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Solutions of Neutral Differential Inclusions

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Abstract

Motivated by the study of neutral differential inclusions, we establish a new fixed point theorem for multivalued countably Meir-Keeler condensing mappings via an arbitrary measure of weak noncompactness which in turn include the fixed point theorems of Krasnoselskii and Dhage as special cases in non separable spaces.

Keywords: Meir-Keeler condensing operators measure of weak noncompactness neutral differential inclusions .
2010 MSC: 47H10, 47H08, 47H09, 47H04.

1. Introduction

Functional integral equations and inclusions of different types play an important and fascinating role in nonlinear analysis and finding various applications in describing of several scientific problems such as transport theory, the theory of radiative transfer, biomathematics, etc (see [16], [21], [23], [26]). Neutral functional differential equations is an important topic of functional differential equations and an exhaustive treatment may be found in Hale [24] and Ntouyas [29]. However, the study of neutral differential inclusions is relatively recent and the study of this problem will definitely contribute immensely to the area of functional differential inclusions. In 1988, Dhage ([19]) initiated the study of nonlinear integral equations in a Banach algebra via fixed point techniques for the strong topology. In the lack of local compactness in infinite dimensional Banach algebras, weak topology is the best environment to investigate the existence of solutions of nonlinear integral equations and nonlinear differential equations. So in recent years, many papers have been devoted to the existence of fixed points for mappings acting on a Banach algebra equipped with its weak topology see ([3], [7], [8], [10]). Some versions of the Krasnoselskii fixed point theorems in the framework of

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the weak topologies and involving multi-valued mappings with weakly sequentially closed graph have been recently stated in [31]. The fixed point theory for multivalued mappings with weakly sequentially closed graph take an important role for the existence of solutions for operator inclusion (see [2],[13],[30]) and others. In 2015, Aghajani and Mursaleen [1] introduced the definition of Meir-Keeler condensing operator and proved a theorem that guarantees the existence of a fixed point for single valued mappings. In [4], the authors introduce the concept of Meir-Keeler condensing operator in a Banach space via an arbitrary measure of weak noncompactness and prove some generalizations of Darbo's fixed point theorem by considering a measure of weak noncompactness which not necessary has the maximum property. They prove some coupled fixed point theorems and they apply them in order to establish the existence of weak solutions for a system of functional integral equations of Volterra type.
In this paper, we introduce the concept of Meir-Keeler condensing multivalued mappings via an arbitrary measure of weak noncompactness and we show that this condition can be relaxed by assuming that it holds only for countable bounded sets. We also establish new fixed point theorems for multivalued operator of type $AB + C$ in a Banach algebra.
Finally, in order to indicate their applicability we study the existence of solutions for a neutral differential inclusion in Banach algebra which a generalization of the work of B.C. Dhage in [17] and the references therein in the weak topology setting.

2. Preliminaries

Let X be a Banach space endowed with the norm $\|.\|$ and with the zero element θ. We denote by

$$\mathcal{P}(X) = \{D \subset X : D \textit{ is nonempty}\},$$

$$\mathcal{P}_{bd}(X) = \{D \subset X : D \textit{ is nonempty and bounded}\},$$

$$\mathcal{P}_{cv}(X) = \{D \subset X : D \textit{ is nonempty and convex}\}$$

$$\mathcal{P}_{cl,bd}(X) = \{D \subset X : D \textit{ is nonempty, closed and bounded}\},$$

Definition 2.1. *Let S be a nonempty subset of X. Let $T : X \to P(X)$ be a multivalued mapping. T is said to have weakly sequentially closed graph if for every sequence $\{x_n\} \subset S$ with $x_n \rightharpoonup x$ in S and for every sequence $\{y_n\}$ with $y_n \in T(x_n)$, for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$ in X implies $y \in T(x)$.*

Definition 2.2. *Let X be a Banach space. An operator $T : X \to X$ is said to be weakly sequentially continuous on X if for every sequence $\{x_n\}_n$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$.*

Note that T is weakly sequentially continuous if and only if $I - T$ is weakly sequentially continuous.

Definition 2.3. *We say that $T : S \to P(X)$ is weakly completely continuous if, T has a weakly sequentially closed graph, and for any bounded subset Ω of S, $T(\Omega)$ is a relatively weakly compact subset of X.*
We say that T is sequentially weakly upper semicompact in S (s.w.u.sco., for short) if for any weakly convergent sequence $\{x_n\} \subset S$ and arbitrary $y_n \in T(x_n)$, the sequence $\{y_n\}$ has a weakly convergent subsequence in X.
If T is a single-valued mapping, T is sequentially weakly upper semicompact if for any weakly convergent sequence $\{x_n\}$ in S, the sequence $\{Tx_n\}$ has a weakly convergent subsequence in X.

For any $\Omega \in \mathcal{P}_{bd}(X)$, let

$$\|\Omega\| = sup\{\|a\|, a \in \Omega\}.$$

For $\Omega_1,\ \Omega_2 \in \mathcal{P}_{cl,bd}(X)$ and $a \in \Omega_1$, let

$$D(a, \Omega_2) = inf\{\|a - b\|, b \in \Omega_2\}$$

$$\rho(\Omega_1, \Omega_2) = sup\{D(a, \Omega_2), a \in \Omega_1\}.$$

The function $d_H : \mathcal{P}_{cl,bd} \times \mathcal{P}_{cl,bd}(X) \to \mathbb{R}^+$ defined by

$$d_H(\Omega_1, \Omega_2) = max\{\rho(\Omega_1, \Omega_2), \rho(\Omega_2, \Omega_1)\}$$

is a metric on $\mathcal{P}_{cl,bd}$ and is called the Hausdorff metric on X (see [22]).
It is clear that $H(\theta, \Omega) = \|\Omega\|$ for any $\Omega \in P_{cl,bd}(X)$.

Proposition 2.4. *(See[22]) If $\Omega_1, \Omega_2 \in \mathcal{P}_{cl,bd}(X)$, then $d_H(\Omega_1, \Omega_2) \leq r$ is equivalent to:*

$$\Omega_1 \subset \Omega_2 + B_r(\theta) \text{ and } \Omega_2 \subset \Omega_1 + B_r(\theta).$$

Definition 2.5.

- *A multivalued mapping $T : S \to \mathcal{P}_{cl,bd}(X)$ is called D-Lipschitzian if there exists a continuous non decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ such that*

$$d_H(T(x), T(y)) \leq \phi(\|x - y\|)$$

for all $x, y \in S$. The function ϕ is called a D-function of T on X.

- *T is called a Lipschitzian multivalued mapping if $\phi(r) = Kr$ for $r > 0$. In particular if $K < 1$, then T is called a multivalued contraction. If ϕ satisfies $\phi(r) < r$, then T is called a nonlinear contraction multivalued mapping with contraction function ϕ.*

- *T is called countably D-Lipschitzian if there exists a continuous non decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ and there exists a countably set S such that*

$$d_H(T(x), T(y)) \leq \phi(\|x - y\|)$$

for all $x, y \in S$.

We recall that a function $\omega : \mathcal{P}_{bd}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness(MWNC, for short) on X if it satisfies the following properties.

(1) For any bounded subset Ω_1, Ω_2 of X, we have $\Omega_1 \subseteq \Omega_2$ implies $\omega(\Omega_1) \leq \omega(\Omega_2)$.

(2) $\omega(\overline{conv}(\Omega)) = \omega(\Omega)$, for all bounded subsets $\Omega \subset X$.

