BUILDING AN INCREASING CONTINUOUS UTILITY FUNCTION

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ABSTRACT

The paper focuses on modern analytical techniques for construction of utility functions over prizes, where the preferences of the decision maker are strictly increasing. If chosen and applied properly these methods facilitate the analysis and guarantee precise description of the decision makers' preferences. The paper discusses modern analytical techniques, such as a modified arctg(.) form of the utility function, which contains prior information for the most typical risk attitude over lotteries, whose prizes can be both profits and losses. Also discussed is a power approximation of the utility.

Keywords. Utility function, lotteries, analytical approximation, risk attitude, local risk aversion

SÜREKLİ ARTAN BİR FAYDA FONKSİYONUNUN MODELLENMESİ

ÖZET

Bu çalışma karar oluşturucuların özellikleri kuvvetli bir şekilde artarken ödüller için kurulmuş fayda fonksiyonunun inşası için modern analitik teknikler üzerine odaklanmaktadır. Eğer bu metotlar seçilir ve uygun bir şekilde uygulanırsa analizi kolaylaştırırlar ve karar oluşturucularının özelliklerini tam anlamıyla garanti ederler. Bu çalışma ödülü hem kar hem zarar olabilen piyangolar üzerine en tipik risk davranışı için ön bilgi içeren fayda fonksiyonunun bir modifiye edilmiş arctg(.) biçimi gibi modern analitik teknikleri tartışmaktadır. Ayrıca fayda yaklaşımı da tartışılmaktadır.

Anahtar Kelimeler: Fayda fonksiyonu, piyangolar, analitik yaklaşım, risk davranışı

1. INTRODUCTION

The systematic approach to analyze subjective or frequency information, namely the decision theory (DT), is one of the most frequently used quantitative analysis techniques under risk. It provides methods for making a rational choice when dealing with multiple possible alternatives. Its purpose is to find sufficient balance between the beliefs of the certain individual or entity making the decision (hence called decision maker), his attitude towards risk, the consequences of his possible actions and their utility.

The basis of DT is the utility theory [1,2]. Utility theory techniques are developed in the context of choices between lotteries. A *lottery* is a set of excluding prizes and the probability to win each one of them. The set of lotteries L and the set of prizes X can be either discrete, or continuous. If only discrete sets are present, the so-called *ordinary lotteries* are defined. *Generalized lotteries* (of I, II or III type) are defined in cases where a continuous set of prizes and/or lotteries is present. The ways to analyze lotteries are set in axioms of rational choice [3,4,5]. The utility function $u(.)$ is used to measure preferences over prizes. Lotteries are ranked in descending order of their expected utility, i.e. the utilities of prizes weighted by their probabilities.

The utility function can either be constructed via linear interpolation or via analytical approximation on the basis of a set of subjective estimates. The paper focuses on the second approach and discusses modern techniques for analytical construction of utilities in the case where the preferences of the decision maker (DM) are strictly increasing, i.e. the higher the prize the more preferred it is. It is emphasized that most of the classical forms only apply to a specific prize interval or to a specific risk attitude. Initial survey on that topic was presented in [6]. However, recent publications have proposed methods, which are much more adequate in approximating the preferences of the DM over arbitrary prize sets. These shall be stressed as well. In what follows, section, the basis of utility theory and the interpretation of risk attitude is presented in section 2. Section 3 discusses classical and modern analytical forms of the utility function and discusses their characteristics and application areas.

2. PREFERENCES AND RISK ATTITUDE IN THE UTILITY FUNCTION

The value system and believes of a decision maker is defined by his preferences over a set of consequences (or prizes) X. Quantitative measurement of these preferences is the *utility function* $u(.)$. The values of $u(x)$ in case of strictly increasing values of $x \in X$, increases with the value of x, if $x_i>x_j$, $u(x_i)>u(x_j)$, for all $x_i \in X$, $x_j \in X$. Building a one-dimensional utility function $u(.)$ is necessary with both one-dimensional and multi-dimensional sets of prizes. In the latter case utility functions are built for each attribute under specific conditions of preferential independency.

