TOOLS FOR DETECTING CHAOS

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Abstract – In this study, useful tools for detecting chaos are explained. To observe the state variables (time series), the phase portrait, the Poincare map, the power spectrum, the Lyapunov exponents and bifurcation diagram are used to detect chaos in dynamical systems. In this paper, the driven pendulum was choosen indicating chaos with the help of these tools. The existence of chaos in driven pendulum was shown with all methods. Simulation results obtained from all tools agree with each other. Also, control of chaos in driven pendulum was realized in this study

Keywords - Chaos, Phase Portrait, Poincare Map, Lyapunov Exponent, Bifurcation Diagram

Özet - Kaosun gösterilmesi için yararlı araçlar bu çalışmada açıklanmıştır. Durum değişkenlerinin gözlenmesi (zaman serileri), faz portresi, Poincare haritası, güç spektrumu, Lyapunov üsteli ve çatallaşma diyagramı, dinamik sistemlerde kaosun gösterilmesi için kullanılmaktadır. Bu çalışmada, bu araçlar yardımıyla kaosun gösterilmesi için sürülen sarkaç örneği seçilmiştir. Sürülen sarkaç'da kaosun varlığı tüm yöntemlerle gösterilmiştir. Tüm yöntemlerle elde edilen benzetim sonuçları birbirleriyle uyuşmaktadır. Ayrıca, sürülen sarkaçtaki kaosun kontrolüde bu çalışmada gerçekleştirilmiştir.

Anahtar Kelimeler - Kaos, Faz Portreleri, Poincare Harita, Lyapunov Üsteli, Çatallaşma Diyagramı

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1. INTRODUCTION

The irregular and unpredictable time evolution of many nonlinear systems has been called chaos. Chaos occurs in many nonlinear systems. Main characteristic of chaos is that system does not repeat its past behavior.

In spite of their irregularity, chaotic dynamical systems follow deterministic equations [1]. The unique characteristic of chaotic systems is dependence on the initial conditions sensitively. Slightly different initial conditions result in very different orbits. But in nonchaotic systems, these differences result in nearly same orbits. In continuous time systems, chaos may occur in system having at least three independent dynamical variables, also system must be nonlinear. These are necessary conditions for detecting chaos in continuous time systems. If system has sensitivity to initial conditions, then we say that system is chaotic. After all of these, the exact definition of chaos is that: A chaotic system is a deterministic system that exhibits random and unpredictable behavior. The defining feature of chaotic system is their sensitive dependence on initial conditions [2].

The asymptotic behavior of autonomous dynamic system is uniquely specified by their initial conditions. The following are four possible types of equilibrium behaviours:

- An equilibrium point,
- A limit cycle,
- A torus
- Chaos.

There are various methods for detecting chaos. We point out most useful ones in this paper. These are:

- Time series (Observe the state variables),
- Phase portraits,
- Poincare maps,
- Power spectrum,
- Lyapunov exponents,
- Bifurcation diagram.

There are other methods for detecting chaos (Lyapunov dimension, correlation dimension, entropy and etc.) but they are rarely used, because they are difficult for detecting chaos in practical systems.

Driven pendulum is a practical system for exploring nonlinear dynamics. First, Galileo noticed its characteristic. Since Galileo has discovered it, pendulum is appropriate system for investigate and teach nonlinear dynamics. So, it is useful for detecting chaos. Figure.1 shows a simple pendulum.

Control of chaotic systems is of current interest and several techniques have been proposed. One fundamental property of chaos is the occurrence of dense orbits. When a chaotic system is to be controlled, the periodic orbit is used such that the desired trajectory of the controlled system is given as a periodic orbit of the chaotic system [3].

In our research, tools for detecting chaos are given and we have applied the Pyragas control method to a chaotic pendulum.

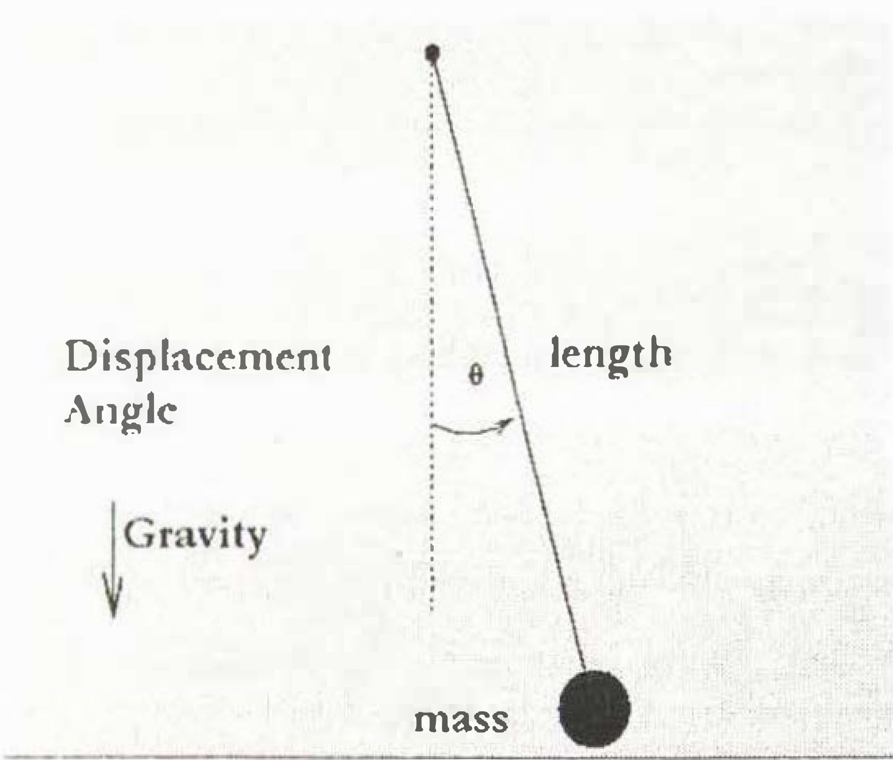


Figure.1 Simple Pendulum

2. DRIVEN PENDULUM

A driven damped pendulum is described by the equation:

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \omega_0^2 I\sin\theta = T\sin\omega_f t \qquad (1)$$

where θ is position of pendulum, I is inertia torque, b is friction parameter, ω_0 is resonance frequency, ω_f is angular velocity of drive, T is magnitude torque of drive. For easiness, we need dimensionless equation [4]. The dimensionless equation is:

$$\frac{d^2\theta}{dt'^2} + \frac{1}{q}\frac{d\theta}{dt'} + \sin\theta = g\cos\omega_D t' \qquad (2)$$

where $t' = \omega_0 t$, $q = \frac{\omega_0 I}{b}$, $g = \frac{T}{\omega_0^2 I}$, $\omega_D = \omega_f/\omega_0$.

We can split a second order differantial equation to two first order equation for solving with computational method (Equation 3.a and 3.b). ω is the angular velocity. Pendulum is a non autonomous system because of external driving force. In chaotic analysis, an nth-order non autonomous system can always be converted into an (n+1)th-order autonomous system by appending an extra state (Equation 3.c) [4]. The final equations are below:

$$\frac{d\omega}{dt} = -\frac{\omega}{q} - \sin\theta + g\cos\phi \qquad (3.a)$$

$$\frac{d\theta}{dt} = \omega \qquad (3.b)$$

$$\frac{d\phi}{dt} = \omega_D \qquad (3.c)$$

For various g values, pendulum exhibits different behaviors. For example at g=1, periodic behavior, g=1.4, period-2 behavior, and for g=1.2, chaotic behavior [1].

