SIMILARITY ANALYSIS OF UNSTEADY THREE DIMENSIONAL BOUNDARY LAYERS OF A NON-NEWTONIAN MODEL

Muhammet YÜRÜSOY, Mehmet YILMAZ, Mehmet PAKDEMİRLİ

Celal Bayar University, Engineering Faculty, Deparment of Mechanical Engineering, Muradiye/Manisa

ABSTRACT

In this study, three dimensional, unsteady, laminar boundary layer equations of a general model of non-Newtonian fluids are treated. In this model, the shear stresses are considered to be arbitrary functions of velocity gradients. A general boundary value problem modeling the flow over a moving surface with suction or injection is considered. Using Similarity Analysis, we showed that equations admit scaling transformation for the arbitrary shear stress case. The specific forms of the stress functions where richer scaling symmetries exist are derived. We reduce the three-independent-variable partial differential system to two-independent-variable partial differential system. Using further translation symmetries of the outcoming equations, the boundary value problem is successfully reduced to an ordinary differential system.

Key Words : Non-Newtonian fluids, Boundary layer

NON-NEWTONYEN BİR MODEL'İN ÜÇ BOYUTLU SINIR TABAKASI DENKLEMLERİNİN BENZERLİK ANALİZİ

ÖZET

Bu çalışmada, non-Newtonyen akışkanların üç boyutlu, laminer sınır tabakası denklemleri incelenmiştir. Bu modelde, kayma gerilmesi hız gradyanının keyfi bir fonksiyonudur. Sınır değer probleminde hareketli yüzey ile beraber emme veya püskürtmeli yüzey üzerindeki akış ele alınmıştır. Benzerlik analizi kullanılarak, keyfi kayma gerilmesi durumunda denklemlerin ölçekleme dönüşümünü kabul ettiği gösterilmiştir. Kayma gerilmesinin özel formlarında daha zengin bir yapının mevcut olduğu yapılar türetilmiştir. Üç bağımsız kısmi diferansiyel denklem sistemi iki bağımsız denklem sistemine indirgenmiştir. Elde edilen denklemler öteleme dönüşümü kullanılarak, sınır değer problemi adi diferansiyel denklem formuna başarıyla indirgenmiştir.

Anahtar Kelimeler : Non-Newtonyen akışkanlar, Sınır tabakası

1. INTRODUCTION

We treat the unsteady boundary layer equations of a general non-Newtonian fluid model first proposed by Hansen and Na (1968). In their model, they take the shear stress as an arbitrary function of the velocity gradient. The model is a generalization of the visco-inelastic behaviour observed in several fluids including Newtonian, Power-Law, Williamson, Prandtl, Powel-Eyring, Eyring, Ellis and Reiner-Philippoff fluids. Hansen and Na (1968) presented a similarity solution for the steady two dimensional case using scaling transformation. Timol and Kalthia (1986) extended the analysis to three dimensions using scaling and spiral group transformations. Pakdemirli (1994) retreated the analysis of references Hansen and Na, (1968) and Timol and Kalthia, (1986) showed that richer similarities exits for some specific forms of the stress function. Recently, Pakdemirli et al. (1996)

used exterior differential forms to determine the general symmetries of the two dimensional steady-state equations of the model. Yürüsoy (1996) and Yürüsoy, Pakdemirli, (1996) calculated the symmetries of the unsteady two-dimensional boundary layer equations by applying Lie Group analysis.

In reference (Yürüsoy, 1996), the classical boundary layer problem, flow with suction or injection and flow over a stretching sheet cases are investigated whereas in reference (Yürüsoy and Pakdemirli, 1996) the combined effects of moving surface and suction or injection are treated. For the boundary value problems of reference (Yürüsoy, 1996), reduction for the partial differential system from three independent variables to two independent variables is possible whereas further reduction to ordinary differential equations is impossible. However, in reference (Yürüsoy and Pakdemirli 1996), it is shown that the three independent variable partial differential system corresponding to moving surface with suction or injection can be reduced to ordinary differential system by successive application of Lie Groups In this work, we treat the three dimensional unsteady boundary layer equations. The boundary value problem is the same as in Yürüsoy and Pakdemirli (1996) with a generalization to three dimensions. We showed that equations admit scaling transformation for the arbitrary shear stress case. The specific forms of the stress functions where richer scaling symmetries exist are derived. By assuming all flow quantities to be independent of z coordinate, we reduce the three-independent-variable partial differential system to two-independent-variable partial differential system. Using further translation symmetries of the outcoming equations, the boundary value problem is successfully reduced to an ordinary differential system.

