ON IDEMPOTENCY OF LINEAR COMBINATIONS OF TWO COMMUTE IDEMPOTENT MATRICES

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Özet - $\mathbf{P}_1$ ve $\mathbf{P}_2$ komutatif idempotent matrislerinin lineer kombinasyonunun da bir idempotent matris olduğu tüm durumları karakterize etme probleminin tam bir çözümü ortaya konulmaktadır. Ayrıca, bu çalışmada ele alınan idempotentlik probleminin bir istatistiksel yorumu da verilmektedir.

Anahtar kelimeler- Köşegenleştirme, Minimal polinom, Eğik izdüşüm, Ortogonal izdüşüm, Kuadratik form, Ki-kare dağılımı

Abstract - A complete solution is established to the problem of characterizing all situations, where a linear combination of two commute idempotent matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ is also an idempotent matrix. A statistical interpretation of the idempotency problem considered in this note is also pointed out.

Keywords- Diagonalization, Minimal polynomial, Oblique projector, Orthogonal projector, Quadratic form, Chi-square distribution

I.INTRODUCTION

It is assumed throughout that c_1, c_2 are any nonzero elements of a field $\Im$ and $\mathbf{P}_1$, $\mathbf{P}_2$ are two different nonzero commute idempotent matrices over $\Im$. The symbols γ_1, γ_2 and

$\mathbf{Q}_1$, $\mathbf{Q}_2$ are used instead of c_1, c_2 and $\mathbf{P}_1$, $\mathbf{P}_2$ when considerations are concerned with complex scalars and matrices.

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In fact, the main aim of this paper is to establish an alternative proof of one part of theorem recently obtained by Baksalary and Baksalary in [6], which deals with the operation of combaning linearly $\mathbf{P}_1$ and $\mathbf{P}_2$ preserves the idempotency property. However, the proof given should be of interest because of the fact that characteristics roots and vectors, and polynomials are useful in statistics. Three such situations are known in the literature, viz., if the combination is either the sum $\mathbf{P}_1+\mathbf{P}_2$ or one of the differences $\mathbf{P}_1-\mathbf{P}_2$, $\mathbf{P}_2-\mathbf{P}_1$, and appropriate additional conditions are fulfilled; cf. Theorem 5.1.2 and 5.1.3 in [4]. The solution obtained asserts that these three situations exhaust the list of all possibilities when attention is restricted to commute idempotent matrices, or complex idempotent matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, and γ_1 and γ_2 complex numbers such that $\mathbf{Q}_1-\mathbf{Q}_2$ is Hermitian.

Quadratic forms with idempotent matrices are used extensively in statistical theory. For this reason, the idempotency problem considered in this note admits a statistical interpretation due to the fact that it a random $n \times 1$ vector $\mathbf{y}$ has a multivariate normal density with covariance matrix equal to $\mathbf{I}$, where $\mathbf{I}$ stands for the identity matrix, then the quadratic form $\mathbf{y}'\mathbf{A}\mathbf{y}$ has a noncentral chi-square density if and only if $\mathbf{A}$ is an idempotent matrix; this is an important result in the analysis of variance; cf. Theorem 5.1.1 in [3] or Lemma 9.1.2 in [4]. (Also see in [8] for details.)

II.MAIN RESULT

As alraedy pointed out the main result deals with the alternative proof of the one part of the theorem in [6] using diagonalization of matrices. Before giving the main result we note that every idempotent matrix is diagonalizable, and has only the eigenvalues 1 and 0.

Let us first give the following lemma.

Lemma. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two commute nxn idempotent matrices and $\mathbf{P}$ be their linear combination of the form

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2$$

with nonzero scalars c_1 and c_2. Then $\mathbf{P}$ is diagonalizable.

Proof. First of all, note that an idempotent matrix is diagonalizable. Since $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent and commute, they are simultaneously diagonalizable (see, e.g., [7], pp.52). Hence there is a single similarity matrix $\mathbf{S}$ such that $\Lambda = \mathbf{S}^{-1}\mathbf{P}_1\mathbf{S}$ and $\mathbf{M} = \mathbf{S}^{-1}\mathbf{P}_2\mathbf{S}$ are diagonal matrices. In addition , their diagonal entries are the eigenvalues of $\mathbf{P}_1$ and $\mathbf{P}_2$ with proper multiplicities. Thus we get

$$\mathbf{P}_1 = \mathbf{S}\Lambda\mathbf{S}^{-1} \text{ and } \mathbf{P}_2 = \mathbf{S}\mathbf{M}\mathbf{S}^{-1} \qquad (1)$$

and hence

$$\mathbf{P} = \mathbf{S}[c_1\Lambda + c_2\mathbf{M}]\mathbf{S}^{-1}.$$

This completes the proof.

Now let us give the thoerem.