3. TOOLS

a. Time Series (Trajectory Plot)

This method is easiest one and it is a visual method. In this method, the state variables of the system are observed and if they exhibit irregular or unpredictable behavior, then it is called chaotic. Otherwise (fixed point, periodic and quasi periodic) it is called nonchaotic. Figure.2.a, 2.b and 2.c show the time series of periodic, period-2 and chaotic behaviors of pendulum respectively.

b. Phase Portraits

Phase portarit is a two-dimensional projection of the phase-space [2]. It represents each of the state variable's instantaneous state to each other. For pendulum example, angular velocity-position graph is a phase portrait. Chaotic and other motions can be distinguished visually from each other according to the Table.1. A fixed point solution is a point in a phase portrait. A periodic solution is a closed curve in phase portrait. Chaotic solutions are distinct curves in phase portrait. Periodic and chaotic phase portrait examples are shown in Figure 3. Figure 3.a shows periodic behavior, Figure 3.b shows chaotic behavior.

Table.1 Solutions of dynamic systems (Phase Portrait)

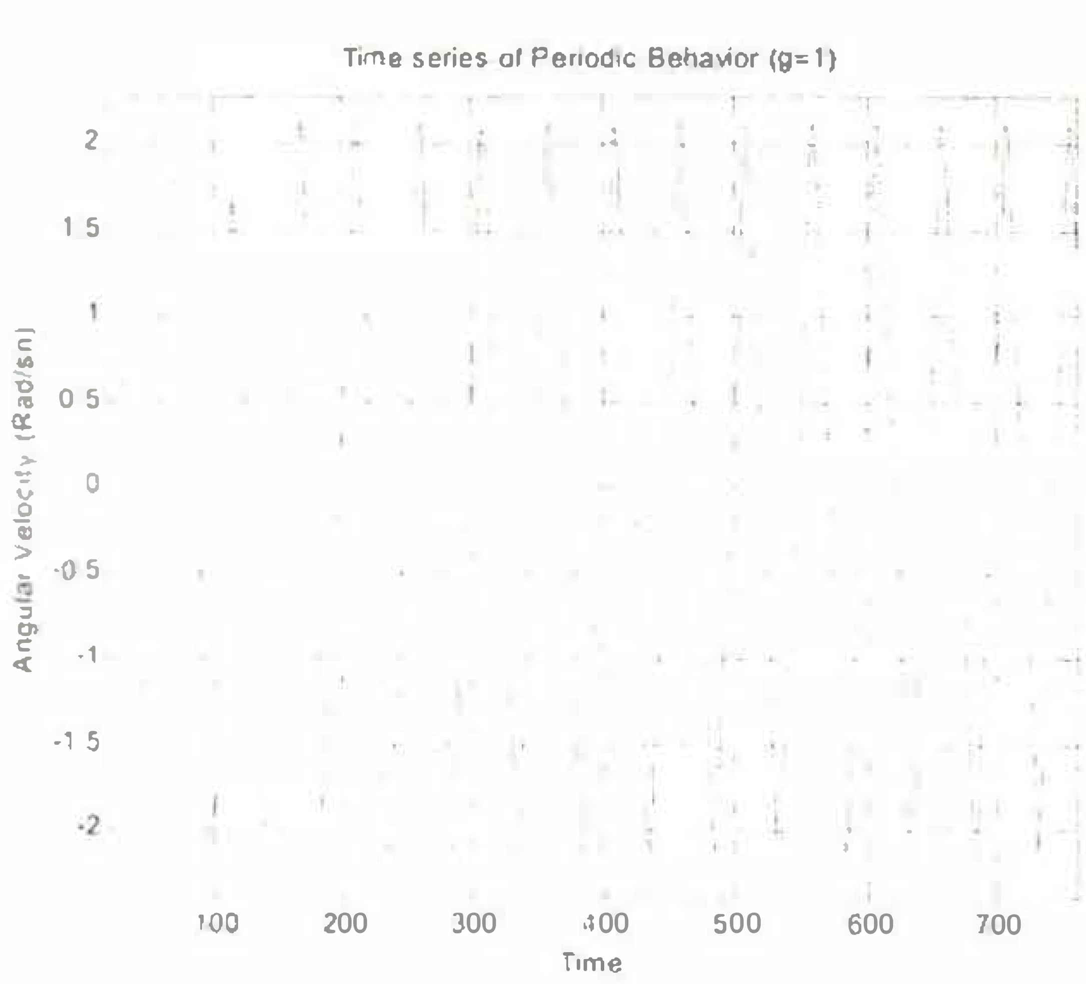
Solution	Fixed	Periodic	Quasi Periodic	Chaos
Phase portrait	Point	Closed curve	Torus	Distinct shapes

c. Poincare Maps

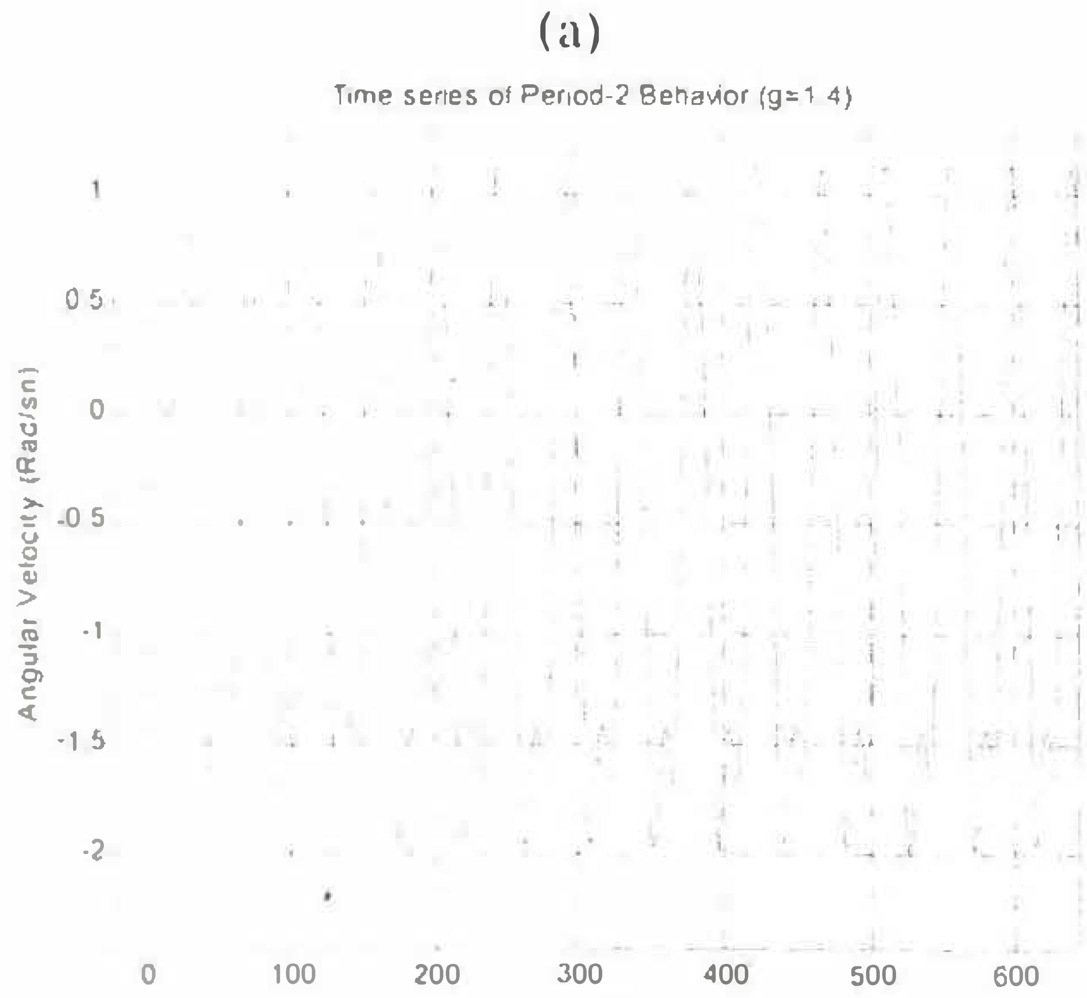
Another method is the Poincare map. The basic is that n^{th}-order continuous time system is replaced with $(n-1)^{th}$-order map. It is constructed by sampling the phase portrait stroboscopically. Its aim is to simplify the complicated systems, and it is useful for stability analysis [5]. Chaotic and other motions can be distinguished visually from each other according to the Table.2. Periodic behavior is a fixed point in Poincare map. A quasi periodic behavior is closed curve or points in Poincare maps. Distinct set of points indicate the chaos in Poincare map. In practice, one chooses an $(n-1)$ dimensional Poincare surface Σ, which divides R^n into two regions. If Σ is choosen properly, then the trajectory under observation will repeatedly pass through Σ. The set of these crossing points is a Poincare map. Figure 4 shows the basic idea of Poincare Map. Figure 5.a and 5.b show the Poincare map of periodic and chaotic behavior, respectively.