2. EQUATIONS OF MOTION

The three dimensional incompressible, laminar, unsteady, boundary layer equations have the following form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial \tau_{xy}}{\partial y}$$
$$+ \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial \tau_{yz}}{\partial y}$$
$$+ \frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} \tag{3}$$

$$F(\tau_{xy}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) = 0 \tag{4}$$

$$G(\tau_{yz}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) = 0 \tag{5}$$

where the shear stresses and velocity gradients are implicitly related through the arbitrary continuous functions F and G. Note that the components of velocity gradients which are largest inside the boundary layer are taken into consideration. U and W denote the x and z components of velocities outside the boundary layer.

3. SCALING SYMMETRIES

In this section, we apply scaling transformation to equations (1)-(5). Two cases are of practical importance: 1) Arbitrary shear stress, 2) Specific forms of stresses where richer symmetries exits.
We scale all the independent and dependent variables as follows,

$$\hat{x} = \lambda^a x, \; \hat{y} = \lambda^b y, \; \hat{z} = \lambda^c z, \; \hat{t} = \lambda^d t, \; \hat{u} = \lambda^e u,$$
$$\hat{v} = \lambda^f v, \hat{w} = \lambda^g w \, \hat{\tau}_{xy} = \lambda^h \tau_{xy}, \; \hat{U} = \lambda^i U, \tag{6}$$
$$\hat{W} = \lambda^j W, \; \hat{\tau}_{yz} = \lambda^k \tau_{yz}$$

Substituting the new variables defined in (6) into (1)-(5), requiring that the new system of equations have equivalent form with the old system results in the invariance conditions

b + e - a - f = 0, c + e - a - g = 0, a-e-d=0, b-d-f = 0
c-d-g = 0, b + e-d-h = 0, e-i = 0 a + e-d-2i = 0
c + e - d - i - j = 0, b + g - d - k = 0, g - j = 0, $\qquad$ (7)
a + g - d - i - j = 0 $\quad$ c+g-d-2j = 0

$$F(\lambda^{-h}\hat{\tau}_{xy}, \lambda^{b-e}\frac{\partial \hat{u}}{\partial y}, \lambda^{b-g}\frac{\partial \hat{w}}{\partial y}) = F(\tau_{xy},$$
$$\frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) \tag{8}$$

$$G(\lambda^{-k}\hat{\tau}_{yz}, \lambda^{b-e}\frac{\partial u}{\partial y}, \lambda^{b-g}\frac{\partial w}{\partial y}) = G(\tau_{yz},$$

$$\frac{\partial u}{\partial y}, \frac{\partial w}{\partial y}) \qquad (9)$$

3. 1. Arbitrary Shear Stress

Requiring F and G to remain arbitrary under the transformation yields

$$b\text{-}e = 0, \; b\text{-}g = 0, \; h = 0, \; k = 0 \qquad (10)$$

Solving (10) and (7) together, we represent all parameters in terms of one parameter

$$a = 3b, \; c = 3b, \; d = 2b, \; e = b, \; f = -b, \; g = b, \\ h = 0 \quad i = b, \; j = b, \; k = 0 \qquad (11)$$

Before defining similarity variables and functions, to avoid three successive reductions, we assume all flow quantities to be independent of z coordinate. The equations determining the similarity variables and functions can now be written as

$$\frac{dx}{3x} = \frac{dy}{y} = \frac{dt}{2t} = \frac{du}{u} = \frac{dv}{-v} = \frac{dw}{w}$$

$$= \frac{d\tau_{xy}}{0} = \frac{dU}{U} = \frac{dW}{W} = \frac{d\tau_{yz}}{0} \qquad (12)$$

The corresponding similarity variables and functions are

$$\xi = \frac{x}{t^{3/2}}, \; \eta = \frac{y}{t^{1/2}}, \; u = t^{1/2}P(\xi,\eta),$$

$$v = t^{-1/2}Q(\xi,\eta) \; w = t^{1/2}R(\xi,\eta), \qquad (13)$$

$$U = t^{1/2}U(\xi), \; W = t^{1/2}W(\xi)$$

τ_{xy} and τ_{yz} are invariants by themselves.