Theorem. Given two different nonzero commute idempotent matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, let $\mathbf{P}$ be their linear combination of the form

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 \qquad (2)$$

with nonzero scalars c_1 and c_2. Then there are exactly three situations, where $\mathbf{P}$ is an idempotent matrix:

(a) c_1=1, c_2=1, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$;
(b) c_1=1, c_2=-1, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$;
(c) c_1=-1, c_2=1, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$.

Proof. Since $\mathbf{P}_1$ and $\mathbf{P}_2$ commute, they are simultaneously diagonaziable. Assume that $\mathbf{S}$ is invertible matrix, which simultaneously diagonalizes $\mathbf{P}_1$ and $\mathbf{P}_2$. Thus, we may write $\mathbf{P}_1$ and $\mathbf{P}_2$ as in (1). So, we have

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 = c_1\mathbf{S}\Lambda\mathbf{S}^{-1} + c_2\mathbf{S}\mathbf{M}\mathbf{S}^{-1}$$
$$= \mathbf{S}\Sigma\mathbf{S}^{-1}$$

where $\Sigma = c_1\Lambda + c_2\mathbf{M}$. Hence, direct calculations show that $\mathbf{P}$ of the form (2) is idempotent if and only if

$$\Sigma^2 = \Sigma$$

or, clearly

$$(c_1\lambda_i + c_2\mu_i)(c_1\lambda_i + c_2\mu_i - 1) = 0,$$
$$i = 1,2,\ldots,n, \qquad (3)$$

where λ_i and μ_i are the diagonal entries of Λ and $\mathbf{M}$, respectively. On the other hand, both $\mathbf{P}_1$ and $\mathbf{P}_2$ have only the eigenvalues 1 and 0 since $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent. Taking into account Lemma and assumtions of the theorem, it is seen from equations (1) that if $\mathbf{P}_1\mathbf{P}_2 = 0$, then (λ_i, μ_i) attains each of the pairs $(0,0)$, $(0,1)$ and $(1,0)$ at least once for at least one value of i. Hence, equations (3) are commonly fulfilled if and only if $c_1 = 1$ (implying $c_2 = 1$), which is the situations (a). Furthermore, in wiew of the assumption that $\mathbf{P}_1 \neq \mathbf{P}_2$, The equalities $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ cannot hold simultaneously. Consequently, under the last assumptions, it is again seen from equations (1) that (λ_i, μ_i) attains each of the pairs $(0,0)$, $(1,0)$ and $(1,1)$ at least once for at least one value of i if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, and each of the pairs $(0,0)$, $(0,1)$ and $(1,1)$ at least once for at least one value of i if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$. And therefore, it can be shown simply that equations (3) are commonly fulfilled if and only if $c_1 = 1$ (implying $c_2 = -1$) for the former case, which is the situations **(b)**, and $c_1 = -1$ (implying $c_2 = 1$) for the latter case, which is the situation **(c)**.

The proof is completed.

Remark. As pointed out at Section 1, it is established an alternative proof (having practical value in statistics) of one part of the theorem recently obtained by Baksalary and Baksalary in [6]. In that paper, a complete solution was established to the problem of characterizing all situations (including noncommute case). Morever, it was given two corollaries:

(1) Under the assumptions (including noncommute case) of the theorem, a necessary condition for $\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2$ to be an idempotent matrix is that each of the products $\mathbf{P}_1\mathbf{P}_2$ and $\mathbf{P}_2\mathbf{P}_1$ is an idempotent matrix; cf. Corollary 1.

(2) Given two different nonzero complex idempotent matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ such that the difference $\mathbf{Q}_1 - \mathbf{Q}_2$ is Hermitian, let $\mathbf{Q}$ be their linear combination of the form $\mathbf{Q} = \gamma_1\mathbf{Q}_1 + \gamma_2\mathbf{Q}_2$ with nonzero complex numbers γ_1 and γ_2. Then there are exactly three situations, where $\mathbf{Q}$ is also idempotent;

(i) $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2$ and $\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{0} = \mathbf{Q}_2\mathbf{Q}_1$,
(ii) $\mathbf{Q} = \mathbf{Q}_1 - \mathbf{Q}_2$ and $\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_2 = \mathbf{Q}_2\mathbf{Q}_1$,
(iii) $\mathbf{Q} = \mathbf{Q}_2 - \mathbf{Q}_1$ and $\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_1$;

cf. Corollary 2.

Other functions of idempotent matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ studied (quite intensively) in the literature; cf. Ref. [1,2,4,5,8].

The main theorem is now supplemented by showing that for the cases (a)-(c) there exist matrices satisfying the required conditions:

Example for the case (a) is provided by

$$\mathbf{P}_1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example for the case (b) is provided by

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example for the case (c) is provided by

$$\mathbf{P}_1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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