(3) $\omega(\Omega \cup \{a\}) = \omega(\Omega)$ for all $a \in X$,$\Omega \in \mathcal{P}_{bd}(X)$.

(4) $\omega(\Omega) = 0$ if and only if Ω is relatively weakly compact in X.

(5) If $(X_n)_{n\geq 1}$ is a decreasing sequence of nonempty bounded and weakly closed subsets of X with $\lim_{n\to+\infty} \omega(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n$ is non empty and $\omega(\cap_{n=1}^{\infty} X_n) = 0$.

The MWNC ω is said to be:

(i) positive homogeneous, if $\omega(\lambda\Omega) = \lambda\omega(\Omega)$, for all $\lambda > 0$ and $\Omega \in P_{bd}(X)$.

(ii) Subadditive, if $\omega(\Omega_1 + \Omega_2) \leq \omega(\Omega_1) + \omega(\Omega_2)$, for all $\Omega_1, \Omega_2 \in P_{bd}(X)$.

As an example of MWNC, we have the De Blasi measure of weak noncompactness [12], defined on $\mathcal{P}_{bd}(X)$ by:

$$\beta(M) = inf\{\varepsilon > 0; \text{ there exists } K \text{ weakly compact such that } M \subset K + B_\varepsilon\},$$

it is well known that β is homogeneous, subadditive, and satisfies the set additivity property:

$$\beta(M \cup N) = max\{\beta(M), \beta(N)\}, \quad \text{for all } M, N \in \mathcal{P}_{bd}(X).$$

For more properties of the MWNC, we refer to [12].
Using the abstract MWNC introduced above, we can formulate some other definitions needed in this paper.

Definition 2.6. *Let S be a subset of a Banach space X, ω be a MWNC on X, and $0 \leq K < 1$. Let a multivalued map $T : S \to \mathcal{P}(X)$. We say that:*

- *T is D-set-Lipshitzian if there exists a continuous non decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ such that*
$$w(T(\Omega)) \leq \phi(\omega(\Omega))$$
for all bounded sets $\Omega \subset X$ such that $T(\Omega) \in \mathcal{P}_{bd}(X)$.
- *If $\phi(r) = kr$, we say that T is k-set-Lipschitzian.*
- *If $k < 1$, then T is called k-set-contraction.*
- *if $\phi(r) < r$ for $r > 0$, then T is called nonlinear D-set-contraction.*
- *T is called countably D-set-Lipchitzian if*
  - *$T(S)$ is bounded.*
  - *$\omega(T(B)) \leq \phi(\omega(B))$ for any countable bounded subset B of S with $\omega(B) > 0$.*
- *T is called ω-condensing if T is bounded and $\omega(T(\Omega)) < \omega(\Omega)$ for all $\Omega \in \mathcal{P}_{bd}(S)$with $\omega(\Omega) \neq 0$.*
- *T is called countably ω-condensing if*
  - *$T(S)$ is bounded.*
  - *$\omega(T(B)) \leq \omega(B)$ for any countable bounded subset B of S with $\omega(B) > 0$.*

Definition 2.7. *Let S be a nonempty subset of a Banach space X. A multivalued map $T : S \to \mathcal{P}(X)$ is called countably Meir-keeler condensing if*

(1) $T(S)$ is bounded.

(2) For all $x \in S, T(x) \cap S \neq \emptyset$.

(3) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \omega(A) < \varepsilon + \delta \Rightarrow \omega(T(A) \cap S) < \varepsilon \tag{2.1}$$

for all countably, bounded subset A of S.

Remark 2.8. *The countably condensing multivalued mapping of Meir-keeler type are more general than countably condensing mappings. Indeed, let T be a countably condensing multivalued mapping, that is,*

$$\omega(T(A)) \leq k\omega(A) \text{ for any countable bounded subset } A \subset S$$

with $0 \leq k < 1$. Suppose for $\delta = (\frac{1}{k} - 1)\varepsilon$ that we have

$$\varepsilon \leq \omega(A) < \varepsilon + \delta$$

then

$$\omega(T(A) \cap S) < k\frac{\varepsilon}{k} = \varepsilon.$$

Thus, T is a countably Meir-keeler condensing multivalued mapping.

Theorem 2.9. *[30] Let X be a metrizable locally convex linear topological space and let Ω be a weakly compact, convex subset of X. Suppose $F : \Omega \to \mathcal{P}_{cl,cv}(\Omega)$ has weakly sequentially closed graph, then F has a fixed point.*

We recall that an algebra X is a vector space endowed with an internal composition law denoted by (.) which is associative and bilinear. A normed algebra is an algebra endowed with a norm $\|.\|$ such that

$$\|xy\| \leq \|x\|\|y\|, \text{ for all } x, y \in X.$$

A complete normed algebra is called a Banach algebra. For basic properties of Banach algebra, refer to [32]. In general, the product of two weakly sequentially continuous mappings on a Banach algebra is not necessarily weakly sequentially continuous.
We say that the Banach algebra X satisfies condition (P) if the operation of multiplication $(x, y) \to x.y$ is sequentially weakly continuous: (P) if $x_n, y_n \subset X$ such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then $x_n.y_n \rightharpoonup x.y$.
Note that every finite-dimensional Banach algebra satisfies condition (P).
We have from Dobrokov's Theorem that if X satisfies condition (P) and K is a Hausdorff compact space, then $C(K, X)$ is also a Banach algebra satisfying condition (P).

Theorem 2.10. *(see [25]) Let S be a Hausdorff compact space and X a Banach space. A bounded sequence $(f_n) \subset C(S, X)$ converges weakly to $f \in C(S, X)$ if and only if, for every $t \in S$, the sequence $(f_n(t))$ converges weakly in X to $f(t)$.*

The following Lemma is useful for the sequel.

Lemma 2.11. *(See[3]). Let X be a Banach algebra with condition (P). Then for any bounded subsets Ω_1 and Ω_2 of X, we have*

$$w(\Omega_1\Omega_2) \leq \|\Omega_1\|w(\Omega_2) + \|\Omega_2\|w(\Omega_1) + w(\Omega_2)w(\Omega_1).$$

In the special case when Ω_2 is weakly compact, we obtain the following useful Lemma.

Lemma 2.12. *Let X be a Banach algebra with condition (P). Then for any bounded subset Ω_1 of X and weakly compact subset Ω_2 of X, we have*

$$w(\Omega_1\Omega_2) \leq \|\Omega_2\|w(\Omega_1).$$

3. Fixed Point Theory

Theorem 3.1. *Let S be a nonempty, closed and convex subset of a Banach space E and ω be an arbitrary measure of weak noncompactness on E. If $F : S \to \mathcal{P}_{cl,cv}(E)$ is a multivalued mapping satisfying the following properties:*

1. *F has weakly sequentially closed graph,*
2. *F is countably Meir-keeler condensing mapping with respect to ω,*

then F has at least one fixed point.

