ON THE ELASTIC-PLASTIC TORSION OF A BAR MADE OF WORK HARDENING MATERIAL

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ABSTRACT

In this study, we reconsider the problem of an elastic-plastic torsion of a bar made of work hardening material. Nonlinear partial differential equation derived is reduced to a well known Laplace equation by means of transformation functions and stresses τ_{xz}, τ_{yz} and the torque T are analytically found for elliptical and circular cross sections. It is further shown that the stresses and the twisting moment do not depend on the value of n in the stress-strain law.

Key Words: Torsion, Non-linear Viscous, Noncircular, Shafts.

UZAMA SERTLEŞMELİ MALZEMEDEN YAPILMIŞ BİR ŞAFTIN ELASTİK-PLASTİK BURULMA PROBLEMİ ÜZERİNE ANALİTİK BİR İNCELEME

ÖZET

Bu çalışmada, uzama serleşmeli malzemeden yapılmış şaftların elastik-plastik burulma problemini ele alınmıştır. Problemin çözümünde ortaya çıkan ve şu ana kadar analitik çözümü bilinmeyen non-lineer kısmi diferensiyel denklem, dönüşüm fonksiyonları yardımıyla çözümleri çok iyi bilinen Laplace diferensiyel denklemine indirgenmekte ve τ_{xy}, τ_{xz} gerilmeleri ile T torku eliptik ve dairesel kesitler için bulunmaktadır. Ayrıca, gerilmeler ile burulma momentinin gerilme uzama baðıntısındaki n sabitine baðlı olmadıðı da ispat edilmektedir.

Anahtar Kelimeler: Nonlineer, Viskos, Burulma, Miller

1. INTRODUCTION

Analytical expressions have many advantages compared to numerical technics because of the easiness of comparison of numerical results with the experimental results and every day use. By their natures, it is usually possible to give analytical results for linear differential equations arising in the formulation of physical events based on many simplifications. But, if the more physical quantities are considered or some other nonlinear effects are included in the theory, then the resulting differential equation becomes nonlinear and therefore unsolvable in many cases (Hodge and Prager, 1951, Shames, 1992). Such differential equations also occur in elastic and plastic analysis of structures. The usual procedures in these cases have been to develop numerical technics for possible solution of the equations. But, in some cases, there may be a way for determining related quantities in an analytical way, and this method can well be extended to include the other problems resulting in nonlinear equations (Pala, 1994). It is therefore the objective of this paper to develop an analytical, but rather simple method when possible for the solution of equations arising in the mathematical formulation of the problem of a elastic-plastic torsion of a bar made of work hardening material. We remind that it is possible to develop numerical methods for the solution of the nonlinear partial differential which is mentioned below (Chakrabarty, 1987, Mendelson, 1968).

2. ANALYSIS

Let us consider a uniform shaft having an arbitrary cross-section and subjected to a torque T (see, Fig.1). The location of axes x, y, z is chosen at the end section of the

bar, the z axis being taken parallel to its generator. But, for convenience, the axes x, y are chosen as principal axes.

With the assumptions made by Saint-Venant (Shames, 1992), the displacements in the x and y directions for small deformation are given by the same equations as were developed for linear elastic shafts.

To analyze the problem of elastic-plastic torsion of a bar made of work hardening material, it is convenient to employ a stress-strain relation that corresponds to no well-defined yield point. The problem can then be simplified by the absence of an elastic-plastic boundary, which permits the same equations in linear elastic torsion problem to apply throughout the cross section (Chakrabarty, 1987). In the derivation of the governing equation, we shall use the Ramberg-Osgood equation (Hodge and Prager, 1951)
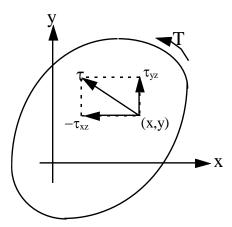

Figure 1. Cross Section of a Bar Made of Work Hardening Material

$$\varepsilon = \frac{\sigma}{E}\left\{1 + m\left(\frac{\sigma}{\sqrt{3}k}\right)^{2n}\right\} \tag{1.1}$$

for the uniaxial stress-strain curve, where m and n are dimensionless constants, and k a nominal yield stress in simple or pure shear, the slope of the stress-strain curve being equal to E when $\sigma=0$.

