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Sesqui-Harmonic Curves in LP-Sasakian Manifolds

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Abstract

In this article, we characterize interpolating sesqui-harmonic spacelike curves in a four-dimensional conformally and quasi-conformally flat and conformally symmetric Lorentzian Para-Sasakian manifold. We give some theorems for these curves.

1. Introduction

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and $\sigma : (M_1, g_1) \to (M_2, g_2)$ be a smooth map. The equation

$$\mathbb{L}(\sigma) = \frac{1}{2}\int_{M_1} |\, d\sigma \,|^2 \, \vartheta_{g_1}$$

gives the critical points of energy functional The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

$$\tau(\sigma) = trace\nabla d\sigma,$$

where $\tau(\sigma)$ is called the tension field of the map σ.
Biharmonic maps between Riemannian manifolds were studied in [1]. Biharmonic maps between Riemannian manifolds $\psi : (M_1, g_1) \to (M_2, g_2)$ are the critical points of the bienergy functional

$$\mathbb{L}_2(\sigma) = \frac{1}{2}\int_{M_1} |\, \tau(\sigma) \,|^2 \, \vartheta_{g_1}.$$

In [2], G.Y. Jiang derived the variations of bienergy formulas and showed that

$$\begin{aligned} \tau_2(\sigma) &= -J^{\sigma}(\tau(\sigma)) \\ &= -\triangle\tau(\Psi) - traceR^N(d\sigma, \tau(\sigma))d\sigma, \end{aligned}$$

where J^{σ} is the Jacobi operator of σ. The equation $\tau_2(\sigma) = 0$ is called biharmonic equation.
Interpolating sesqui-harmonic maps were studied by Branding [3]. The author defined an action functional for maps between Riemannian manifolds that interpolated between the actions for harmonic and biharmonic maps. Ψ is interpolating sesqui-harmonic if it is critical point of $_{\delta_1,\delta_2}(\Psi)$,

$$\mathbb{L}_{\delta_1,\delta_2}(\Psi) = \delta_1 \int_{M_1} |d\Psi|^2 v_{g_1} + \delta_2 \int_{M_1} |\tau(\Psi)|^2 v_{g_1}, \tag{1.1}$$

where $\delta_1, \delta_2 \in \mathbb{R}$ [3].

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For $\delta_1, \delta_2 \in \mathbb{R}$ the equation

$$\tau_{\delta_1,\delta_2}(\Psi) = \delta_2 \tau_2(\Psi) - \delta_1 \tau(\Psi) = 0, \tag{1.2}$$

is the interpolating sesqui-harmonic map equation [3].
An interpolating sesqui-harmonic map is biminimal if variations of (1.1) that are normal to the image $\Psi(M_1) \subset M_2$ and $\delta_2 = 1$, $\delta_1 > 0$ [4]. In a 3-dimensional sphere, interpolating sesqui-harmonic curves were studied in [3]. Interpolating sesqui-harmonic Legendre curves in Sasakian space forms were characterized in [5]. Recently, Yüksel Perktaş et all. introduced biharmonic and biminimal Legendre curves in 3-dimensional f-Kenmotsu manifold [6]. Moreover, spacelike and timelike curves characterized in a four dimensional manifold to be proper biharmonic in [7]. Motivated by the above studies, in this paper, we examine interpolating sesqui-harmonic curves in 4-dimensional LP-Sasakian manifold.

2. Preliminaries

2.1. Lorentzian almost paracontact manifolds

Let M be an n-dimensional differentiable manifold equipped with a structure (ϕ, ζ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form on M such that [8]

$$\phi^2 = Id + \eta \otimes \zeta \tag{2.1}$$

$$\eta(\zeta) = -1. \tag{2.2}$$

Also, we have

$$\eta \circ \phi = 0, \quad \phi\zeta = 0, \quad rank(\phi) = n-1.$$

If M admits a Lorentzian metric g, such that

$$g(\phi V, \phi W) = g(V,W) + \eta(V)\eta(W), \tag{2.3}$$

then M is said to admit a *Lorentzian almost paracontact structure* (ϕ, ζ, η, g).
The manifold M endowed with a Lorentzian almost paracontact structure (ϕ, ζ, η, g) is called a *Lorentzian almost paracontact manifold* [8,9]. In equations (2.1) and (2.2) if we replace ζ by $-\zeta$, we obtain an almost paracontact structure on M defined by I. Sato [10].
A Lorentzian almost paracontact manifold $(M, \phi, \zeta, \eta, g)$ is called a *Lorentzian para-Sasakian manifold* [8] if

$$(\nabla_V \phi)W = g(V,W)\zeta + \eta(W)V + 2\eta(V)\eta(W)\zeta. \tag{2.4}$$

It is well konown that, conformal curvature tensor $\tilde{C}$ is given by

$$\tilde{C}(V,W)Z \;=\; R(V,W)Z - \frac{1}{n-2}\{\, S(W,Z)V - S(V,Z)W + g(W,Z)V - g(V,Z)QW \,\} + \left(\frac{r}{(n-1)(n-2)}\right)\{g(W,Z)V - g(V,Z)W\},$$