Table.2 Solutions of dynamic systems (Poincare Maps)

Solution	Fixed	Periodic	Quasi Periodic	Chaos
Poincare Maps	-	Point	Closed Curve	Distinct points



(a)



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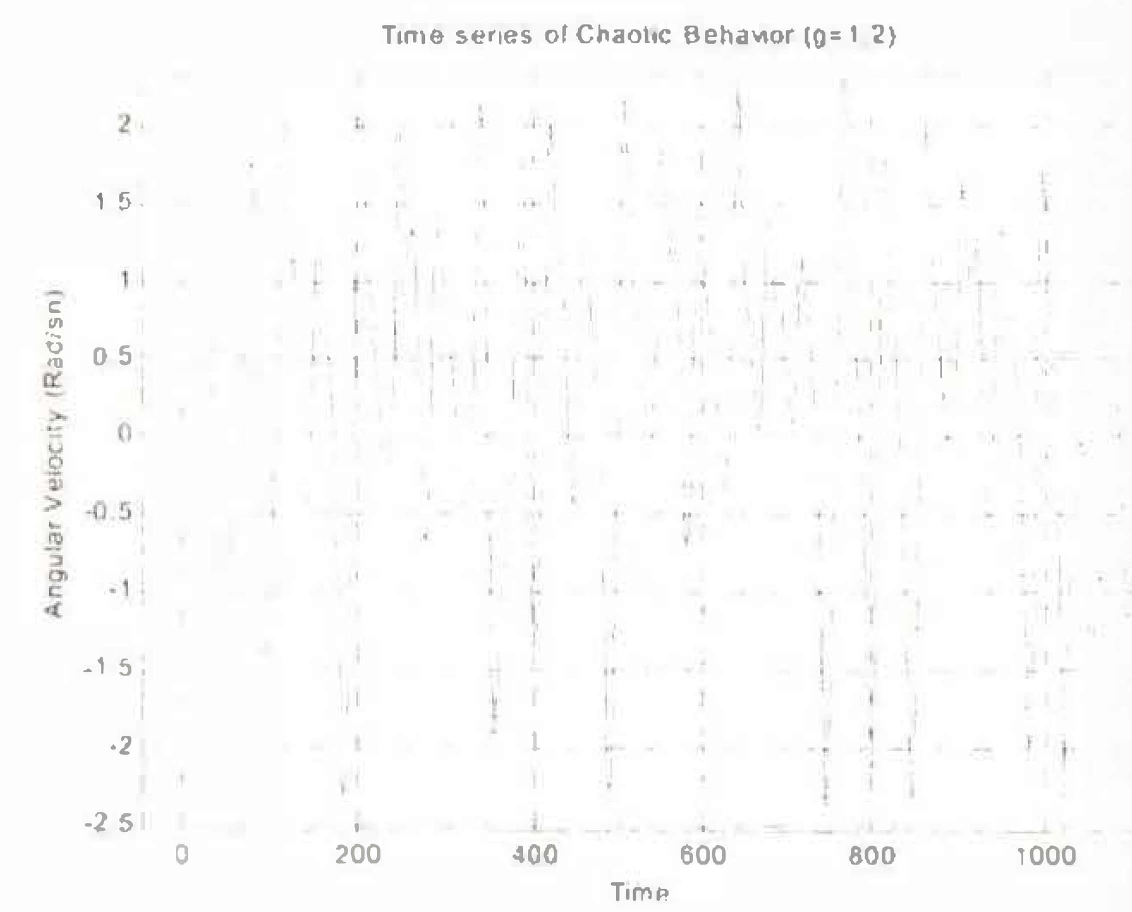
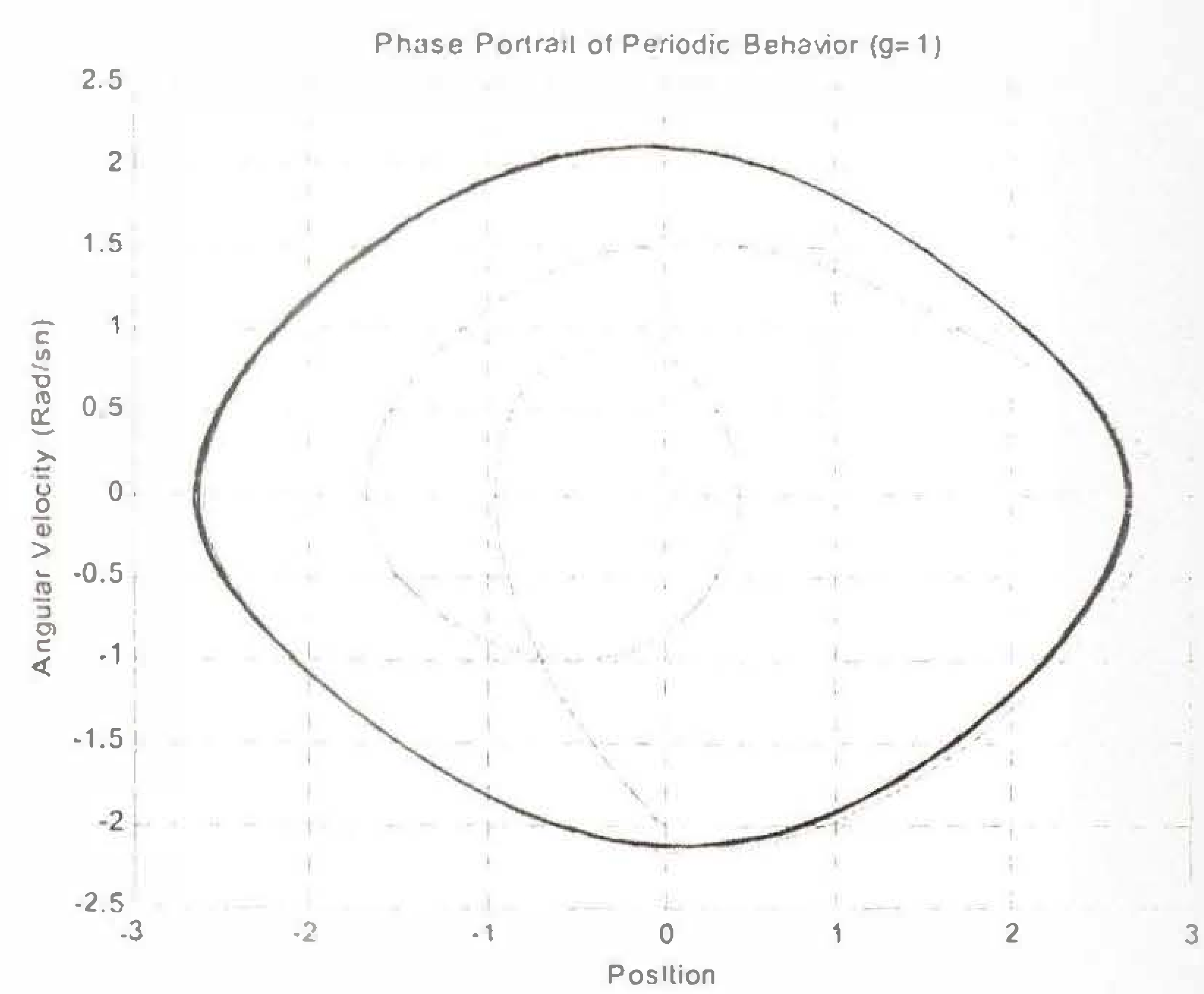
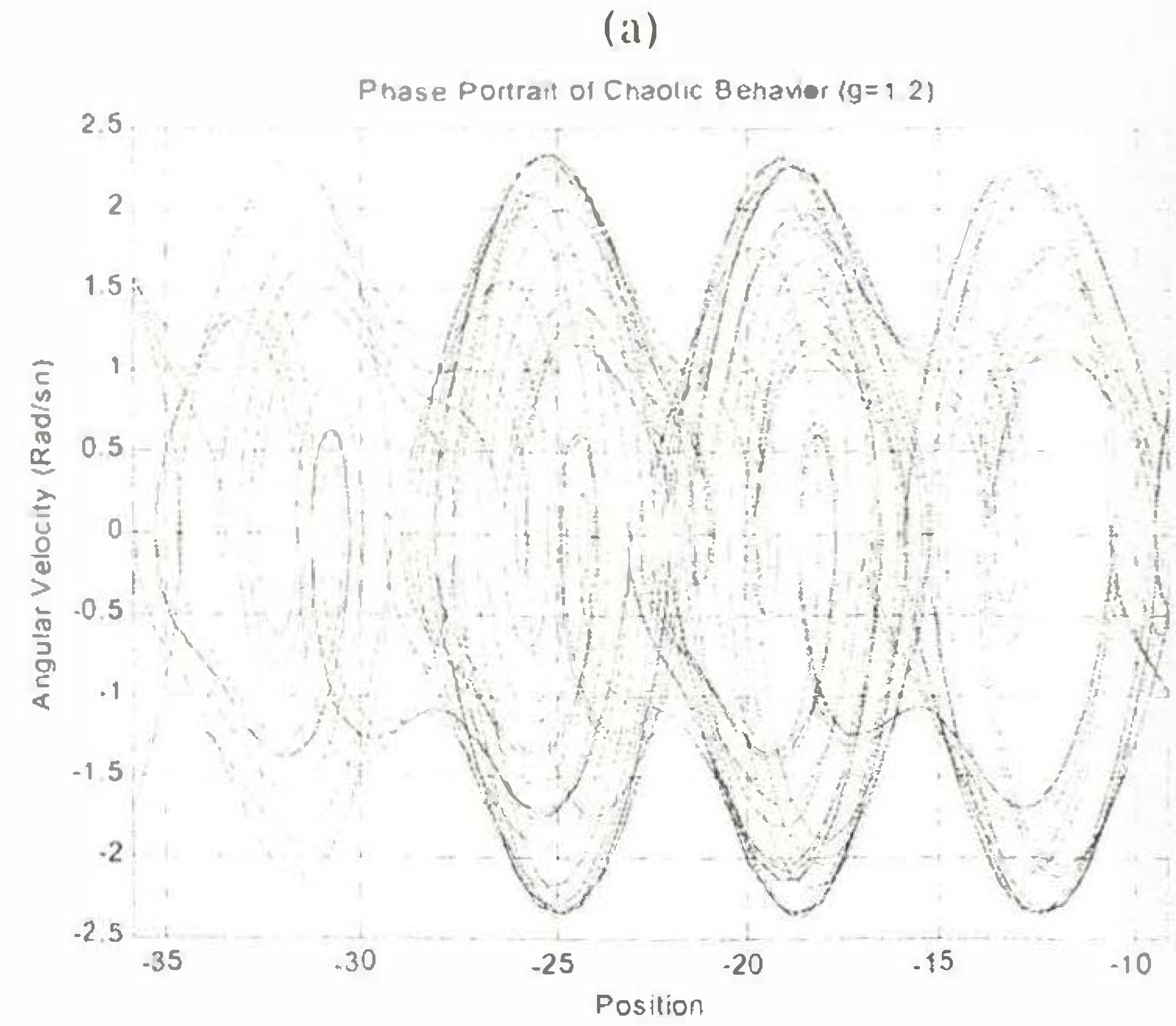