The elicitation of utilities of discrete and continuous prizes follows a similar procedure. It solves a preferential equation over lotteries using the bisection method [7] or its modifications (i.e. the triple bisection method) [8]. The DM solves the preferential equation by changing a given parameter in one of the lotteries until he is indifferent between those. Methods such as the probability equivalent or lottery equivalent change the probability in one of the lotteries [9]. Other methods, such as the certainty equivalent or the uncertain equivalent methods change the value of a prize [10,11]. The resulting estimates are then used to build the utility function $u(.)$.

The form of the utility function describes not only preferences, but also risk attitude [12,13,14]. When analyzing risk attitude, the set X usually contains only monetary prizes.

Choosing an alternative, in terms of the utility theory, boils down to calculating the *expected utility* of each ordinary or generalized lottery, and picking the one with the highest result. The formula for expected utility is as follows:

$$E(u \mid p)=\begin{cases} \sum_{i=1}^{r} u(x_i)\, p(x_i), \text{ for discrete prizes} \\ \int_{-\infty}^{+\infty} u(x)\, f(x)\, dx, \text{ for continuous prizes} \end{cases} \quad (1)$$

The formula $\sum_{i=1}^{r} u(x_i)\, p(x_i)$ applies for ordinary lotteries and generalized lotteries of II type, while $\int_{-\infty}^{+\infty} u(x)\, f(x)\, dx$ applies to generalized lotteries of I and III type, where the acquired prize x is a realization of a continuous random variable with density $f(x)$. If a lottery contains only monetary prizes, then it is possible to calculate the expected value of profit:

$$E(x \mid p)=\begin{cases} \sum_{i=1}^{n} x_i\, p(x_i), \text{ for discrete prizes} \\ \int_{-\infty}^{+\infty} x\, f(x)\, dx, \text{ for continuous prizes} \end{cases} \quad (2)$$

The expected value of prizes $E(x \mid p)$ can be used to rank lotteries, but that suggests the DM is risk neutral, which is rarely the case. Utility theory is based on rationality axioms, which suggest that $E(u \mid p)$ is to be used for rational choice between lotteries, because it takes into account the true risk attitude of the DM.

Let x_c be the sum of money whose utility equals the expected utility of a lottery:

$$u(x_c)=E(u \mid p). \quad (3)$$

The sum x_c is the price of the lottery according to the DM and is called *certainty equivalent* (CE). It is an alternative way to rank lotteries in descending order of x_c. The CE may be elicited subjectively using the bisectional algorithms or one of its modifications.

The difference between the expected value and the CE, called *risk premium*, can best describe the risk attitude. In [14] it is denoted as RP:

$$RP = E(x \mid p) - x_c. \quad (4)$$

In [3,14] the close relation of risk attitude and the sign of the risk premium is proven in theorems. An individual is risk prone if $RP<0$, risk neutral if $RP=0$, and risk averse if $RP>0$.

Probably the most appropriate measure of DM's risk attitude is the *local risk aversion* $r(x)$ proposed in [12]. It uses only the utility function (often the only source of information). It is defined as

$$r(x)= -u''(x)/\, u'(x). \quad (5)$$

It has been proven that if the utility function is transformed into its strategic equivalent $w(x)=\alpha\, u(x)+\beta$, the local risk aversion would not change. The first derivative of $w(x)$ is $w'(x)=\alpha\, u'(x)$, while the second derivative is $w''(x)=\alpha\, u''(x)$, so $r(x)$ remains the same. The reverse statement also holds and is proven in a theorem in [14]. The practical importance of the local risk aversion is proven in a theorem in [12, 14]. Let $u_1(.)$ and $u_2(.)$ be two utility functions of two DMs over the continuous set of prizes X, while $r_1(x)$ and $r_2(x)$ are represent their local risk aversions. The risk premium of the first DM $(RP)_1$ will be higher than the one of the second DM $(RP)_2$, if $r_1(x)> r_2(x)$.