We now impose the boundary conditions for the equations of motion

$$u(x,0,t) = A(x,t), \; v(x,0,t) = \pm V(x,t) \; w \\ (x,0,t) = B(x,t), \; u(x,\infty,t) = U(x,t) \; w(x,\infty,t) \\ = W(x.t) \qquad (14)$$

The boundary conditions imply that the surface is moving and there is suction or injection through the surface. Requiring that the functions A(x,t), V(x,t) and B(x,t) possess scaling properties yield.

$$A(x,t) = t^{1/2}A(\xi), \; V(x,t) = t^{-1/2}V(\xi), \\ B(x,t) = t^{1/2}B(\xi) \qquad (15)$$

Substituting the new variables (13) and (15) into (1)-(5) and (14) and remembering that all flow quantities are independent of z coordinate, we finally obtain a partial differential system with two independent variables

$$P_\xi + Q_\eta = 0 \qquad (16)$$

$$\frac{1}{2}P - \frac{3}{2}\xi P_\xi - \frac{1}{2}\eta P_\eta + PP_\xi + QP_\eta = (\tau_{xy})_\eta$$

$$+ \frac{1}{2}U - \frac{3}{2}\xi U_\xi + UU_\xi \qquad (17)$$

$$\frac{1}{2}R - \frac{3}{2}\xi R_\xi - \frac{1}{2}\eta R_\eta + PR_\xi + QR_\eta =$$

$$(\tau_{yz})_\eta + \frac{1}{2}W - \frac{3}{2}\xi W_\xi + UW_\xi \qquad (18)$$

$$F(\tau_{xy}, P_\eta, R_\eta) = 0 \qquad (19)$$

$$G(\tau_{yz}, P_\eta, R_\eta) = 0 \qquad (20)$$

P (ξ,0) = A (ξ), Q (ξ,0) = ± V (ξ), R (ξ,0) = B (ξ), P (ξ,∞) = U (ξ), R (ξ,∞) = W (ξ) (21)

3. 2. Specific Forms of Stresses

If we require that F and G functions in (8) and (9) possess scaling properties, then we obtain some special forms of the functions. For those special forms, obviously the scaling symmetries would be enriched. To manage this, we differentiate (8) and (9) with respect to λ, return to original variables, solve the outcoming first order partial differential system, solve (7) and finally obtain

$$a = d + e, \; b = d + f, \; c = d + g, \; h = f+e, \; i = \\ e, \quad j = g, \; k = f + g \qquad (22)$$

$$F = F(\alpha_1, \alpha_2), \; G = G(\alpha_3, \alpha_4) \qquad (23)$$

$$\alpha_1 = \frac{\tau_{xy}}{(\frac{\partial u}{\partial y})^{\frac{f+e}{e-d-f}}}, \; \alpha_2 = \frac{\tau_{xy}}{(\frac{\partial w}{\partial y})^{\frac{f+e}{g-d-f}}}, \; \alpha_3$$

$$= \frac{\tau_{yz}}{(\frac{\partial u}{\partial y})^{\frac{f+g}{e-d-f}}}, \; \alpha_4 = \frac{\tau_{yz}}{(\frac{\partial w}{\partial y})^{\frac{f+g}{g-d-f}}} \qquad (24)$$

Note that for the specific forms of F and G defined in (23) and (24) we have four parameter Lie Group scaling transformations as seen in (22) whereas for arbitrary F and G, we only have one parameter Lie

Group scaling transformation (see eq. (11)). Therefore symmetries are richer in this case compared to the arbitrary shear stress case. Note that Newtonian and Power-Law fluids obey the general form given in (23) and (24). Applications of Newtonian and Power-Law fluids for the symmetries given in (22)-(24) would be similar to those given in reference (Pakdemirli, 1994) with the exception that shear stresses are explicit functions of velocity gradients in Pakdemirli (1994) whereas in our case, they are implicit functions.