(c)

Figure. 2 Time series of a state variable (angular velocity) of pendul (a) periodic behavior (g=1), (b) period-2 behavior (g=1.4), (c) cha behavior (g=1.2).



(a)



(b)

Figure 3. Phase portraits of pendulum (Angular velocity-position) periodic behavior (g=1) (b) chaotic behavior (g=1.2).

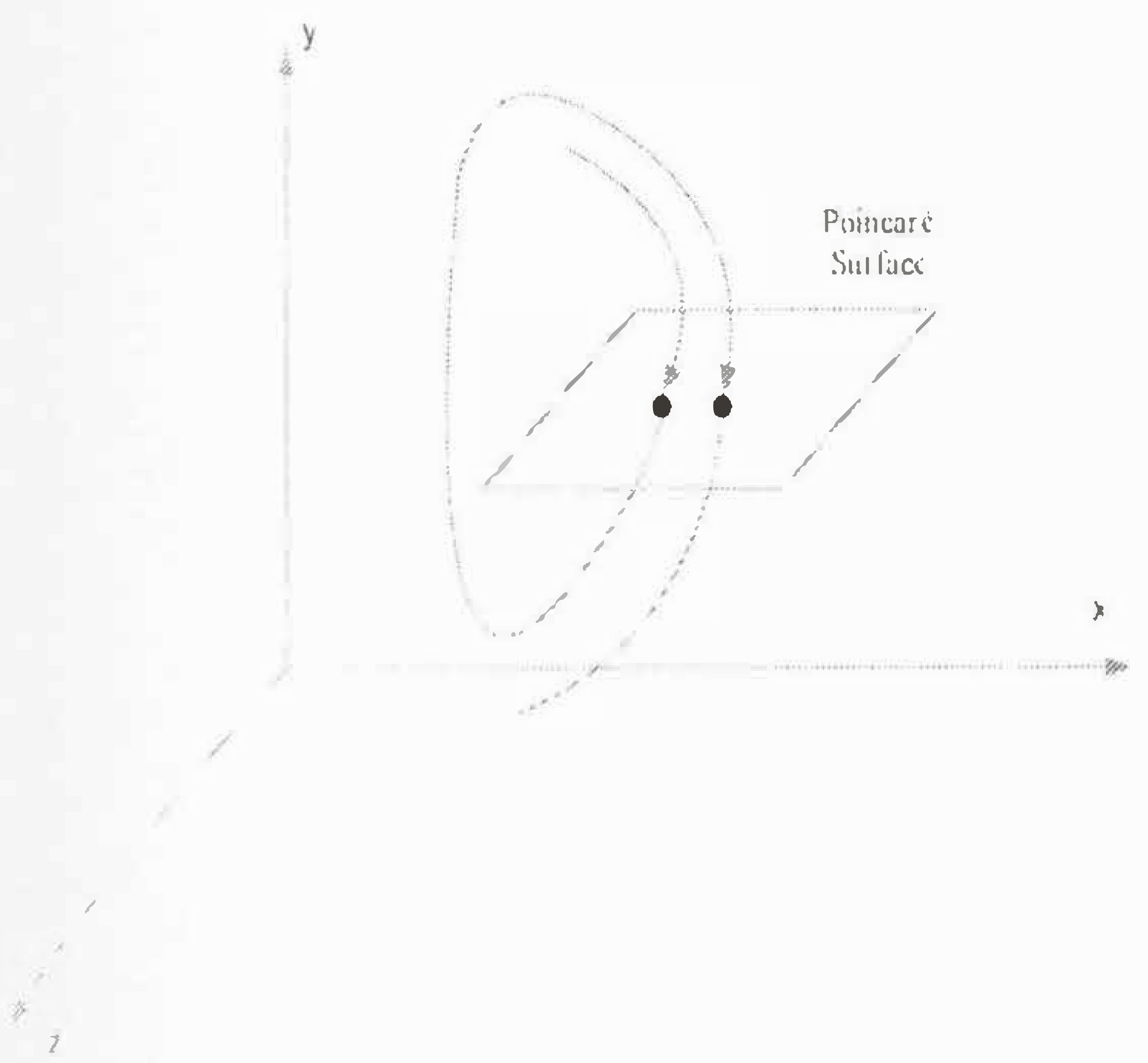
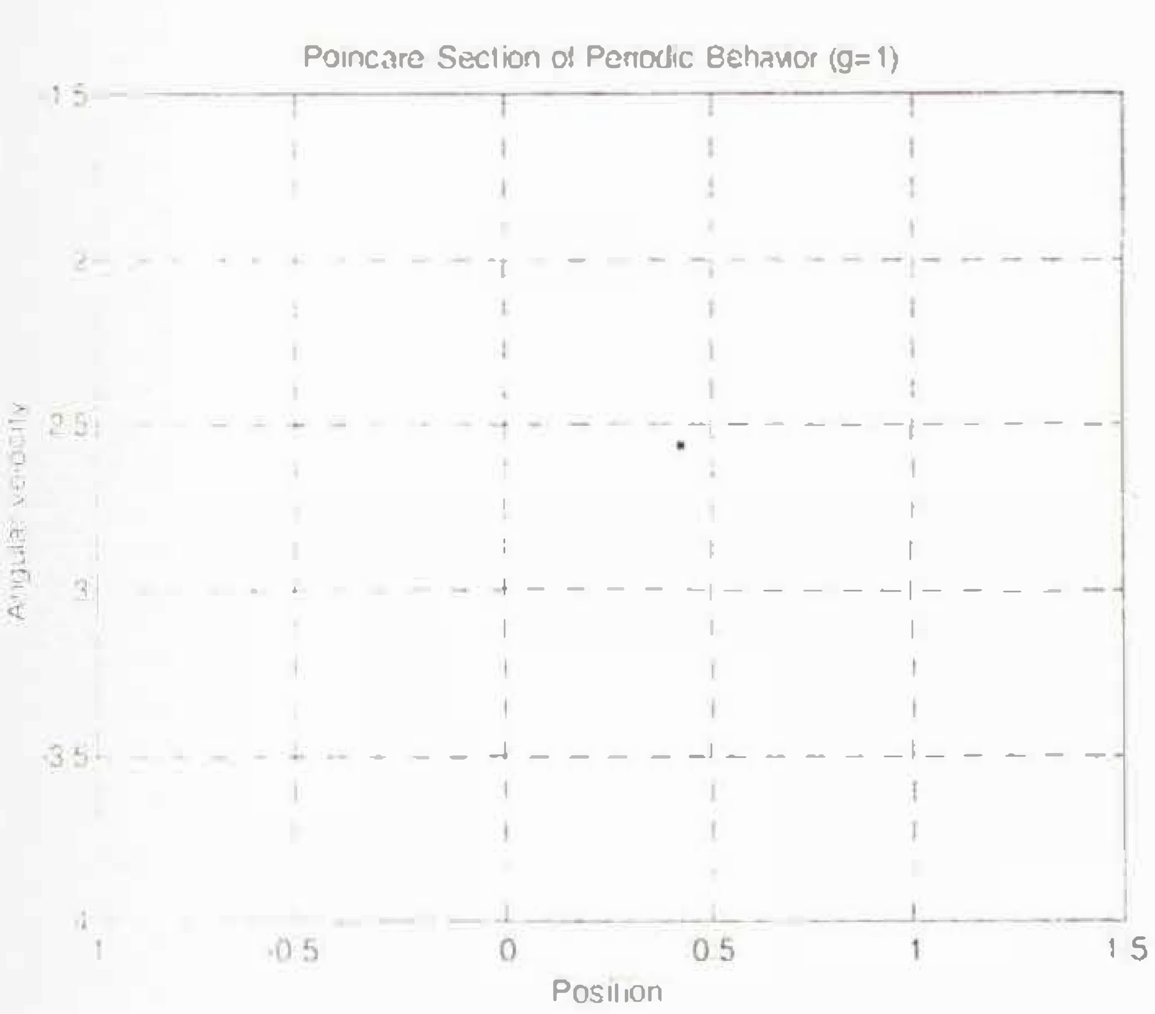
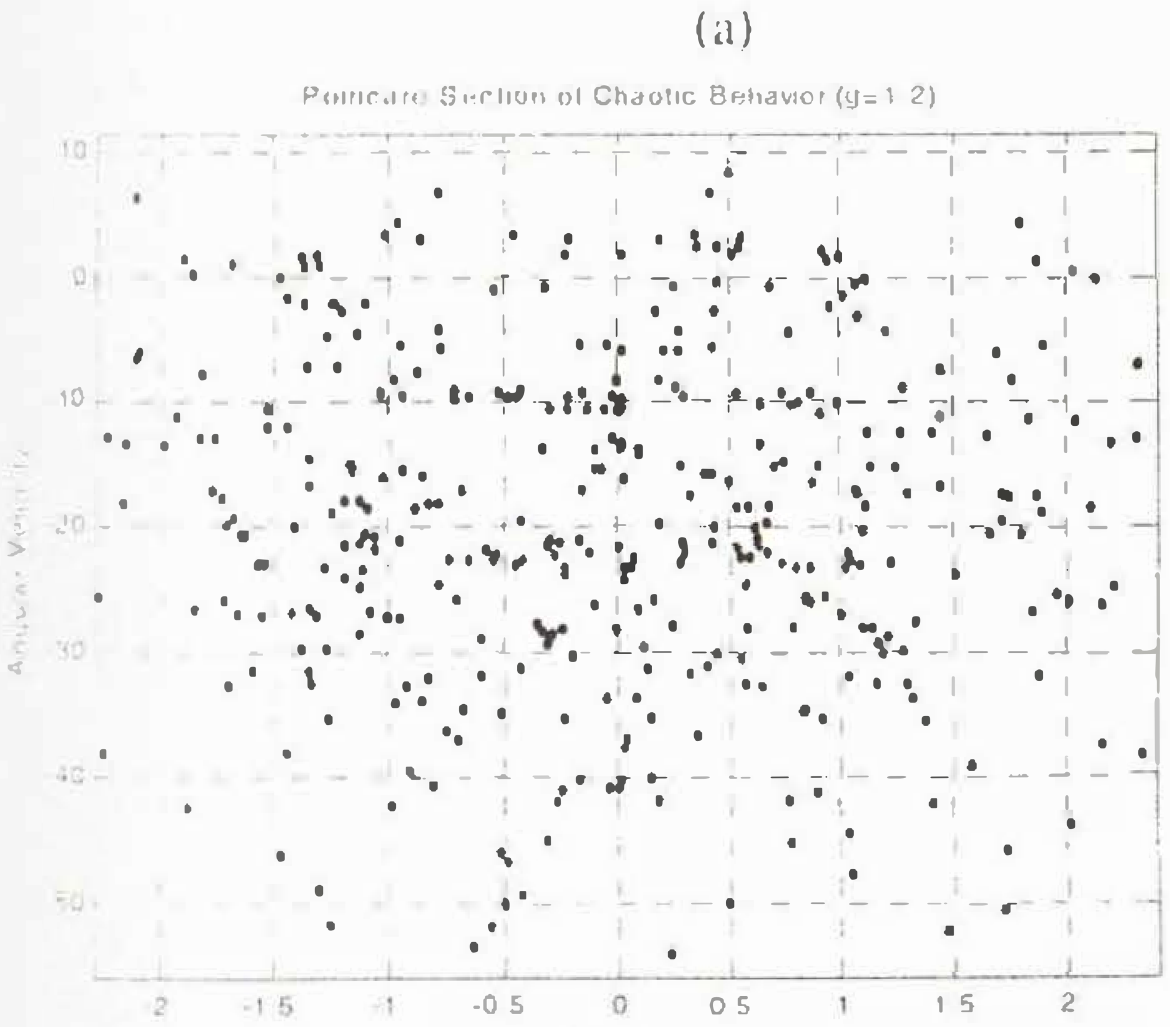