Let us now consider the formulation of the torsion problem by using the Hencky stress-strain relations (Chakrabarty, 1987). Since the velocity field in the twisted bar (Chakrabarty, 1987) is given by the partial derivative of

$$u = -\theta yz, \quad v = \theta xz , \quad w = w(x,y,\theta) \tag{1.2}$$

with respect to θ, which is taken as the time scale, the components of the shear-strain increment are expressed as

$$2d\,\gamma_{xz} = \left(\frac{\partial^2 w}{\partial x \partial \theta} - y\right)d\theta, \quad 2d\,\gamma_{yz} = \left(\frac{\partial^2 w}{\partial y \partial \theta} + x\right)d\theta \tag{1.3}$$

The elimination of w from these equations leads to the strain compatibility equation

$$\frac{\partial}{\partial x}\left(d\,\gamma_{yz}\right) - \frac{\partial}{\partial y}\left(d\,\gamma_{xz}\right) = d\theta \tag{1.4}$$

If we use Hencky stress-strain relations, which may be written as (Ckakrabarty, 1987).

$$2G\gamma_{xz} = k(1+\lambda)\frac{\partial\phi}{\partial y},$$
$$2G\gamma_{yz} = -k(1+\lambda)\frac{\partial\phi}{\partial x} \tag{1.5}$$

in the case of monotonic loading, we have from Eq.(1.4) that

$$\frac{\partial}{\partial x}\left[(1+\lambda)\frac{\partial\phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[(1+\lambda)\right]$$
$$= \frac{-2G\theta}{k} \tag{1.6}$$

with the boundary condition ϕ=constant on the boundary curve of the cross section (Chakrabarty, 1987), where ϕ is the stress function and λ is a positive quantity given by

$$\lambda = \frac{3Gm}{E}\left(\frac{\tau}{k}\right)^{2n}$$
$$= \frac{3m}{2(1+\nu)}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}^n \tag{1.7}$$

in which τ and ϕ are the resultant shear stress and stress function, respectively.

Since Eq.(1.4) also holds in the plastic range with τ_{xz} and τ_{yz} replaced by $2G\gamma_{xz}$ and $2G\gamma_{yz}$, respectively (Chakrabarty, 1987), we can write the non-zero stresses as

$$\tau_{xz} = k(1+\lambda)\phi_y, \quad \tau_{yz} = -k(1+\lambda)\phi_x \tag{1.8}$$

Although this formulation seems to be identical to the linear elastic torsion problem, stresses and the governing Eq.(1.6) are given in completely different forms.

3. SOLUTION

It is almost impossible to find the open form of ϕ satisfying both the equation (1.6) and the boundary condition ϕ=0. Instead of that, we will try to find the shear stresses in terms of the partial derivatives of a special function ψ which depends on ϕ.

Let us make use of the transformations

$$\psi_x = (1+\lambda)\phi_x \ , \ \psi_y = (1+\lambda)\phi_y \tag{1.9}$$

from which we can write

$$\frac{\psi_x}{\psi_y} = \frac{\phi_x}{\phi_y} \tag{1.10}$$

where ψ_x and ψ_y are the partial derivatives of the function ψ with respect to x and y, respectively. Then, Eq.(1.6) reduces to well known Laplace equation

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0 \tag{1.11}$$

after making a second transformation in the form of $\psi = \psi_1 - (A/4)(x^2+y^2)$, where $A=2G\theta/k$.

Now, on the other hand, considering the boundary condition ϕ=constant, we can also write $\phi_x/\phi_y = -dy/dx$ on the boundary curve. Employing equation (1.10), we have

$$\frac{\psi_x}{\psi_y} = \frac{\phi_x}{\phi_y} = -\frac{dy}{dx} \tag{1.12}$$

Shear stresses τ_{xz} and τ_{yz} and the twisting moment T (Chakrabarty, 1987) can be written in terms of ψ_x and ψ_y without using ϕ_x and ϕ_y since it is not possible to find an expression for ϕ_x and ϕ_y because of the nonlinear relation between ψ_x, ψ_y and ϕ_x, ϕ_y (Chakrabarty, 1987):

$$\tau_{xz} = k\,\psi_y, \quad \tau_{yz} = -k\,\psi_x$$
$$T = \iint (x\,\tau_{yz} - y\,\tau_{xz})dxdy \tag{1.13}$$
$$= -k\iint (x\,\psi_x + y\,\psi_y)dxdy$$

Our aim is to find the functions ψ_x and ψ_y both satisfying Eqs.(1.6) and (1.12).

