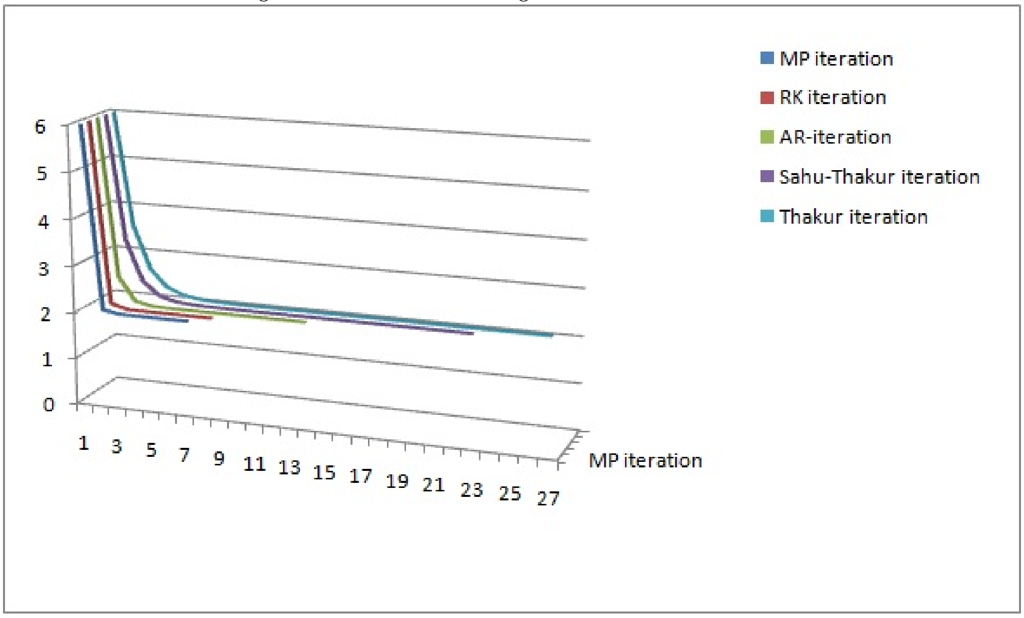
Example 4.2. *Let the real number system $\mathbb{R}$ be the space modulared as $\rho(u) = |u|$. It follows that ρ is satisfying (UUC1) and Δ_2-conditions. Define $B_\rho = \{u \in L_\rho : 0 \leq u < \infty\}$ and $T : B_\rho \to B_\rho$ as $T = \frac{2u+3}{3}$. Clearly, B_ρ is a nonempty ρ-compact, ρ-bounded, ρ-convex subset of $L_\rho = \mathbb{R}$ and T is a ρ-nonexpansive mapping with $F_\rho(T) \neq \phi$. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function by $\psi(r) = \frac{r}{4}$.*
Now, we show that T satisfies condition (I), that is, $\rho(u - Tu) \geq \psi(d_\rho(u, F_\rho(T)))$, $\forall\ u \in B_\rho$. If $u \in (0, 3)$ then

$$\begin{aligned}\rho(u - Tu) &= \rho\left(u - \frac{2u+3}{3}\right)\\ &= \left|u - \frac{2u+3}{3}\right| = \left|\frac{u-3}{3}\right|.\\ \psi(d_\rho(u, F_\rho(T))) &= \psi(d_\rho(u, \{3\}))\\ &= \psi(|u-3|) = \left|\frac{u-3}{4}\right|.\end{aligned}$$

Table 1: Comparison of rate of convergence various iterations

n	MP iteration	RK iteration	AR-iteration	Sahu-Thakur iteration	Thakur iteration
1	6	6	6	6	6
2	2.064453125	2.10546875	2.5859375	3.3125	3.5625
3	2.001038551	2.002780914	2.085830688	2.430664063	2.610351563
4	2.000016734	2.000073325	2.012572855	2.141311646	2.238418579
5	2.00000027	2.000001933	2.001841727	2.046367884	2.093132257
6	2.000000004	2.000000051	2.000269784	2.015214462	2.036379788
7	2	2.000000001	2.000039519	2.004992245	2.014210855
8	2	2	2.000005789	2.00163808	2.005551115
9	2	2	2.000000848	2.000537495	2.002168404
10	2	2	2.000000124	2.000176366	2.000847033
11	2	2	2.000000018	2.00005787	2.000330872
12	2	2	2.000000003	2.000018989	2.000129247
13	2	2	2	2.000006231	2.000050487
14	2	2	2	2.000002044	2.000019722
15	2	2	2	2.000000671	2.000007704
16	2	2	2	2.00000022	2.000003009
17	2	2	2	2.000000072	2.000001175
18	2	2	2	2.000000024	2.000000459
19	2	2	2	2.000000008	2.000000179
20	2	2	2	2.000000003	2.00000007
21	2	2	2	2.000000001	2.000000027
22	2	2	2	2	2.000000011
23	2	2	2	2	2.000000004
24	2	2	2	2	2.000000002
25	2	2	2	2	2.000000001
26	2	2	2	2	2