4. TRANSLATION SYMMETRIES OF REDUCED SYSTEM

In this section, we treat equations (16)-(21) obtained for the arbitrary shear stress case. From the results given in reference (Yürüsoy and Pakdemirli 1996), we expect the reduced system (16)-(21) to possess translation symmetries only. We therefore write

$$\hat{\xi} = \xi + \varepsilon a, \hat{\eta} = \eta + \varepsilon b, \hat{P} = P + \varepsilon c,$$
$$\hat{Q} = Q + \varepsilon d, \hat{R} = R + \varepsilon e, \hat{\tau}_{xy} = \tau_{xy} + \varepsilon f$$
$$\hat{U} = U + \varepsilon g, \hat{W} = W + \varepsilon h, \hat{\tau}_{yz} = \tau_{yz} + \varepsilon i, \quad (25)$$
$$\hat{A} = A + \varepsilon j, \hat{V} = V + \varepsilon k, \hat{B} = B + \varepsilon l$$

Substituting (25) into (16)-(21) and remembering that F and G are arbitrary, we obtain the following invariance conditions

$$g = c, c = \frac{3}{2}a, d = \frac{b}{2}, g = \frac{3}{2}a, h = e,$$
$$f = 0, i = 0, b = 0, j = c, k = d, l = e \quad (26)$$

The above equations can be represented in terms of two arbitrary parameters m and p as follows

$$a = 2m, b = 0, c = 3m, d = 0, e = p, f = 0,$$
$$g = 3m, h = p, I = 0, j = 3m, k = 0, l = p \quad (27)$$

Choosing m = 1 and p = 3, we write the differential system for the similarity variables and functions

$$\frac{d\xi}{2} = \frac{d\eta}{0} = \frac{dP}{3} = \frac{dQ}{0} = \frac{dR}{3} = \frac{d\tau_{xy}}{0} =$$
$$\frac{dU}{3} = \frac{dW}{3} = \frac{d\tau_{yz}}{0} = \frac{dA}{3} = \frac{dV}{0} = \frac{dB}{3}$$
(28)

Solving (28), we find

$$\mu = \eta, P = \frac{3}{2}\xi + L(\mu), Q = M(\mu),$$
$$R = \frac{3}{2}\xi + N(\mu), U = \frac{3}{2}\xi + C_1$$
$$W = \frac{3}{2}\xi + C_2, A = \frac{3}{2}\xi + C_3, V = C_4, \quad (29)$$
$$B = \frac{3}{2}\xi + C_5$$

where τ_{xy} and τ_{yz} are absolute invariants again. Substituting (29) into (16)-(21), we have

$$M' + \frac{3}{2} = 0 \quad 30)$$

$$2L + L'(M - \frac{1}{2}\mu) = (\tau_{xy})_\mu + 2C_1 \quad (31)$$

$$\frac{1}{2}N + \frac{3}{2}L + N'(M - \frac{1}{2}\mu) = (\tau_{yz})_\mu$$
$$+ \frac{1}{2}C_2 + \frac{3}{2}C_1 \quad (32)$$

$$F(\tau_{xy}, L', N') = 0 \quad (33)$$

$$G(\tau_{yz}, L', N') = 0 \quad (34)$$

$$L(0) = C_3, M(0) = \pm C_4, N(0) = C_5, L(\infty) = C_1 \quad N(\infty) = C_2 \quad (35)$$

We therefore succesfully reduced the partial differential system of three independent variables to an ordinary differential system by applying first scaling and then translation symmetries. A closed form solution of ordinary differential system (30)-(35) cannot be achieved unless we specify F and G. Even for specific F and G, however, a numerical treatment of equations might be inevitable.

5. REFERENCES

Hansen, A. G. and Na, T. Y. 1968. Similarity Solutions of Laminar, Inompresible Boundary Layer Equations of Non-Newtonian ASME J. Basic Engng., 90, 71.

Pakdemirli, M. 1994. Similarity Analysis of Boundary Layer Equations of a Class of Non-Newtonian Fluids. Int. J. Non-Linear Mec. 29, 187.

Pakdemirli, M., Yürüsoy M. and Küçükbursa, A. 1996. Symmetry Groups of Boundary Layer Equations of a Class of Non-Newtonian Fluids. Int. J. Non-Linear Mec. 31, 267.

Timol M. G. and Kalthia, N. L. 1986. Similarity Solutions of Three-Dimensional Doundary Layer Equations of Non-Newtonian Fluids. Int. J. Non-Linear. Mec., 21, 475.

Yürüsoy, M. 1996. Lie Group Analysis of Unsteady Boundary Layer Equations of Non-Newtonian Fluids, M. S. Thesis, Celal Bayar University, (in Turkish).

Yürüsoy, M. and Pakdemirli, M. 1996. Group Theoretic Approach to Unstaedy Boundary Layer Equations of Some Non-Newtonian Fluids, Modern Group Analysis VI Symposium, 15-20 January, Johannesburg, South Africa.