Elliptical and Circular Cross Sections:

We assume that the boundary condition of the cross section is an ellipse, whose equation is given by $b^2x^2+a^2y^2=a^2b^2$ where a and b are semiaxes. Then, we have for the slope of the curve that $-dy/dx=(b^2/a^2)(x/y)$. Substituting this expression into (1.12), we have

$$\frac{\psi_x}{\psi_y} = \frac{b^2 x}{a^2 y} \tag{1.14}$$

On the other hand, since the function ψ_1 satisfies Laplace equation, it can be written as the real parts of the complex function $z=(x+iy)^\beta$, which will be found for the integer

values of β. Among these solutions, the unique one which is compatible with the boundary condition (1.10) is x^2-y^2. Therefore, ψ must be taken in the form of

$$\psi = \alpha(x^2 - y^2) - \frac{A}{4}(x^2 + y^2) \tag{1.15}$$

from which we find that

$$\psi_x = \left(2\alpha - \frac{A}{2}\right)x, \ \ \psi_y = -\left(2\alpha + \frac{A}{2}\right)y \tag{1.16}$$

Substituting Eq.(1.16) in Eq.(1.14) and finding m out, we obtain

$$\alpha = \frac{A}{4}\frac{a^2 - b^2}{a^2 + b^2} \tag{1.17}$$

and

$$\psi = \frac{A}{4}\left[\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) - (x^2 + y^2)\right] \tag{1.18}$$

Then, Eqs.(1.16) give

$$\psi_x = \frac{-A b^2}{a^2 + b^2} x, \ \ \psi_y = \frac{A a^2}{a^2 + b^2} y \tag{1.19}$$

The stresses τ_{xz}, τ_{yz} and the torque T are given by

$$\tau_{xz} = \frac{kA a^2}{a^2 + b^2} y, \ \ \tau_{yz} = \frac{kA b^2}{a^2 + b^2} x$$
$$T = \frac{\pi kA a^3 b^3}{2(a^2 + b^2)} \tag{1.20}$$

Using Eqs.(1.5), strain components γ_{xz} and γ_{yz} are obtained as

$$\gamma_{xz} = \frac{k}{2G}\frac{-A a^2}{a^2 + b^2} y \ ,$$
$$\gamma_{yz} = \frac{k}{2G}\frac{A b^2}{a^2 + b^2} x \tag{1.21}$$

It is observed from Eqs.(1.20) that the elastic-plastic boundary on which $\tau_{xz}^2 + \tau_{yz}^2 = k^2$, where k is the yield stress, is again an ellipse for an elliptical cross section. For circular cross section where a=b, elastic-plastic boundary is circle, as is clear from Eqs.(1.20).

4. RESULTS AND CONCLUSIONS

It has been shown here that, in a bar made of work hardening material, it is possible to solve the governing equation and to find the stress distribution for the elliptical and circular cross sections in an analytical way. One important point that has been observed in this analysis is that stresses τ_{xz}, τ_{yz} and the torque T do not depend on the constant n in the stress strain law (see, Eq.(1.3)) for a work hardening material. These results are also supported by numerical technics (Chakrabarty, 1987). Indeed, it is seen in the case of square cross section that the twisting moment gives very near values for the values between 1 and 9 of n (Chakrabarty, 1987). However, It must be reminded here that it is not possible to find the value of λ in an analytical way (Chakrabarty, 1987, Hodge and Prager, 1951, Prager, 1947). It is also beneficial to say a few words about square cross section. Since, by Eq.(1.10), the function ψ is also constant on the surface, the conditions in this problem are identical with those in the torsion problem of linear elastic shafts and therefore ϕ_x and ϕ_y in there can directly be taken as ψ_x and ψ_y in this problem (Shames, 1992, Pala, 1994).

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