Figure 4.1: Rate of convergence of various iterations



If $u \in F_\rho(T) = \{3\}$ then clearly $\rho(u - Tu) = 0 = \psi(d_\rho(u, F_\rho(T)))$.
When $u \in (3, \infty)$ then

$$\begin{aligned}\rho(u - Tu) &= \rho\left(u - \frac{2u+3}{3}\right)\\ &= \left|u - \frac{2u+3}{3}\right| = \frac{u-3}{3}.\\ \psi(d_\rho(u, F_\rho(T))) &= \psi(d_\rho(u, \{3\}))\\ &= \psi(|u-3|) = \frac{u-3}{3}.\end{aligned}$$

From above calculations, $\rho(u - Tu) \geq \psi(d_\rho(u, F_\rho(T)))$ for all $u \in B_\rho$. We now prove that T is a ρ-nonexpansive mapping.

$$\rho(Tu - Tv) = |Tu - Tv| = \left|\frac{2u+3}{3} - \frac{2v+3}{3}\right| = \left|\frac{2(u-v)}{3}\right|.$$
$$\rho(u - v) = |u - v| \Longrightarrow \rho(Tu - Tv) \leq |u - v|.$$

Therefore, T is ρ-nonexpansive mapping. In the end, we show that $\{u_n\}$ is ρ-convergent to the fixed point of T by constructing a table. In table 2, the rate of convergence of MP iteration (1.11) *with RK iteration* (1.9)*, AR-iteration* (1.10)*, Sahu-Thakur iteration* (1.7) *and Thakur iteration* (1.6) *is compared and figure 4.2 shows the graphical representation of comparison of rate of convergence of iterative processes* (1.6), (1.7), (1.9) *and* (1.11).

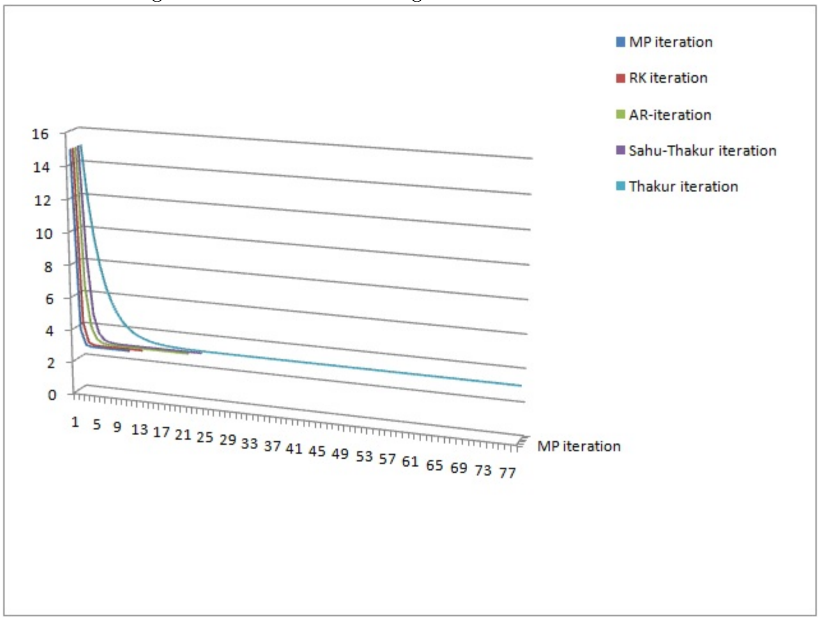
5. Stability Results

In the following of section, first we define the concept of T-stability, almost T-stability, summably almost T-stability of an iterative process in modular function spaces. Then we prove some stability results for our newly defined MP iterative process (1.11). In the end, we construct an example to establish the concept of summably almost T-stability for MP iterative process.