Proof. Define the sequence $\{S_n\}$ of subsets of S by $S_0 = S$ and $S_n = conv(FS_{n-1} \cap S), n \geq 1$.
Let $n \geq 0$ be fixed and let

$$a_n = \sup\{\omega(K); K \text{ is a countable subset of } S_n\}.$$

Now let K_i^n be a sequence of countable subsets of S_n with $\omega(K_i^n) \to a_n$ as $i \mapsto \infty$. Let $K^n = \cup_{i\geq 1} K_i^n$, and since K^n is a countable subset of S_n, we obtain

$$a_n \geq \omega(K^n) \geq \omega(K_i^n) \to a_n.$$

Then $\omega(K^n) = a_n$. Let $x \in K^n$. There exist $p_x \in \mathbb{N}^*$ and $z_1, \cdots, z_{p_x} \in (FS_{n-1} \cap S)$ such that $x = \sum_{k=1}^{p_x} \lambda_k z_k$ with $\lambda_k \geq 0, \forall k \in [1, p_x]$ and $\sum_{k=1}^{p_x} \lambda_k = 1$. For every $k \in [1, p_x], z_k = F(b_k) \in S$ with $b_k \in S_{n-1}$. Let $\mathcal{M}_x = \{b_k \in S_{n-1}; k \in [1, p_x]\}$, and $\mathcal{M}_n = \cup_{x\in K^n} \mathcal{M}_x$. Since K^n is a countable subset of S_n, we have $\mathcal{M}_n$

is a countable subset of S_{n-1} and then $\mathcal{M}_n \subset K^{n-1}$. Note $x = \sum_{k=1}^{p_x} \lambda_k F(b_k) \in conv(F\mathcal{M}_n \cap S)$. Then, $K^n \subset conv(F\mathcal{M}_n \cap S)$. Note since $F(S)$ is bounded, then so also are the sets $S_n, \mathcal{M}_n$ and K^n. Note that $\{a_n\}_{n\geq 1}$ is a positive decreasing sequence of real numbers. We pose $a = \lim_{n\mapsto +\infty} a_n$. If $a > 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$a \leq \omega(K^{n_0}) < a + \delta(a)$$

where $\delta(a)$ is chosen according to (2.1). By the definition of a_n, we have

$$a_{n_0+1} = \omega(K^{n_0+1}) \leq \omega(conv(F\mathcal{M}_{n_0+1} \cap S)) \leq \omega(conv(FK^{n_0} \cap S)) < a,$$

which is a contradiction. Then we deduce that $a = 0$. For $n \geq n_0$, we let $\{x_k^n\}_{k\geq 1} \subset S_n$. Since $\{x_k^n\}_{k\geq 1}$ is a countable subset of S_n, we have $\omega(\{x_k^n, k \geq 1\}) \leq a = 0$. Then S_n is relatively weakly sequentialy compact and now the Eberlein-Smulian Theorem argument guaranties that S_n is relatively weakly compact. Consequently, by condition (5), (in the definition of the measure of weak noncompactness) we deduce that the set $S_\infty = \cap_{n\geq n_0} \overline{S_n}^w$ is nonempty, weakly closed convex and $S_\infty \in ker\omega$. We pose

$$\mathcal{U} = \{X \subset S_\infty, X \text{is weakly compact, convex and} FX \cap S \subset X\}.$$

Since

$$FS_\infty \cap S \subset FS_n \cap S \subset \overline{\text{conv}^\omega}(FS_n \cap S) = S_{n+1} \subset S_n$$

for all $n \geq n_0$, we have

$$FS_\infty \cap S \subset \bigcap_{n\geq n_0} S_n = S_\infty.$$

We deduce from this that $FS_\infty \in \mathcal{U}$, then $\mathcal{U} \neq \emptyset$. On the other hand $\mathcal{U}$ is partially ordered by

$$X_1 \preceq X_2 \Leftrightarrow X_1 \supset X_2, \text{ for } X_1, X_2 \in \mathcal{U}.$$

Every chain $\mathcal{J}$ in $\mathcal{U}$ has the finite intersection property, so as S_∞ is weakly compact the intersection of all members of any chain in $(\mathcal{U}, \preceq)$ is nonempty, that is, $B = \bigcap_{X\in\mathcal{J}} X \neq \emptyset$. Since

$$FB \cap S \subset FX \cap S \subset X,$$

for all $X \in \mathcal{J}$, we get

$$FB \cap S \subset \bigcap_{X\in\mathcal{J}} X = B,$$

that is, $B \in \mathcal{U}$, and it is an upper bound of $\mathcal{J}$. By Zorn's lemma, we deduce that $\mathcal{U}$ has a maximal element say X_0. Let $x \in X_0$, we have

$$F(x) \cap S \subset FX_0 \cap S \subset X_0;$$

we deduce that $Fx \cap X_0 \neq \emptyset$, for all $x \in X_0$. Put $X_1 = \overline{\text{conv}^\omega}(TX_0 \cap X_0)$. Therefore

$$FX_0 \cap X_0 \subset FX_0 \cap S \subset X_0;$$

so $X_1 \subset X_0$. By the maximality of X_0, we have $X_1 = X_0$. It is clear that X_1 is convex, weakly compact and $FX_1 \subset FX_0 \cap X_0 \subset X_1$. Thus, it suffices to apply Theorem 2.9, for $F : X_1 \to \mathcal{P}_{cl,cv}(X_1)$ and we obtain a fixed point. □

Now, we will define the notion of L-functions which was introduced by Lim [27] and Suzuki [33].

Definition 3.2. *[27] A function φ from $\mathbb{R}_+$ into itself is called L-function (resp.strictly L-function) if $\varphi(0) = 0$, $\varphi(s) > 0$, for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ (resp. $\varphi(t) < s$), for any $t \in [s, s + \delta]$*

Similar to Proposition 1 in [33] and Theorem 2.6 in [1], we can prove the following characterization of Multivalued Meir-Keeler condensing operators.

Proposition 3.3. *Let S be a nonempty and bounded subset of a Banach space X, ω a measure of weak noncompactness, then every multivalued mapping $T : S \to P(X)$ is a countably Meir-Keeler condensing operator if and only if $T(S)$ is bounded, $T(x) \cap S \neq \emptyset$ for every $x \in S$ and there exists an L-function φ (respectively strictly L-function) such that*

$$\omega(T\Omega \cap S) < \varphi(\omega(\Omega)),\ (resp.\ \omega(T\Omega \cap S) \leq \varphi(\omega(\Omega)))$$

for all countable and bounded subset Ω of X with $\omega(\Omega) \neq 0$.

As a consequence of Theorem 3.1 and Proposition 3.3, we obtain the following result.

Corollary 3.4. *Let S be a nonempty, bounded, closed and convex subset of a Banach space X, ω a measure of weak noncompactness and $T : S \to P_{cl,cv}(X)$ a multivalued mapping satisfying the following properties:*

1. *T has weakly sequentially closed graph,*
2. *$\omega(T(\Omega)) < \varphi(\omega(\Omega))$, (resp. $\omega(T(\Omega)) \leq \varphi(\omega(\Omega))$) for all countable and bounded subset Ω of S, where φ is an L-function (resp. strictly strictly L-function) for all $\Omega \in \mathcal{P}_{bd}(S)$ with $\omega(\Omega) \neq 0$,*
3. *$T(x) \cap S \neq \emptyset$ for all $x \in S$,*
4. *$T(S)$ is bounded.*

Then, T has at least one fixed point.

In the particular case where $T(S) \subset S$, we conclude the following result

Corollary 3.5. *Let S be a nonempty, bounded, closed and convex subset of a Banach space X, ω a measure of weak noncompactness and $T : S \to P_{cl,cv}(S)$ a multivalued mapping satisfying the following properties:*

1. *T has weakly sequentially closed graph,*
2. *$\omega(T(\Omega)) < \varphi(\omega(\Omega))$, (resp. $\omega(T(\Omega)) \leq \varphi(\omega(\Omega))$) for all countable and bounded subset Ω of S with $\omega(\Omega) \neq 0$, where φ is an L-function (resp. strictly strictly L-function),*
3. *$T(S)$ is bounded.*

Then, T has at least one fixed point.