The risk attitude of a single DM over different sets of prizes can also be analyzed through the local risk aversion. Most people are risk prone over large losses and small profits, and that proneness decreases with the increase of the losses. In the same time, people are risk averse over large profits and small losses, and their aversion decreases with the increase of the profits. Those are rather descriptive statements, but all possible deviations are caused only by incorrect utility elicitation process [3].

Figure 1 presents the typical utility function [15] and its local risk aversion. The S-shape function clearly demonstrates why it is important to describe all possible consequences referring to a zero point relevant to the actual state of the world at the decision moment.

3. APPROXIMATED UTILITY FUNCTIONS

The utility function can either be constructed via linear interpolation or via analytical approximation on the basis of a set of subjective estimates. The second approach gives better results especially when a multi-dimension utility function is present and there is a small amount of subjectively elicited utility points.

In [13] the exponential utility function $u(x)=1–e^{-x/R}$ is proposed and analyzed. That is a concave function and may represent risk averseness. The parameter R is called *risk tolerance* and determines how risk averse the utility function is. Large values of R make the exponential utility function flatter, while smaller values make it more concave or more risk-averse. That is why if an individual can tolerate more risk, he would assess larger R and vise versa.

R may be subjectively elicited. A bet is proposed, giving the chance to win Y with probability 0.5, and to lose $Y/2$ with probability 0.5. The largest value of Y for which an individual would prefer to take the gamble rather than not take it is approximately equal to his risk tolerance R. Once R is assessed, it is easy to determine the CE of a lottery. The formula $x_c \approx \mu - \frac{0.5\sigma^2}{R}$ is proposed in [12,16] in order to calculate CE for the case of many outcomes. The parameters μ and σ are respectively the expected value and variance,.

It is important to notice that the function $u(x)=1 – e^{-x/R}$ is appropriate for the case of *constant risk aversion*, where the risk premium of a gamble does not depend on the initial wealth and is a positive constant. A *decreasing risk aversion* situation is also possible, when the risk premium of a gamble decreases with the increase in the initial wealth. The utility function for such cases can be assessed as $u(x) = \ln(x)$ [Keeney, Raiffa, 1993].

The graphics of $u(x)=1 – e^{-x/R}$ and $u(x)=\ln(x)$ are presented in [13] and show no significant difference. Still, decreasing risk aversion appears to provide a more appropriate model of preferences that does constant risk aversion.

Other analytical forms under different risk attitudes are also discussed in [14]. Constant risk aversion is described with the exponential function $u(x) \sim –e^{-cx}$, for $c>0$, risk neutrality - with the linear function $u(x)=x$, while decreasing risk proneness is described with the exponential function $u(x)\sim e^{-cx}$, for $c<0$. Other exponential and logarithmic expressions to reflect decreasing risk aversion are also defined, under certain parameter values. For the case of decreasing risk proneness the expression $u(x)=x^2$ is proposed (decreasing risk proneness occurs when the risk premium for a gamble increases with the increase in the initial wealth).

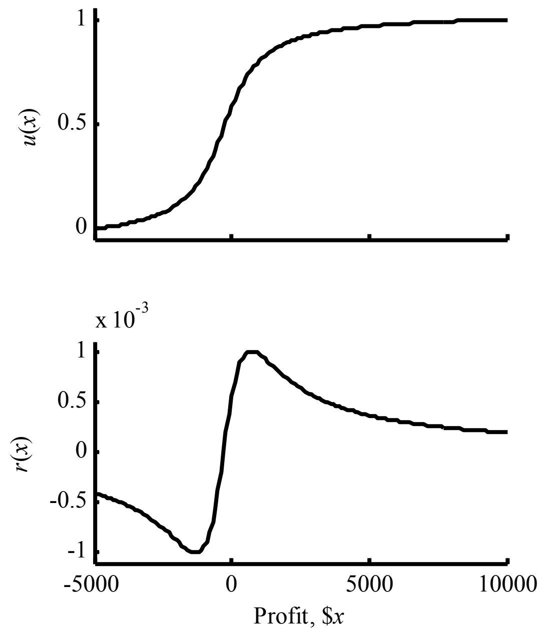

Fig. 1. Typical utility function $u(x)$ over profits in the interval from –\$5000 to \$10 000 (up) and local risk aversion $r(x)$ of the typical utility function (down)