Figure. 4 Basic of Poincare Map



(a)



(b)

Figure. 5 Poincare maps of pendulum (Angular velocity-Position) (a) periodic behavior (g=1). (b) chaotic behavior (g=1.2).

d. Power Spectrum

Chaotic signals are wideband signals, so they can be easily distinguished from periodic signals by looking at their frequency spectra. Figure 6.a and 6.b show the power spectrum of periodic behavior and chaotic behavior, respectively. If the behavior is chaotic, then power spectra of system is expressed in terms of oscillations with a continuum of frequencies.

e. Lyapunov Exponents

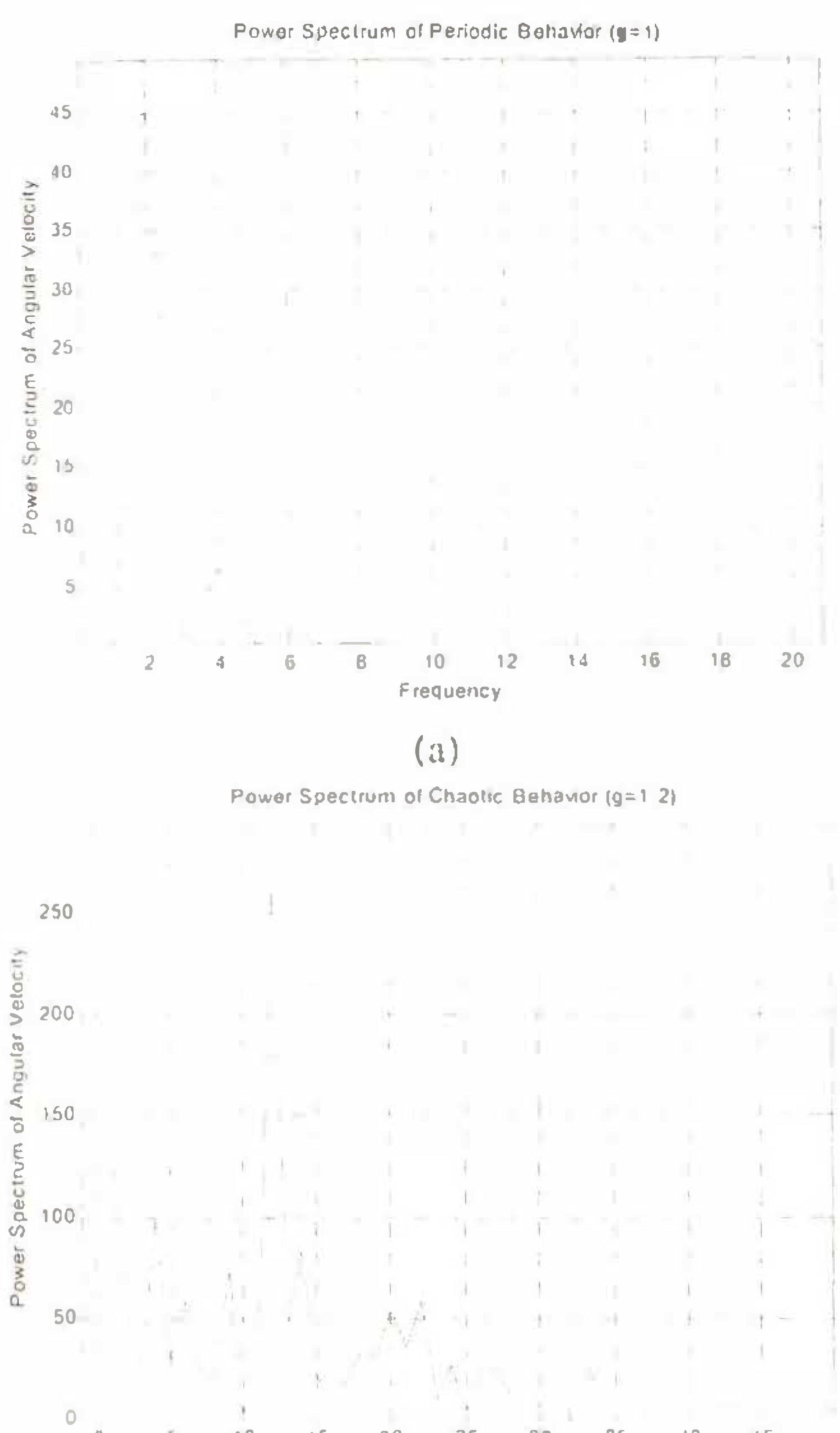
The Lyapunov exponents (λ_i i=1,2,..n) are the numbers that measure the exponential attraction or separation in time of two adjacent orbits in the phase space with close initial conditions. n dimensional system has n Lyapunov exponents. If original system is non autonomous as forced pendulum, one Lyapunov exponent is zero. If the system has at least one positive Lyapunov exponent, it indicates the chaos. If the largest Lyapunov exponent is negative then the orbits converge in time and system is insensitive to initial conditions. If it is positive, then the distance between adjacent orbits grows exponentially and system exhibits sensitive dependence on initial conditions, so it is chaotic we say. Main idea of Lyapunov exponent is below:
Suppose x is a point at time t, and consider a nearby point, say $x+\delta$, where δ is a tiny separation vector of initial length. In numerical studies, one Lyapunov finds as $\delta = \delta_0 e^{\lambda t}$. If $\lambda > 0$, neighboring trajectories separate exponentially rapid. So positive λ indicates the sensitivity to initial conditions.

Assume an initial condition x_0 is chosen arbitrarily. The Lyapunov exponents are

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln|m_i(t)| \qquad (4)$$

$i = 1,2,\ldots,n$ and m be the eigenvalues of Jacobian matrix of system [5].

Lyapunov exponents of pendulum for periodic and chaotic behaviors are shown in Figure. 6.a and 6.b respectively. The pendulum is 3rd-order so, it has three exponents. One of them is zero because of equation 3.c. as seen. In Figure 7.a system is periodic, so exponents are negative. But in Figure 7.b one exponent is positive, so it is chaotic [Matlab/LET Toolbox] [6].