We deduce now the following result wich is a generalization of Theorem 3.1 in [6].

Corollary 3.6. *Let S be a nonempty, bounded, closed and convex subset of a Banach space X, ω a measure of weak noncompactness and $T : S \to P_{cl,cv}(X)$ a multivalued mapping satisfying the following properties:*

1. *T has weakly sequentially closed graph,*
2. *there exists a non linear contraction φ such that*

$$\omega(T(\Omega)) \leq \varphi(\omega(\Omega)),$$

   for all countable and bounded subset Ω of S with $\omega(\Omega) \neq 0$,
3. *$T(x) \cap S \neq \emptyset$ for all $x \in S$*
4. *$T(S)$ is bounded.*

Then, T has at least one fixed point.

In the following, we state some fixed point theorems for multivalued mappings in Banach algebras.

Theorem 3.7. *Let X be a Banach algebra with condition (P) and let S be a nonempty closed convex subset of X. Let $A, B, C : S \to \mathcal{P}(X)$ be three multivalued mappings satisfying the following properties:*

1. *A, B and C are s.w.u.sco,*
2. *A, B and C have weakly sequentially closed graphs,*
3. *A, B and C are countably D-set Lipchitzian,*
4. *for all $x \in S$, $A(x)B(x) + C(x) \in \mathcal{P}_{cl,cv}(S)$,*
5. *$A(S), B(S)$ and $C(S)$ are bounded,*
6. *for $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|A(S)\|\phi_B(r) + \|B(S)\|\phi_A(r) + \phi_A(r)\phi_B(r) + \phi_C(r) < \varepsilon$$

for all $r \in [\varepsilon, \varepsilon + \delta[$ where ϕ_A, ϕ_B, and ϕ_C are the D-functions of A, B and C (respectively).

Then the equation $x \in A(x)B(x) + C(x)$ has at least one fixed point in S.

Proof. It is clear that the mapping $AB + C : S \to \mathcal{P}_{cl,cv}(S)$ is well defined.
Let $\{x_n\} \subset S$, $x_n \rightharpoonup x \in S$ and $y_n \in A(x_n)B(x_n) + C(x_n) \rightharpoonup y \in S$. We have that $y_n = a_n b_n + c_n$ such that $a_n \in A(x_n)$, $b_n \in B(x_n)$, $c_n \in C(x_n)$. Since A, B and C are s.w.u.sco., then $a_n \rightharpoonup a$, $b_n \rightharpoonup b$ and $c_n \rightharpoonup c$. The (P) condition guarantees that

$$y_n = a_n b_n + c_n \rightharpoonup a.b + c \in A(x)B(x) + C(x).$$

Hence $AB + C$ has weakly sequentially closed graph. Let Ω be a countably bounded subset of S with $\omega(\Omega) \neq 0$. Define $\varepsilon = \omega(\Omega)$ and let $\delta = \delta(\varepsilon) > 0$ be chosen according to (2.1). It is clear that $(AB + C)(\Omega)$ is bounded and we have

$$\begin{aligned} \omega((AB + C)(\Omega)) &\leq \|A(\Omega)\|\omega(B(\Omega)) + \|B(\Omega)\|\omega(A(\Omega)) \\ &\quad + \omega(A(\Omega))\omega(B(\Omega)) + \omega(C(\Omega)) \\ &\leq \|A(\Omega)\|\phi_B(\omega(\Omega)) + \|B(\Omega)\|\phi_A(\omega(\Omega)) + \\ &\phi_A(\omega(\Omega))\phi_B(\omega(\Omega)) + \phi_C(\omega(\Omega)) \\ &= \phi(\omega(\Omega)), \end{aligned}$$

where $\phi(r) = \|A(\Omega)\|\phi_B(r) + \|B(\Omega)\|\phi_A(r) + \phi_A(r)\phi_B(r) + \phi_C(r)$ is a strictly L-function on $\mathbb{R}_+$. Thus, the multivalued mapping $AB + C$ is countably Meir-Keeler condensing. From Corollary 3.4, the mapping $AB + C$ has a fixed point. □

Theorem 3.8. *Let X be a Banach algebra with condition (P) and let S be a nonempty closed convex subset of X. Let $A, B, C : S \to \mathcal{P}(X)$ be three multivalued mappings satisfying the following properties:*

1. *A and C have weakly sequentially closed graphs,*
2. *B is weakly completely continuous,*
3. *A and C are countably D-set Lipchitzian,*
4. *A and C are s.w.u.sco,*
5. *For all $x \in S$, $A(x)B(x) + C(x)$ is a colsed convex subset of S,*
6. *$A(S), B(S)$ and $C(S)$ are bounded,*
7. *For $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|B(S)\|\phi_A(r) + \phi_C(r) < \varepsilon$$

for all $r \in [\varepsilon, \varepsilon + \delta[$ where ϕ_A, and ϕ_C are the D-functions of A and C (respectively).

Then the equation $x \in A(x)B(x) + C(x)$ has at least one solution in S.

Proof. We show that B is s.w.u.sco. Let $(x_n)_n$ be a sequence of X such that $x_n \rightharpoonup x \in S$ and $y_n \in B(x_n)$. Since the set $\{x_n,\ n \in \mathbb{N}\}$ is bounded , we obtain that $B(\{x_n,\ n \in \mathbb{N}\})$ is relatively weakly compact (since B is weakly completely continuous) and, so, the sequence $\{y_n\} \subset B\{(x_n)\}$ has a weakly convergent subsequence. As in the proof of Theorem 3.7, it is clear that the mapping $AB + C : S \longrightarrow P_{cl,cv}(S)$ is well defined and has weakly sequentially closed graph.
Let Ω be a countably bounded subset of S with $\omega(\Omega) \neq 0$. Define $\varepsilon = \omega(\Omega)$ and let $\delta = \delta(\varepsilon) > 0$ be chosen according to (2.1). It is clear that $(AB + C)(\Omega)$ is bounded and we have from Lemma 2.12

$$\begin{aligned} \omega((AB + C)(\Omega)) &\leq \|B(S)\|\omega(A(\Omega)) + \omega(C(\Omega)) \\ &\leq \|B(\Omega)\|\phi_A(\omega(\Omega)) + \phi_C(\omega(\Omega)) \\ &= \phi(\omega(\Omega)), \end{aligned}$$

where $\phi(r) = \|B(\Omega)\|\phi_A(r) + \phi_C(r)$ is a strictly L-function on $\mathbb{R}_+$. Thus, the multivalued mapping $AB + C$ is countably Meir-Keeler condensing. From Theorem 3.1 the mapping $AB + C$ has a fixed point. □

The previous result is a generalization of Theorem 4.3 in [6]. In the sequel, we will need the following Lemma.

Lemma 3.9. *Let X be a Banach space and $T : X \longrightarrow \mathcal{P}_{cl,bd}(X)$ a multivalued mapping satisfying the following properties:*

1. *T is countably $D-Lipschitzian$ with $D-$function ϕ*
2. *T is s.w.u.sco.*

Then for any countable, bounded subset Ω of X, we have $\beta(T(\Omega)) \leq \phi(\beta(\Omega))$, where ω is the De Blasi measure of weak noncompactness.