Let X be a one-dimensional continuous set of prizes in a certain interval, where $x_i \succ x_j \Leftrightarrow x_i>x_j$, for $x_i, x_j \in X$ [17]. The most desirable price is defined as $x_{best} =sup(X)$, while the least desirable one is $x_{worst} =inf(X)$. They are appointed utilities as follows: $u(x_{best})=1$ and $u(x_{worst})=0$. In order to build $u(x)$ for an individual DM over X, it is necessary to elicit several appropriate inner nodes between (x_{worst}; x_{best}) in dialog with the DM. Ideally the nodes represent a single value. The real DM is usually inconsistent with the axioms of rationality, and his subjective estimates are in an interval form, which in turn leads to partially transitive preferences. DMs with such preferences are called fuzzy rational DMs [4].

Some methods generate uncertainty intervals of the utility [$\hat{u}_l^d$; $\hat{u}_l^u$] (where $\hat{u}_l^d$ and $\hat{u}_l^u$ are respectively the lower and upper bounds of the uncertainty interval). Other methods generate uncertainty intervals of the prize [$\hat{x}_{u_l}^d$; $\hat{x}_{u_l}^u$] (where $\hat{x}_{u_l}^d$ and $\hat{x}_{u_l}^u$ are respectively the lower and upper bounds of the uncertainty interval). As a result, the available information can be either one of the following:

$$\{(x_l;\ \hat{u}_l^d\ ;\ \hat{u}_l^u) \mid l=1, 2,\dots, z\}, \text{ where}$$
$$\begin{aligned} & x_1 < x_2 < \dots < x_z, \\ & 0 = \hat{u}_1^d \le \hat{u}_2^d \le \dots \le \hat{u}_z^d = 1, \\ & 0 = \hat{u}_1^u \le \hat{u}_2^u \le \dots \le \hat{u}_z^u = 1, \\ & \hat{u}_l^d < \hat{u}_l^u,\ l=2, 3,\dots, z-1, \end{aligned} \tag{6}$$

$$\{(\hat{x}_{u_l}^d\ ;\hat{x}_{u_l}^u\ ; u_l) \mid l=1, 2,\dots, z\}, \text{ where}$$
$$\begin{aligned} & \hat{x}_{u_1}^d \le \hat{x}_{u_2}^d \le \dots \le \hat{x}_{u_z}^d, \\ & \hat{x}_{u_1}^d = \hat{x}_{u_1}^u \le \hat{x}_{u_2}^u \le \dots \le \hat{x}_{u_z}^u = \hat{x}_{u_z}^d, \\ & \hat{x}_{u_l}^d < \hat{x}_{u_l}^u,\ l=2, 3,\dots, z-1, \\ & 0 = u_1 < u_2 < \dots < u_z = 1. \end{aligned} \tag{7}$$

The use of an analytical approximated utility function is appropriate in cases of low number of subjectively elicited nodes or the presence of large uncertainty intervals. Let the analytical dependency approximating the utility function of a fuzzy rational (or real) DM has the following form:

$$u=u(x,\vec{\mathbf{p}}) \tag{8}$$

The parameter $\vec{\mathbf{p}}$ = ($p_1, p_2, p_3,\dots, p_n$) is an n-dimensional vector of unknown parameters belonging to the n-dimensional set Π and defining the form of the utility function.