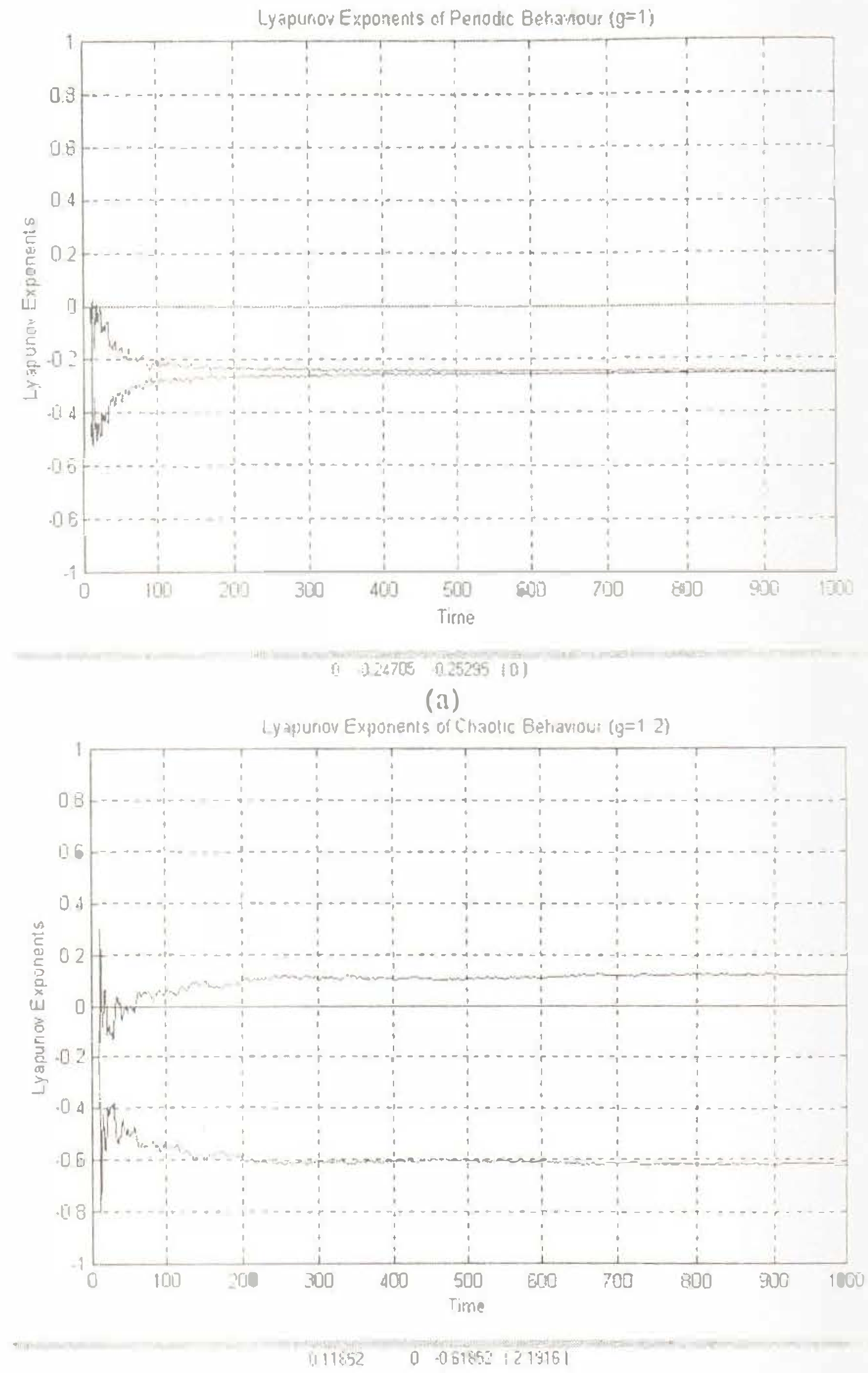
(a)

(b)

Figure. 6 Power spectrum of pendulum (Angular velocity) (a) periodic behavior (g=1), (b) chaotic behavior (g=1.2).

f. Bifurcation Diagram

Phase portraits, Poincare maps, time series, power spectrum and lyapunov exponents provide information about the dynamics of the pendulum for specific values of the parameters (g, q, ω_D). The dynamics may also be viewed more globally over a range of parameter values, thereby allowing simultaneous comparison of periodic and chaotic behavior. For some values of the parameters, a pendulum will have only one long term motion, while for other slightly different choices; two or more motions may be possible. If several of them are stable, the actual behavior will be depending on initial conditions. In differential equations, if a change in the number of solution is depending on parameter variation, it is called bifurcation [1].



(a)

(b)

Figure. 7 Lyapunov exponents of pendulum for (a) periodic behavi (g=1), (b) chaotic behavior (g=1.2)

For the pendulum, bifurcations can be easily detected b examining a graph of ω versus the drive amplitude g Several examples of these graphs called bifurcatio diagrams. A bifurcation diagram of pendulum is shown i Figure. 8 [7]. As shown in Figure.8, the interval $[0,9-1$ in horizontal axis (g), exhibits periodic behavior, interva $[1-1.1]$ and $[1.3-1.4]$ exhibit period-2 behavior, an intervals $[1.13-1.3]$ and $[1.7-1.8]$ exhibit chaoti behavior.

4. CONTROL OF CHAOS IN DRIVEN PENDULULUM

Control of chaos is an important task for scientist. Ther are a few methods for chaos control: OGY method, Pyraga methods (two methods), taming chaos method and etc. [Pyragas methods use continuous time feedback [8]. Th block diagrams of these methods are shown in Fig.9.