Proof. Let Ω be a countable bounded subset of X. Then there exists a constant r such that $\|x\| \leq r$ for all $x \in \Omega$. Since now T is countably $D - Lipschitzian$ with $D-$function ϕ, we have for a fix $y \in \Omega$

$$\begin{aligned} \|T(x)\| &\leq d_H(T(x), 0) \\ &\leq d_H(T(x), T(y)) + d_H(T(y), 0) \\ &\leq \phi(2\|\Omega\|) + \|T(y)\|. \end{aligned}$$

Hence $T(\Omega)$ is bounded. Suppose now that $\beta(\Omega) < t$ for some $t > 0$, then there exists a weakly compact $K \subset X$ such that $\Omega \subset K + B_t(0)$. Then, for all $x \in \Omega$, there exists $y \in K$ such that $\|x - y\| \leq t$. We have that $x, y \in \Omega \cup y$, so

$$d_H(T(x), T(y)) \leq \phi(\|x - y\|) \leq \phi(t).$$

From Proposition 2.4, we have

$$T(x) \subset T(y) + B_{\phi(t)}(0).$$

and so,

$$T(\Omega) \subset T(K) + B_{\phi(t)}(0) \subset \overline{T(K)}^w + B_{\phi(t)}(0).$$

let $(x_n)_n$ be a sequence of K and $y_n \in T(x_n)$. From the Eberlein-Smulian's theorem, there exists a subsequence (x_{n_k}) weakly convergent to $x \in K$, then $(y_n)_n$ has a subsequence weakly convergent to y in $T(x)$, and so, $\overline{T(K)}^w$ is weakly compact. Then,

$$\beta(T(\Omega)) \leq \phi(t).$$

Letting $t \longrightarrow \beta(\Omega)$ and by the right continuity of ϕ, we deduce that

$$\beta(T(\Omega)) \leq \phi(\beta(\Omega)).$$

□

Theorem 3.10. *Let X be a Banach algebra with condition (P) and let S be a nonempty closed convex subset of X. Let $A, B, C : S \to \mathcal{P}(X)$ be three multivalued mappings satisfying the following properties:*

1. *A and C have weakly sequentially closed graphs,*
2. *B is weakly completely continuous,*
3. *A and C are countably D-Lipchitzian,*
4. *A and C are s.w.u.sco,*
5. *for all $x \in S$, $A(x)B(x) + C(x)$ is a closed convex subset of S,*
6. *$A(S), B(S)$ and $C(S)$ are bounded,*
7. *for $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|B(S)\|\phi_A(r) + \phi_C(r) < \varepsilon$$

for all $r \in [\varepsilon, \varepsilon + \delta[$ where ϕ_A, and ϕ_C are the D-functions of A and C (respectively).

Then the equation $x \in A(x)B(x) + C(x)$ has at least one solution in S.

Proof. From Lemma 3.9, the multivalued mappings A and C are countably D-set-Lipschitzian with D-function ϕ_A and ϕ_C, respectively. The proof follows reasoning similar to that in the proof of Theorem 3.8. □

Remark 3.11. *In [5, 8, 7], the authors introduced and used the operator $\frac{I-C}{A}$, for single valued mappings, to establish some fixed point results in Banach algebras. After that, in [6], the authors introduced a new defintion of the operator $\frac{I-C}{A}$ for multivalued mappings. They say that the mapping $\frac{I-C}{A}$ is well defined on $x \in E$ and they write $y \in (\frac{I-C}{A})(x)$ if $x \in A(x)y+C(x)$ with E is a Banach algebra and A and $C : E \to \mathcal{P}(X)$ and used it in order to prove a generalization of the results in [5, 8, 7]. In the following result, we use the multivalued operator $\frac{I-C}{A}$ in order to obtain a fixed point result where the Banach algebra may not satisfy condition (P).*

Theorem 3.12. *Let X be a Banach algebra, and let S be a nonempty closed convex subset of X. Let $A, C : X \to \mathcal{P}(X)$ and $B : S \rightharpoonup \mathcal{P}(X)$ be three multivalued mappings satisfying the following properties:*

1. *$\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.*
2. *B and $\frac{I-C}{A}$ are s.w.u.sco and have weakly sequentially closed graph.*
3. *B is countably $D-$set-Lipchitzian and $\left(I - \frac{I-C}{A}\right)$ is countably $k-$set-Lipchitzian.*
4. *$A(S)$, $B(S)$ and $C(S)$ are bounded.*
5. *$\left(\frac{I-C}{A}\right)^{-1} Bx$ is a convex subset of S a.e. $x \in S$*
6. *for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\frac{1}{k}\phi_B(r) < \varepsilon$ for all $r \in [\varepsilon, \varepsilon + \delta[$.*

Then $x \in A(x)B(x) + C(x)$ has at least one solution.

Proof. From assumption (5), it is clear that the multivalued mapping

$$\begin{aligned} H :& S \to \mathcal{P}_{cv}(S) \\ & x \mapsto H(x) = \left(\frac{I-C}{A}\right)^{-1} Bx \end{aligned}$$

is well defined. By Corollary 3.4, it suffices to prove that H has weakly sequentially closed graph and it is countably Meir-Keeler condensing. For this, let $\{x_\delta\}_\delta$ and $\{y_\delta\}_\delta$ be nets in S such that $x_\delta \rightharpoonup x \in S$, $y_\delta \rightharpoonup y$ and $y_\delta \in H(x_\delta)$. Since $\left(\frac{I-C}{A}\right) y_\delta = Bx_\delta$, we obtain $\frac{I-C}{A} y_\delta \rightharpoonup \frac{I-C}{A} y$ and $Bx_\delta \rightharpoonup Bx$, it follows that $\frac{I-C}{A} y = Bx$ and then $y \in Hx$, and so H has weakly sequentially closed graph. Let M be countably and bounded subset of S with $\omega(M) > 0$. Define $\varepsilon = \omega(M)$ and let $\delta = \delta(\varepsilon) > 0$ be chosen according to (2.1). It is clear that $H(M)$ is bounded and we have

$$\frac{I-C}{A}(Hx) = \{Bx\}, \quad for\ all\ x \in M,$$

then, for all $y \in Hx$ we have:

$$\left(\frac{I-C}{A}\right) y = Bx;$$

hence:

$$y = Bx + \left(I - \frac{I-C}{A}\right) y$$

consequently

$$Hx \subset Bx + \left(I - \frac{I-C}{A}\right)(Hx), \quad for\ all\ x \in M,$$

then

$$H(M) \subset B(M) + \left(I - \frac{I-C}{A}\right)(H(M)),$$

and

$$\begin{aligned}\omega(H(M)) &\leq \omega(B(M)) + \omega(\left(I - \frac{I-C}{A}\right)(H(M))) \\ &\leq \phi_B(M) + k\omega(H(M))\end{aligned}$$

It follows that

$$\omega(H(M)) \leq \frac{1}{k}\phi_B(M) = \phi(\omega(M));$$

where $\phi(r) = \frac{1}{k}\phi_B(r)$, and then, H is countably Meir-Keeler condensing. □

Now, let us recall the notion of w-weakly closed graph which was introduced by Cardinali and Papalini in [14].

Definition 3.13. *Let S be a nonempty subset of X.*

1. *We say that a multivalued map $T : S \to \mathcal{P}(X)$ has weakly closed graph if for every net $(x_\delta)_\delta$, $x_\delta \in S$; $x_\delta \longrightarrow x$ in S, and for every $(y_\delta)_\delta, y_\delta \in T(x_\delta)$, $y_\delta \longrightarrow y$ then $T(x) \cap S(x,y) \neq \emptyset$, where $S(x,y) := \{\lambda y + (1-\lambda)x : \lambda \in [0,1]\}$.*

2. *We say that $T : S \to \mathcal{P}(X)$ has ω-weakly closed graph in $S \times X$, if it has weakly closed graph in $S \times X$ with respect to the weak topology.*

Using this definition we obtain the following Theorem.