Table 2: Comparison of rate of convergence of various iterations

n	MP iteration	RK iteration	AR-iteration	Sahu-Thakur iteration	Thakur iteration
1	15	15	15	15	15
2	4.024234111	4.371742112	6.588477366	8.277777778	11.77777778
3	3.087421293	3.156806369	4.073097484	4.731770833	9.420781893
4	3.007461656	3.017924825	3.320898836	3.568237305	7.696683051
5	3.000636874	3.00204902	3.095961518	3.186452866	6.435536676
6	3.000054359	3.000234227	3.028696311	3.061179847	5.513031458
7	3.00000464	3.000026775	3.008581339	3.020074637	4.838235974
8	3.000000396	3.000003061	3.002566162	3.00658699	4.344635573
9	3.000000034	3.00000035	3.000767384	3.002161356	3.983576021
10	3.000000003	3.00000004	3.000229478	3.000709195	3.719467645
11	3	3.000000005	3.000068623	3.000232705	3.526277259
12	3	3.000000001	3.000020521	3.000076356	3.384962069
13	3	3	3.000006137	3.000025054	3.281592625
⋮	⋮	⋮	⋮	⋮	⋮
20	3	3	3.000000001	3.00000001	3.031553288
21	3	3	3	3.000000003	3.023080646
22	3	3	3	3.000000001	3.016883065
23	3	3	3	3	3.01234965
⋮	⋮	⋮	⋮	⋮	⋮
35	3	3	3	3	3.000289798
⋮	⋮	⋮	⋮	⋮	⋮
45	3	3	3	3	3.000012709
⋮	⋮	⋮	⋮	⋮	⋮
55	3	3	3	3	3.000000557
⋮	⋮	⋮	⋮	⋮	⋮
65	3	3	3	3	3.000000024
⋮	⋮	⋮	⋮	⋮	⋮
77	3	3	3	3	3.000000001
78	3	3	3	3	3

Figure 4.2: Rate of convergence of various iterations



Definition 5.1. Let B_ρ be a nonempty ρ-bounded, ρ-closed and ρ-convex subset of a modular function space L_ρ and $T : B_\rho \to B_\rho$ be an operator. Assume that $u_1 \in B_\rho$ and $u_{n+1} = f(T, u_n)$ defines an iteration process which generate a sequence $\{u_n\}_{n=1}^{\infty} \subset D$. Suppose that $\{u_n\}_{n=1}^{\infty}$ converges to $u^* \in F_\rho(T) \neq \phi$ and $\{v_n\}_{n=1}^{\infty}$ is any bounded sequence in B_ρ. Put $\epsilon_n = \rho(v_{n+1} - f(T, v_n))$.

1. The iterative process $\{u_n\}_{n=1}^{\infty}$ defined by $u_{n+1} = f(T, u_n)$ is said to be T-stable on B_ρ if $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} v_n = u^*$.
2. The iterative process $\{u_n\}_{n=1}^{\infty}$ defined by $u_{n+1} = f(T, u_n)$ is said to be almost T-stable on B_ρ if $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} v_n = u^*$.
3. The iteration process $\{u_n\}_{n=1}^{\infty}$ defined by $u_{n+1} = f(T, u_n)$ is said to be summably almost T-stable on B_ρ if and only if $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\sum_{n=1}^{\infty} \rho(v_n - u^*) < \infty$.

Remark 5.2. Any fixed point iteration $\{u_n\}$ which is almost T-stable is also summably almost T-stable, since

$$\sum_{n=1}^{\infty} \rho(y_n - x) < \infty \Rightarrow \lim_{n\to\infty} y_n = x.$$

But converse need not be true (see Example 4.1 [15]).

Now, we show that MP iteration process (1.11) is summably almost T-stable.

Theorem 5.3. *Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition. Let $B_\rho \neq \phi$ be a ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : B_\rho \to B_\rho$ be ρ-nonexpansive mapping. Let $\{u_n\} \subset B_\rho$ be defined by the iterative process* (1.11). *Then, $\{u_n\}$ is summably almost T-stable.*