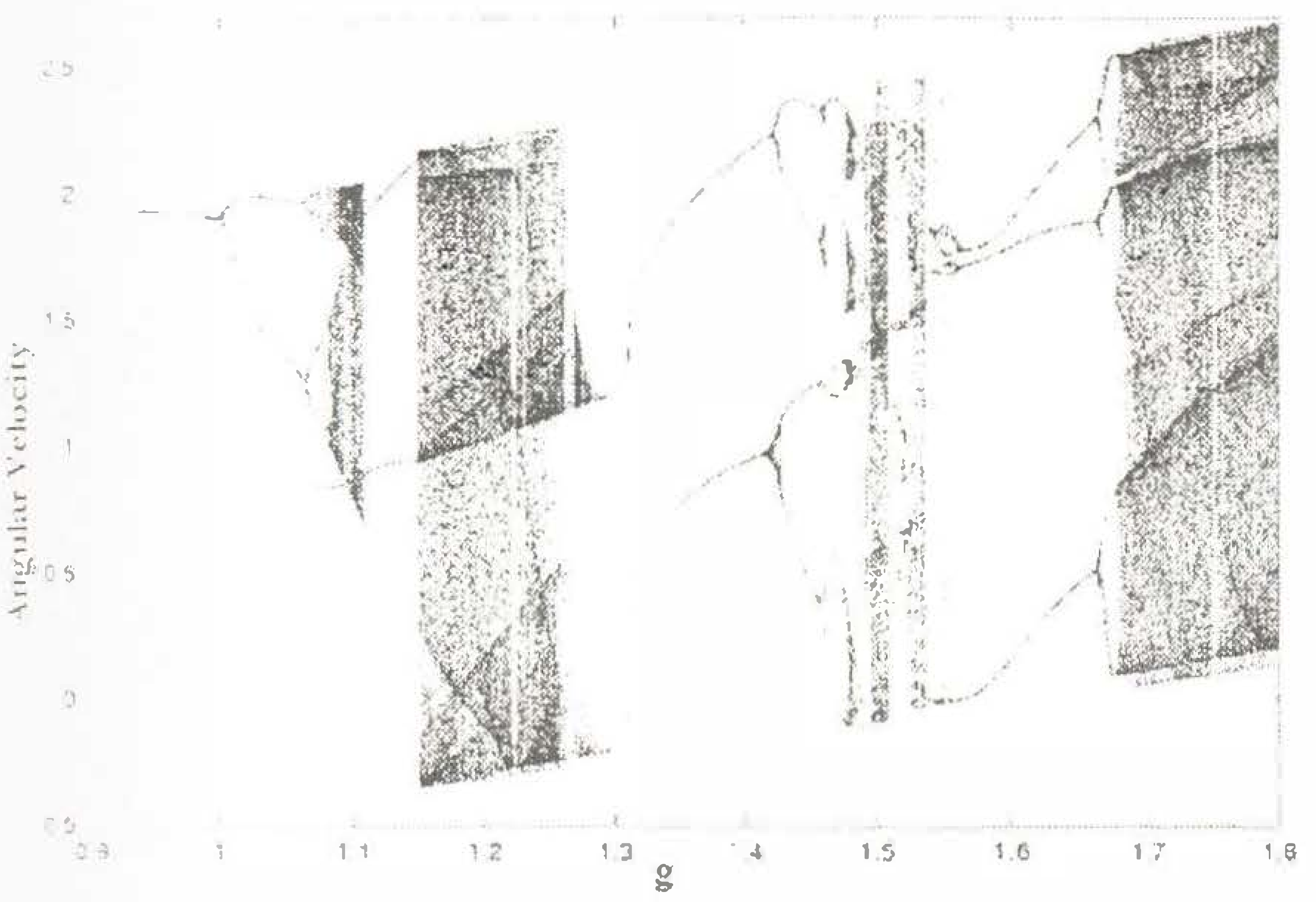


Figure. 8 Bifurcation diagram of pendulum

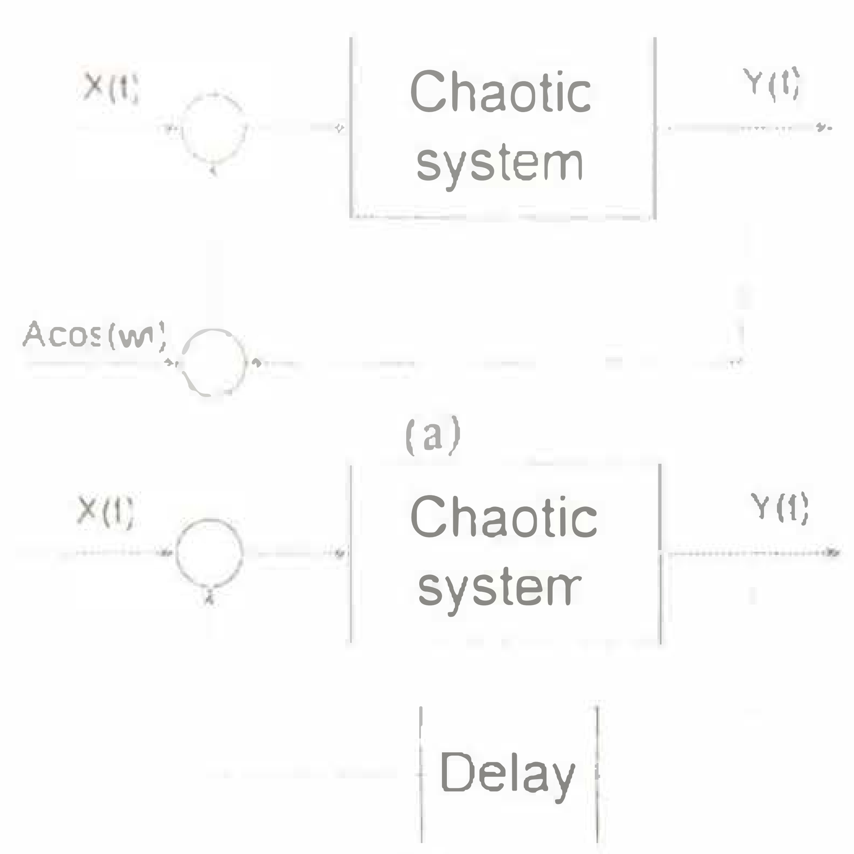


Figure. 9 Block diagram of Pyragas methods.

In this study, Pyragas control method (second one) is applied to chaotic pendulum (g=1.2). Chaotic behaviour of pendulum is shown in Figure 10.a. After control, behaviour of controlled pendulum is shown in Figure 10.b

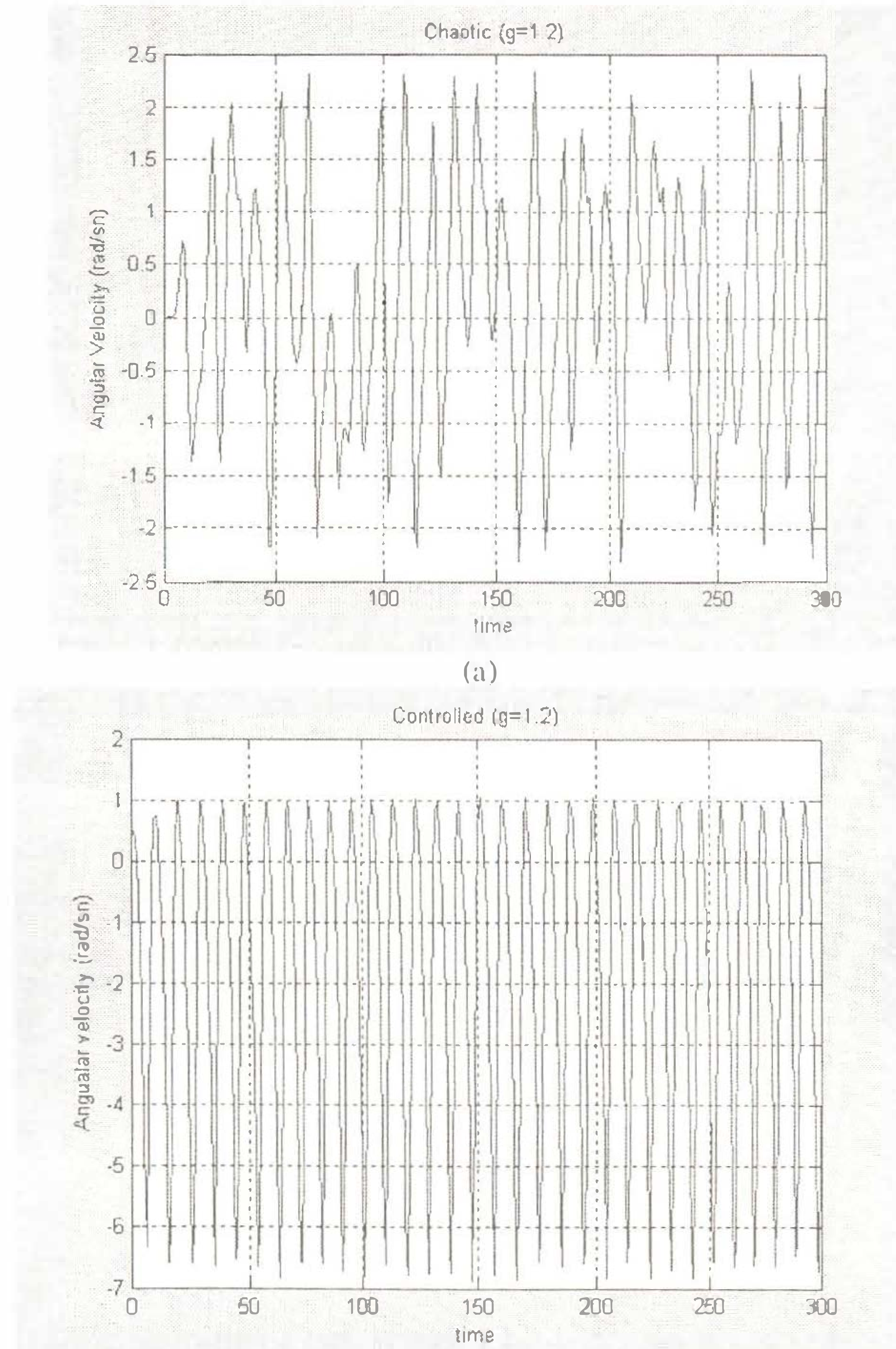


Figure. 10 Behaviour of chaotic pendulum (g=1.2) (a)- Before control (b)- After control

5. CONCLUSIONS

A non autonomous dynamical system (driven pendulum) has been simulated by MATLAB. Driven pendulum has different behaviours, when g=1, driven pendulum exhibits periodic behavior, and at g=1.2 it exhibits chaotic behavior. So, tools for detecting chaos are used in this system. All these results of methods are in full agreement and confirm the correctness of these methods. Also, control of chaos was implemented in this study.

6. REFERENCES

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