Theorem 3.14. *let S be a closed convex subset of a Banach space X, ω is a MWNC on X and let $T : S \to \mathcal{P}_{cl,cv}(S)$ a multivalued satisfying the following conditions*

1. *T is countably Meir-keeler condensing.*
2. *T maps weakly compact sets into relatively weakly compact sets.*
3. *T has a ω-weakly closed graph in $S \times S$.*

Then T has a fixed point.

Proof. We produce like in the proof of Theorem 3.1 to show that there exist a convex relatively weakly compact subset X_1 of S such that $T(X_1) \subset X_1$. For the rest of the proof, it follows the same ideas as in Theorem 3.18 in [9]. □

Remark 3.15. *Theorem 3.14 generalizes Theorem* 3.18 *in [9] and Theorem* 4.1 *in [15] to the case of countably Meir-keeler condensing mapping. The condition that X is separable in the statement of this Theorem is not needed.*

Now, we can establish a new version of Krasnoselskii-Daher fixed point Theorem in Banach algebra.

Theorem 3.16. *Let X be a Banach algebra satisfying condition (P) and let S be a nonempty closed convex subset of X. Let $A, B, C : S \to P_{cl}(X)$ be three multivalued mappings satisfying the following properties:*

1. *C, B are countably D-set-Lipschitzian, and they maps weakly compact sets into relatively weakly compact sets.*
2. *A maps bounded sets into relatively weakly compact sets.*
3. *$(A.B + C)(S)$ is a bounded convex subset of S.*
4. *C has w-weakly closed graph.*
5. *A, B have w-closed graphs.*
6. *$\theta \in A.B$.*
7. *for $\varepsilon > 0$, there exists a $\delta > 0$,such that*

$$\|A(S)\|\phi_B(r) + \phi_C(r) < \varepsilon$$

for all $r \in [\varepsilon, \varepsilon + \delta[$, where ϕ_B and ϕ_C are the D-functions of B and C (respectively).

then $A.B + C$ has at least a fixed point in S.

Proof. Let $H : S \to \mathcal{P}(X)$ be $H(x) = A(x)B(x) + C(x)$.

- From assumption (3) we have that $H(S)$ is a bounded subset of S.
- Consider a countable bounded subset D of S with $\varepsilon = \omega(D) > 0$ and let $\delta = \delta(\varepsilon) > 0$ be chosen according to (2.1). We have

$$\begin{aligned}\omega(H(D)) &= \omega(A(D)B(D) + C(D)) \\ &\leq \|A(D)\|\omega(B(D)) + \|B(D)\|\omega(A(D)) \\ &\quad + \omega(A(D))\omega(B(D)) + \omega(C(D))\end{aligned}$$

  from assumption (2) we have $\omega(A(D)) = 0$ then

$$\begin{aligned}\omega(H(D)) &\leq \|A(D)\|\omega(B(D)) + \omega(C(D)) \\ &\leq \|A(D)\|\phi_B(\omega(D)) + \phi_C(\omega(D)) \\ &= \phi(\omega(D)),\end{aligned}$$

  where $\Phi(r) = \|A(D)\|\phi_B(r) + \phi_C(r)$ is a strictly $L-$function on $\mathbb{R}_+$. Thus, H is countably Meir keeler-condensing.
- From assumption (3), we have that $H(x)$ is convex, and since $C(x)$ is relatively compact and $A(x)$, $B(x)$ are closed, we have $H(x)$ is closed. Thus $H(x) \in \mathcal{P}_{cl,cv}(S)$.
- We now show that H has w-weakly closed graph. Consider a net $(x_\delta)_\delta$ with $x_\delta \in S$ and $x_\delta \rightharpoonup x$, and consider $(y_\delta)_\delta$, $y_\delta \in H(x_\delta)$, for any δ, such that $y_\delta \rightharpoonup y$, then there exists three nets $(h_\delta)_\delta$, $(k_\delta)_\delta$ and $(t_\delta)_\delta$ such that $h_\delta \in A(x_\delta)$, $k_\delta \in B(x_\delta)$, $t_\delta \in C(x_\delta)$ and $y_\delta = h_\delta k_\delta + t_\delta$. From assumption (1) and (2) we may assume that $h_\delta \rightharpoonup h$, $k_\delta \rightharpoonup k$ and from condition (P), we have $h_\delta k_\delta \rightharpoonup hk$. Since A and B have w-closed graph, then $hk \in AB(x)$. Thus $t_\delta \rightharpoonup y - hk$. Since now C has w-weakly closed graph, then

$$S(x, y - hk) \cap C(x) \neq \emptyset.$$

Then, there exists a $z \in S(x, y - hk) \cap C(x)$. Thus, there exists a $\lambda \in [0,1]$ such that $z = (1-\lambda)x + \lambda(y - hk) \in C(x)$. So $(1-\lambda)x + \lambda y \in C(x) + \lambda hk$.
We have

$$\begin{aligned}\lambda h.k &\in conv(A.B(x) \cup \{\theta\}) \\ &\subset conv(A.B(x) \cup A.B(x)), \\ &= A.B(x)\end{aligned}$$

so, $(1-\lambda)x+\lambda y \in C(x)+A.B(x)$. Moreover, we have $(1-\lambda)x+\lambda y \in S(x,y)$, therefore $S(x,y)\cap H(x) \neq \emptyset$. Thus H has w-weakly closed graph.

- We now show that if M is a weakly compact subset of S, then $H(M)$ is relatively weakly compact. From assumption (1) we have $\omega(C(M)) = \omega(B(M)) = 0$, and from assumption (2), we have $\omega(A(M)) = 0$, and so

$$\begin{aligned}\omega(H(D)) &= \omega(A(D).B(D) + C(D)) \\ &\leq \|A(D)\|\omega(B(D)) + \|B(D)\|\omega(A(D)) \quad +\omega(A(D))\omega(B(D)) + \omega(C(D)) \\ &= 0\end{aligned}$$

Thus $H(M)$ is relatively weakly compact.
From Theorem 3.14 we deduce that there exists $x \in S$ such that $x \in A(x).B(x) + C(x)$.