From $x_i \succ x_j \Leftrightarrow x_i > x_j$, for $x_i \in X$, $x_j \in X$ follows that:

$$u(x_i, \vec{\mathbf{p}}) > u(x_j, \vec{\mathbf{p}}) \Leftrightarrow x_i > x_j,$$
$$\text{for } x_i \in [x_{worst}; x_{best}],\ x_j \in [x_{worst}; x_{best}] \text{ and } \vec{\mathbf{p}} \in \Pi. \tag{9}$$

The approximation nodes $u(x_{best}, \vec{\mathbf{p}})$ and $u(x_{worst}, \vec{\mathbf{p}})$ are error-free, so:

$$u(x_{worst}, \vec{\mathbf{p}})=0,\ u(x_{best}, \vec{\mathbf{p}})=1, \text{ for } \vec{\mathbf{p}} \in \Pi \tag{10}$$

If the elicited nodes are consistent with (6), the unknown parameters can be calculated through the weigthed least square method [Press, et al., 1992], where the deviation of the model from the best subjective estimate of a node is weighted by the width of the uncertainty interval of the utility quantile index . In [18] the following form is proposed:

$$\chi_u^2 = \sum_{l=2}^{z-1} \left(\frac{2u(x_l, \vec{\mathbf{p}}) - \left(\hat{u}_l^d + \hat{u}_l^u\right)}{2\left(\hat{u}_l^u - \hat{u}_l^d\right)} \right)^2 \tag{11}$$

The optimal parameters $\vec{\mathbf{p}}_{\text{opt}}$ can be calculated using n-dimensional minimization of χ_u^2 on $\vec{\mathbf{p}}$:

$$arg\left\{ \min_{\vec{\mathbf{p}}} \left\{ \chi_u^2(\vec{\mathbf{p}}) \right\} \right\}. \tag{12}$$

From (9) it follows that an inverse function of (8) exists for which:

$$x=x(u, \vec{\mathbf{p}})= u^{-1}(x, \vec{\mathbf{p}}), \tag{13}$$

$$x(u_i, \vec{\mathbf{p}}) > x(u_j, \vec{\mathbf{p}}) \Leftrightarrow u_i > u_j, \text{ for } u_i \in [0; 1],\ u_j \in [0; 1] \text{ and } \vec{\mathbf{p}} \in \Pi \tag{14}$$

$$x(0, \vec{\mathbf{p}})=x_{worst},\ x(1, \vec{\mathbf{p}})= x_{best}, \text{ for } \vec{\mathbf{p}} \in \Pi \tag{15}$$

In cases where the elicited nodes are consistent with (7) a similar approach exists:

$$\chi_x^2 = \sum_{l=2}^{z-1} \left(\frac{2x(u_l, \vec{\mathbf{p}}) - \left(\hat{x}_{u_l}^d + \hat{x}_{u_l}^u\right)}{2\left(\hat{x}_{u_l}^u - \hat{x}_{u_l}^d\right)} \right)^2, \tag{16}$$

$$\vec{\mathbf{p}}_{\text{opt}} = arg\left\{ \min_{\vec{\mathbf{p}}} \left\{ \chi_x^2(\vec{\mathbf{p}}) \right\} \right\} \tag{17}$$

The calculation of optimal parameters will be facilitated if the inverse function is also analytical.

In [18] the following analytical dependency is proposed:

$$u(x) = \frac{\text{arctg}[a(x-x_0)] - \text{arctg}[a(x_1-x_0)]}{\text{arctg}[a(x_z-x_0)] - \text{arctg}[a(x_1-x_0)]} \tag{18}$$

The parameters a and x_0 allow the form of $u(.)$ to be adapted to different DM. Risk sensitivity is defined by the parameter a, while x_0 defines the inflex point of $u(.)$, which divides the prizes of the risk averse part (above x_0) and those of the risk prone part (under x_0). This form is referred to in the same source as *arctg-approximated utility function*. The following dependencies hold for the form (24):

$$\Pi = \{ (a, x_0) \mid a \in (0, \infty) \wedge x_0 \in (-\infty, \infty) \}, \quad (19)$$

where Π is the two-dimensional set of uncertain parameters a and x_0.