Proof. Let $\epsilon_n = \rho(u_{n+1} - f(T, u_n))$. Suppose $p^* \in F_\rho(T)$ and $\{u_n\}$ is any arbitrary sequence. Using definition of ρ-nonexpansive mapping and convexity of ρ, we have the following computations

$$\begin{aligned}
\rho(u_{n+1} - p^*) &= \rho(u_{n+1} - f(T, u_n) + f(T, u_n) - p^*) \\
&\leq \rho(u_{n+1} - f(T, u_n)) + \rho(f(T, u_n) - p^*) \\
&\leq \epsilon_n + \rho(T^2 v_n - p^*) \\
&\leq \epsilon_n + \rho(T v_n - p^*) \\
&\leq \epsilon_n + \rho(v_n - p^*)
\end{aligned}$$

$$\rho(u_{n+1} - p^*) \le \epsilon_n + \rho(v_n - p^*). \tag{5.1}$$

$$\begin{aligned}\rho(v_n - p^*) &= \rho(T((1-\alpha_n)Ts_n + \alpha_n Tw_n) - p^*) \\ &\le \rho((1-\alpha_n)Ts_n + \alpha_n Tw_n - p^*) \\ &\le (1-\alpha_n)\rho(Ts_n - p^*) + \alpha_n \rho(Tw_n - p^*) \\ &\le \rho(s_n - p^*) + \rho(w_n - p^*)\end{aligned}$$

$$\rho(v_n - p^*) \le \rho(s_n - p^*) + \rho(w_n - p^*). \tag{5.2}$$

$$\begin{aligned}\rho(w_n - p^*) &= \rho(T((1-\beta_n)s_n + \beta_n Ts_n) - p^*) \\ &\le \rho((1-\beta_n)s_n + \beta_n Ts_n - p^*) \\ &\le (1-\beta_n)\rho(s_n - p^*) + \beta_n(Ts_n - p^*) \\ &\le \rho(s_n - p^*)\end{aligned}$$

$$\rho(w_n - p^*) \le \rho(s_n - p^*). \tag{5.3}$$

$$\begin{aligned}\rho(s_n - p^*) &= \rho(T((1-\gamma_n)u_n + \gamma_n Tu_n) - p^*) \\ &\le \rho((1-\gamma_n)u_n + \gamma_n Tu_n - p^*) \\ &\le (1-\gamma_n)\rho(u_n - p^*) + \gamma_n(Tu_n - p^*) \\ &\le \rho(u_n - p^*)\end{aligned}$$

$$\rho(s_n - p^*) \le \rho(u_n - p^*). \tag{5.4}$$

Using (5.2), (5.3) and (5.4) in (5.1), we obtain

$$\rho(u_{n+1} - p^*) \le \epsilon_n + \rho(u_n - p^*). \tag{5.5}$$

By lemma 1 [4], the iteration process (1.11) is summably almost T-stable. □

Example 5.4. *Let the real number system $\mathbb{R}$ be space modulared as $\rho(u) = |u|$. It follows that ρ satisfies (UUC1) and Δ_2-conditions. Let $B_\rho = \{u \in L_\rho : 0 \le u < \infty\}$. Define $T : B_\rho \to B_\rho$ as $T = \frac{u}{4}$. Clearly, T is a ρ-nonexpansive mapping with $F_\rho(T) \ne \phi$ and B_ρ is a nonempty ρ-compact, ρ-bounded and convex subset of $L_\rho = \mathbb{R}$. Now, we show that the iteration process* (1.11) *is summably almost T-stable. Let $\{v_n\} = \left\{\frac{1}{n}\right\}$ be any bounded sequence. For convenience, taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$. Therefore,*

$$w_n = \frac{5}{8}u_n,\ v_n = \frac{25}{256}u_n,\ u_{n+1} = \frac{125}{8192}u_n.$$

$$\begin{aligned}\epsilon_n &= \rho(u_{n+1} - f(T, u_n)) \\ &= \rho\left(\frac{1}{n+1} - \frac{125}{8192}\cdot\frac{1}{n}\right) \\ &= \left|\frac{1}{n+1} - \frac{125}{8192}\cdot\frac{1}{n}\right| \\ &= \frac{1}{n+1} - \frac{125}{8192}\cdot\frac{1}{n} < \frac{1}{n}\end{aligned}$$

Hence, $\lim_{n\to\infty}\epsilon_n = 0$. *Therefore, the iterative process* (1.11) *is summably almost T-stable.*

6. Conclusion

We have proved some fixed point convergence results for ρ-nonexpansive mappings using MP iteration in modular function spaces and provided two numerical examples to empathize the validity of our results. With the help of some graphical representations (see figures 4.1 and 4.2), we can conclude that MP iterative process converges faster than that of some well known iterative processes. Summably almost T-stability of MP iteration is also discussed with a supporting example. We may suggest to the readers that one can prove some fixed point convergence results for generalized nonexpansive mappings using MP iteration.

7. Declarations

Availability of data and material

Not applicable

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

All authors contributed equally and approved the final manuscript.

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