□

Remark 3.17. *If we take $B = Id$, then we obtain a generalization of Theorem 3.19 in [9] to the case of countably $D-$set-Lipschitzian functions.*

4. Application

Let X be a real Banach algebra satisfying condition (P), and let $I_0 = [-r, 0]$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$.
Let $C = C(I_0, X)$ denote the Banach space of all continuous functions on I_0, with the usual supremum norm $\|.\|_C$ given by: $\|\phi\|_E = sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. For any continuous function x defined on the interval $J = [-r, T] = I_0 \cup I$ and for any $t \in I$ we denote by x_t the element of X defined by

$$x_t(\theta) = x(t+\theta); \quad \theta \in I_0$$

where the argument θ represents the delay in the argument of solutions and $t + \theta \in J = [-r, T]$. Clearly $C(J, X)$ is a Banach algebra with the norm $\|x\| = sup\{|x(t)| : t \in J\}$ and the multiplication "." defined by $(x.y)(t) = x(t)y(t)$ for all $t \in J$.
Given a function $\phi \in C$ and consider the first order neutral functional differential inclusion (In short NFDI)

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t) - h(t, x_t)}{f(t, x_t)}\right] \in G(t, x(t)); \quad t \in I \\ x(t) = \phi(t); \quad t \in I_0 \end{cases} \tag{4.1}$$

where $f, h : I \times X \to X, G : I \times X \to \mathcal{P}_{cv}(X)$.
By a solution of NFDI we mean a function $x \in C(J, X) \cap AC(I, X) \cap C(I_0, X)$ (where $AC(I, X)$ is the space of all absolutely continuous functions on I) that satisfies:

1. The mapping $t \longrightarrow \dfrac{x(t) - h(t, x_t)}{f(t, x_t)}$ is differentiable,
2. There exists a $v \in L^1(I, X)$ with $v(t) \in G(t, x(t))$ satisfying $\dfrac{d}{dt}\left(\dfrac{x(t) - h(t, x_t)}{f(t, x_t)}\right) = v(t)$ for all $t \in I$ and $x_0 = \phi \in C$

We shall seek the solution of NFDI in the space $E = C(J,X)$ of continuous functions on J. Note that $C(J,X) \subset AC(J,X)$.
Under the following hypotheses, we could reach the solution of (4.1):

(H_0) (i) for all $t \in [0,T]$, $f(t,.) : X \longrightarrow X$ is weakly sequentially continuous,
(ii) $f(0,\phi) = 1$.
(iii) there is an $L > 0$ and $k > 0$ such that

$$\|f(t,x_t) - f(t,y_t)\| \leq \frac{L\|x(0) - y(0)\|}{k + \|x(0) - y(0)\|}$$

for all $x, y \in D \subset E$ and $t \in J$ with D is a bounded countable subset of E.

(H_1) (i) for all $t \in [0,T]$, $h(t,.) : X \longrightarrow X$ is weakly sequentially continuous,
(ii) $h(0,\phi) = 0$.
(ii) there is an $L_1 > 0$ and $k_1 > 0$ such that

$$\|h(t,x_t) - h(t,y_t)\| \leq \frac{L_1\|x(0) - y(0)\|}{k_1 + \|x(0) - y(0)\|}$$

for all $x, y \in D \subset E$ and $t \in J$ with D is a bounded countable subset of E.

(H_2) (i) For any $r > 0$, there exists $\varphi_r \in L^1([0,T])$ with $\|G(t,u)\| \leq \varphi_r(t)$ for a.e. $t \in [0,T]$ and all $u \in X$ with $|u| \leq r$. We let

$$M_r = \phi(0) + \int_0^T \varphi_r(s)ds$$

(ii) For all $t \in [0,T]$, $G(t,.)$ is weakly completely continuous.
(iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $M_R \dfrac{Lr}{(k+r)} + \dfrac{L_1 r}{k_1 + r} < \varepsilon$ for all $r \in [\varepsilon, \varepsilon + \delta[$
(iv) There exist $R > 0$ such that $M_R \dfrac{LR}{(k+R)} + \dfrac{L_1 R}{(k_1 + R)} + M_R\|f(t,0)\| + \|h(t,0)\| \leq R$

Theorem 4.1. *Assume that hypotheses (H_0)-(H_2) hold. Then (4.1) has a solution in $C(J,X)$.*

Proof. Now NFDI is equivalent to the integral inclusion :

$$\begin{cases} x(t) \in f(t,x_t)\left(\phi(0) + \displaystyle\int_0^t G(s,x(s))ds\right) + h(t,x_t); & t \in I \\ x(t) = \phi(t); \quad t \in I_0. \end{cases} \tag{4.2}$$

This integral equation may be written in the form:

$$x(t) \in Ax(t)Bx(t) + Cx(t); \quad t \in J$$

where $A, C : E \longrightarrow E$ and $B : E \longrightarrow \mathcal{P}(E)$ defined by:

$$Ax(t) = \begin{cases} f(t,x_t), & if\ t \in I \\ 1, & if\ t \in I_0 \end{cases} \tag{4.3}$$

$$Bx(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t G(s,x(s))ds, & if\ t \in I \\ \phi(t), & if\ t \in I_0 \end{cases} \tag{4.4}$$

$$Cx(t) = \begin{cases} h(t,x_t), & if\ t \in I \\ 0, & if\ t \in I_0 \end{cases} \tag{4.5}$$

Let $S = \{x \in E, \|x\| \leq R\}$. Now S is a closed convex subset of E. We shall show that the operators A, B and C satisfy all the condition of Theorem 3.10. For that we need the following definition:

Definition 4.2. *A mapping $f : [0,T] \longrightarrow X$ is said to be scalarly measurable if for every φ in the topological dual space X^* the function $\varphi \circ f(<\varphi, f>)$ is measurable.*
We say that f is scalarly integrable if $\varphi \circ f(<\varphi, f>)$ is integrable for each $\varphi \in X^*$.

Now for a multivalued mapping $G : [0,T] \times X \longrightarrow \mathcal{P}(X)$, we define

$$\int_0^t G(s,x(s))ds = \left\{ \int_0^t v(s)ds,\ v \text{ is Pettis integrable},\ v(t) \in G(t,x(t)) \right\}$$

Then we have the following Lemma due to Musial (see[28]).

Lemma 4.3. *Let $f : [0,T] \longrightarrow X$ be a function satisfying the following conditions:*

(1) There exists a sequence of Pettis integrable functions (f_n) weakly convergent to f.
(2) There exists $h \in L^1([0,T])$ such that for each $\varphi \in X^*$ and each $n \in \mathbb{N}$, the inequality $|<\varphi, f_n>| \leq h$ holds.

Then f is Pettis integrable and $\int_0^t f_n(s)ds$ converges weakly to $\int_0^t f(s)ds$.

We will show that A, B and C verify all the conditions of Theorem 3.10.

Step 1. We show that A and C are bounded operators on E and countably D-Lipchitzian. Now for any $x \in E$, one has

$$\begin{aligned}
\|Ax_t\|_E &\leq \|A(0)\|_E + \|A(x_t)_{t\in I} - A(0)\|_E \\
&\leq sup_{t\in J}|f(t,0)| + |f(s,x_s) - f(s,0)| \\
&\leq F_0 + \frac{L\|x_t(0) - 0\|}{k + \|x_t(0) - 0\|} \\
&\leq F_0 + \frac{L\|x(0)\|}{k + \|x(0)\|} \\
&\leq F_0 + L
\end{aligned}$$

for all $x \in E$, where $F_0 = sup_{t\in J}|f(t,0)|$, which shows that A is bounded with bound $F_0 + L$.
Next, we show that A is countably D-Lipschitzian. Let $x, y \in D \subset E$ bounded and countable. Then

$$\begin{aligned}
\|Ax - Ay\| &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\
&\leq |f(t,x_t) - f(t,y_t)| \\
&\leq \frac{L|x_t(0) - y_t(0)|}{k + |x_t(0) - y_t(0)|} \\
&= \phi(\|x - y\|_E)
\end{aligned}$$

Then A is countably D-lipschitzian with D-function $\phi_A(r) = \dfrac{Lr}{K + r}$.
In the same why we show that C is bounded and countably D-lipschitzian with D-function $\phi_C(r) = \dfrac{L_1 r}{K_1 + r}$.