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{2a^2(x-x_0)}{1+a^2(x-x_0)^2} \qquad (20)$$

Of importance in this approach is the analytically inverse function of (18):

$$x(u) = \frac{\mathrm{tg}\{u \times \mathrm{arctg}[a(x_z - x_o)] + (1-u)\mathrm{arctg}[a(x_1 - x_0)]\}}{a} + x_0 \qquad (21)$$

whose existence significantly facilitates the calculation of the optimal parameters.

The arctg-approximation can be applied over a set of prizes X, consisting of both profits and losses, and its corresponding local risk aversion $r(.)$ represents the most common risk attitudes. The existence of an analytically inverse function of (21) allows for fast calculation of optimal parameters for the utility function. The analytical construction of the utility function allows for error filtering of the subjective measurement of utility, given that the chosen mathematical form is capable of correctly describing the attitude towards risk of the decision maker in question. If the optimal approximated curve passes through the intervals of uncertainty of the evaluated nodes of the utility function, there exists a possibility that the arctg-approximation will decrease the errors, because it uses correct prior information for the risk attitude. In case of great deviations of the optimal approximation curve from the intervals of uncertainty, the arctg-approximation should be replaced by another, because the attitude towards risk of the given DM is not typical.

In [14] the *power approximation* is proposed to build an utility function over a set of prizes consisting of both profits and losses. The power-approximated utility function has the following form:

$$u(x) = \frac{(x - x_d + x_0)^a - x_0{}^a}{(x_u - x_d + x_0)^a - x_0{}^a} \qquad (22)$$

where Π is defined as

$$\Pi = \{ (a, x_0) \mid a \in (0, \infty) \cap x_0 \in (0, \infty) \} \quad (23)$$

The parameter x_0 defines the deviation of the beginning of the coordinate system left of x_d. The parameter a, which represents the risk sensitivity, affects the sign of local risk aversion function:

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{1-a}{x - x_d + x_0} \qquad (24)$$

The power-approximated utility function also has an analytical inverse function:

$$x(u) = \sqrt[a]{u(x_u - x_d + x_0)^a + (1-u)x_0{}^a} + x_d - x_0 \quad (25)$$

In [17] arctg-approximation and power approximation are compared on the basis of an empirical study, in which participants constructed their utility function in the interval of monetary prizes using different elicitation techniques. It proved that arctg-approximation better describes the data than power approximation since it flexibly accounts for the risk attitude of the decision maker according to the prizes.

In [18] an empirical experiment has been conducted, in which 104 decision makers have elicited 9 inner nodes from a one-dimensional utility function over monetary prizes using three elicitation methods. Due to the fuzzy rationality of real DMs, the resulting estimates are in the form of uncertainty intervals. Those are used in a weighted least square method to find the parameters and to build a utility function of the form (18) or (21) for each elicitation method and for each DM. The resulting 312 utility functions are constructed another three times, using respectively 3, 4 and 5 inner elicited nodes. The analysis empirically proves that 5 inner nodes are sufficient to provide satisfactory approximation of the utility function that reflects typical risk attitude of the DM.

4. CONCLUSIONS

This paper discussed the modern analytical approximation functions and their appropriate use. It presented the arctg-approximated utility function and its advantages over previously proposed approximated functions. This approximation requires few parameters, as well as a low number of subjectively elicited points of the utility function's graphic, simplifying and facilitating the process of utility function building.

Al comments presented here hold for the case of monotonic utility functions over monetary prizes. The analysis follows the same path in the case of non-monetary prizes. It must also be pointed out that most utility functions are not monotonic, since each real-life parameter has an optimum value, which is most preferred, while deviating from it leads to decrease in preferences. Yet, each decision is analyzed within a certain context (the part of the world, over which the influence of the decision is investigated). This shortens the parameter intervals and makes the utility function monotonic.

It is also important to point out at existing computer programs to assess utility functions of different forms that are consistent with various input data specifying both qualitative and quantitative characteristics of the utility function (see [14,19]).

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