Step 2. We show that A is weakly sequentially continuous.
Let $(x_n)_n \subset E$ such that $x_n \rightharpoonup x$, so for all $t \in [0,T]$, $x_n(t+\theta) \rightharpoonup x(t+\theta)$ a.e $\theta \in J$. Since $f(t,.)$ is weakly sequentially continuous, then from Dobrokov's Theorem, we deduce that $Ax_n \rightharpoonup Ax$, and so A is weakly sequentially continuous. Similarly we prove that C is weakly sequentially continuous. Then A and C are w.s.u.sc.

step 3. Prove that $Bx \in \mathcal{P}(C(J,X))$. Let $x \in E$ and $r > 0$ such that

$$\|x\| = sup_{t\in[0,T]}\|x(t)\| < r.$$

Let

$$u(t) = \phi(0) + \int_0^T v(s)ds$$

where v is pettis integrable and $v(s) \in G(s, x(s))$.

- Let $t, t' \in [0, T]$ such that $u(t) \neq u(t')$ and $t \leq t'$. From the Hahn-Banach Theorem, there exists $\phi \in E^*$ with $\|\phi\| = 1$ and

$$\|u(t) - u(t')\| = \phi(u(t) - u(t')).$$

  From assumption (H_2), we get

$$\|u(t) - u(t')\| = \phi\left(\int_t^{t'} v(s)ds\right) \leq \int_t^{t'} \varphi_r(s)ds.$$

- If $t, t' \in I_0$, then

$$\|u(t) - u(t')\| = \|\phi(t) - \phi(t')\|.$$

- For the case where $t \leq 0 \leq t'$ we have that

$$\begin{aligned}\|u(t) - u(t')\| =& \|\phi(t) - \phi(0) - \int_0^{t'} v(s)ds\| \\ &\leq \|\phi(t) - \phi(0)\| + \|\int_0^{t'} v(s)ds\| \\ &\leq \|\phi(t) - \phi(0)\| + \int_0^{t'} \varphi_r(s)ds.\end{aligned}$$

Hence, in all cases, we have $\|u(t) - u(t')\| \rightharpoonup 0$. Consequently, $u \in E$ and $Bx \in \mathcal{P}(E)$.
Let $x_n \in E$, $x_n \rightharpoonup x$ and $y_n \in Bx_n$ such that $y_n \rightharpoonup y$. There exists a sequence $v_n : [0, T] \to X$ such that $v_n(s) \in G(s, x_n(s))$ for each $s \in [0, T]$, and for each $t \in [0, T]$, we have

$$y_n(t) = \phi(0) + \int_0^t v_n(s)ds.$$

So, if $x_n(s) \rightharpoonup x(s)$ for all $s \in [0, T]$, then

$$\Omega = \{x_n(s), n \in \mathbb{N}\}$$

is relatively weakly compact and, so, it is a bounded subset of X. We know that $v_n(s) \in G(s, x_n(s))$ for all $s \in [0, T]$, so we have

$$\{v_n(s),\ n \in \mathbb{N}\} \subset G(s, \Omega).$$

From assumption $(H_2)(ii)$, the set $\{v_n(s), n \in \mathbb{N}\}$ is relatively weakly compact for all $s \in [0, T]$. From Tychonoff's Theorem, the set

$$\{v_n,\ n \in \mathbb{N}\} \equiv \prod_{s \in [0,T]} \{v_n(s),\ n \in \mathbb{N}\}$$

is relatively weakly compact. Hence, $v_n \rightharpoonup v \in E$. From assumption (H_2) and the Lemma 4.3, v is Pettis integrable and we have

$$y_n(t) = \phi(0) + \int_0^T v_n(s)ds \rightharpoonup \phi(0) + \int_0^T v(s)ds \quad for\ all\ t \in [0, T] .$$

Since $G(s,.)$ has a weakly sequentially closed graph, we conclude that

$$v(s) \in G(s, x(s)) \;\; for \; all \; s \in [0,T]$$

and

$$y(t) = \phi(0) + \int_0^T v(s)ds \in Bx(t).$$

Now, if $t \in I_0$, then $y_n(t) = \phi(t) = y(t)$.
Now,we show that $B(S)$ is relatively weakly compact. Let $\{y_n\}$ be a sequence in $B(S)$ and let $\{x_n\}$ be a sequence in E such that $y_n \in B(x_n)$. There exists a sequence of Pettis integrable mappings $\{v_n\}$ such that

$$v_n(s) \in G(s, x_n(s)) \;\; for \;\; all \;\; s \in [0,T]$$

and

$$y_n(t) = \phi(0) + \int_0^T v_n(s)ds \;\; for \;\; all \;\; t \in [0,T]$$

Since $\{x_n\}$ is bounded, the subset $\{v_n(s)\} \subset G(s, \{v_n(s)\})$ is relatively weakly compact for all $s \in [0,T]$. Then the set

$$\{v_n, \; n \in \mathbb{N}\} \equiv \prod_{s \in [0,T]} \{v_n(s), \; n \in \mathbb{N}\}$$

is relatively weakly compact in $X^{[0,T]}$. So, there exists a subsequence $v_{n_k} \rightharpoonup v$. By the Lemma 4.3, we deduce that

$$y_{n_k}(t) \rightharpoonup y(t).$$

By Dobrokov's Theorem, we deduce that $y_{n_k} \rightharpoonup y$.
If $t \in I_0$, then $y_n(t) = \phi(t) = y(t)$.

Step 5. Finally, we show that $A(x)B(x) + C(x)$ is a convex subset of S for each $x \in S$. Let $x \in E$ and $y, y_1 \in A(x)B(x) + C(x)$ such that

$$y(t) = \begin{cases} f(t, x_t)(\phi(0) + \int_0^t v(s)ds) + h(t, x_t), & if \; t \in I \\ \phi(t), \;\; if \; t \in I_0 \end{cases}$$

$$y_1(t) = \begin{cases} f(t, x_t)(\phi(0) + \int_0^t v_1(s)ds) + h(t, x_t), & if \; t \in I \\ \phi(t), \;\; if \; t \in I_0 \end{cases}$$

where $v, \; v_1$ are Pettis integrable with $v(s), v_1(s) \in G(s, x(s))$ for all $s \in [0,T]$. For all $\lambda \in (0,1)$, we have

$$\lambda y(t) + (1-\lambda)y_1(t) = f(t, x_t)(\phi(0) + \int_0^t (\lambda v(s) + (1-\lambda)v_1(s))ds) + h(t, x_t).$$

Since $G(t, x(t)$ is a convex subset of X, it is clear that

$$\lambda y + (1-\lambda)y_1 \in A(x)B(x) + C(x).$$

For all $y \in A(x)B(x) + C(x)$, we have

$$\begin{aligned} \|y\| \leq & \|f(t, x_t)\|(\phi(0) + \int_0^T \varphi_R(s)ds) + \|h(t, x_t)\| \\ & \leq (\|f(t, x_t) - f(t, 0)\| + \|f(t, 0)\|)(\phi(0) + \int_0^T \varphi_R(s)ds) + \|h(t, x_t)\| \\ & \leq (\phi_f(\|x\|) + \|f(t, 0)\|)M_R + \phi_h(\|x\|) + \|h(t, 0)\| \\ & \leq R. \end{aligned}$$

If $t \in I_0$, then we have

$$\lambda y(t) + (1-\lambda) y_1(t) = \phi(t) \in A(x)(t)B(x)(t) + C(x)(t)$$

and $\|y\| = \|\phi\| \leq R$. Hence, $A(x)B(x) + C(x)$ is a convex subset of S.

□

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