where S is the Ricci tensor and r is the scalar curvature. If $C = 0$, then Lorentzian para-Sasakian manifold is called *conformally flat*.
Also, quasi conformal curvature tensor $\hat{C}$ is defined by

$$\hat{C}(V,W)Z \;=\; \alpha R(V,W)Z - \beta\{\, S(W,Z)V - S(V,Z)W + g(W,Z)QV - g(V,Z)QW \,\} - \left(\frac{r}{n}\left(\frac{\alpha}{(n-1)} + 2\beta\right)\right)\{g(W,Z)V - g(V,Z)W\},$$

where α, β constants such that $\alpha\beta \neq 0$. If $\hat{C} = 0$, then Lorentzian para-Sasakian manifold is called quasi conformally flat.
A conformally flat and quasi conformally flat LP-Sasakian manifold M^n $(n > 3)$ is of constant curvature 1 and also a LP-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation $R(V,W) \cdot C = 0$ holds [11]. For a conformally symmetric Riemannian manifold [12], we have $\nabla C = 0$. So, for a conformally symmetric space $R(V,W) \cdot C = 0$ satisfies. Therefore a conformally symmetric LP-Sasakian manifold is locally isometric to a Lorentzian unit sphere [11].
In this case, for conformally flat, quasi conformally flat and conformally symmetric LP-Sasakian manifold M, for every $V,W,Z \in TM$ [11], we have

$$R(V,W)Z = g(W,Z)V - g(V,Z)W. \tag{2.5}$$

3. Main results

In this section, we give our main results about interpolating sesqui-harmonic curves in a conformally flat, quasi conformally flat and conformally symmetric LP-Sasakian manifold $\tilde{M}$. From now on, we will consider such a manifold as $\tilde{M}$.

Theorem 3.1. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t, n, b_1, b_2\}$ orthonormal Frenet frame such that first binormal vector b_1 is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$,
or
ii) γ is a helix with $\rho_1^2 - \rho_2^2 = 1 - \frac{\delta_1}{\delta_2}$
where $\frac{\delta_1}{\delta_2} < 1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the first binormal vector b_1 of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & \rho_2 & 0 & \rho_3 \\ 0 & 0 & \rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.1}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve γ [13].
By using (3.1) and equation (2.5), we obtain

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1\rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 + \rho_1\rho_2^2)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1\rho_2^2 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \quad \rho_2 = const.$$

$$\rho_1^2 - \rho_2^2 = 1 - \frac{\delta_1}{\delta_2},$$

$$\rho_2\rho_3 = 0.$$

So, we get the proof. □

Theorem 3.2. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t,n,b_1,b_2\}$ orthonormal Frenet frame such that second binormal vector b_2 is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$,
or
ii) γ is a helix with $\rho_1^2 + \rho_2^2 = 1 - \frac{\delta_1}{\delta_2}$
where $\frac{\delta_1}{\delta_2} < 1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the vector b_2 of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & -\rho_2 & 0 & \rho_3 \\ 0 & 0 & \rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.2}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
From (3.2) and (2.5), we get

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1\rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 - \rho_1\rho_2^2)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 - \rho_1\rho_2^2 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

In this case, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \quad \rho_2 = const.$$

$$\rho_1^2 + \rho_2^2 = 1 - \frac{\delta_1}{\delta_2},$$

$$\rho_2\rho_3 = 0.$$

This equation proves our assertion. □

Theorem 3.3. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t, n, b_1, b_2\}$ orthonormal Frenet frame such that binormal vector b_1 is null. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) $\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$ and
and
ii) $\rho_2 = 0$ or $|ln|\rho_2(s) = -\int \rho_3(s)ds$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the first binormal vector b_1 of $\{t, n, b_1, b_2\}$ orthonormal Frenet frame is a null(lightlike) vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & \rho_2 & 0 & -\rho_3 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.3}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
By use of (3.3) and equation (2.5), we have

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1\rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 + \rho_1)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

In view of (1.2), we arrive at

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1\rho_1' = 0$$

$$(\rho_1'' - \rho_1^3 + \rho_1)\delta_2 - \rho_1\delta_1 = 0,$$

$$2\rho_1'\rho_2 + \rho_1\rho_2' + \rho_1\rho_2\rho_3 = 0.$$

If we consider non-geodesic solution, we obtain

$$\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}},$$

$$\rho_2' + \rho_2\rho_3 = 0,$$

where $\frac{\delta_1}{\delta_2} < 1$. □

Theorem 3.4. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t, n, b_1, b_2\}$ orthonormal Frenet frame such that normal vector n is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1 = \sqrt{\frac{\delta_1}{\delta_2} - 1}$,
or
ii) γ is a helix with $\rho_1^2 + \rho_2^2 = \frac{\delta_1}{\delta_2} - 1$
where $\frac{\delta_1}{\delta_2} > 1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the normal vector n of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ \rho_1 & 0 & \rho_2 & 0 \\ 0 & \rho_2 & 0 & \rho_3 \\ 0 & 0 & -\rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.4}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
By using (3.4) and equation (2.5), we obtain

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1 \rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t \quad = \quad -(3\rho_1 \rho_1')t + (\rho_1'' + \rho_1^3 + \rho_1 \rho_2^2 + \rho_1)n + (2\rho_1' \rho_2 + \rho_1 \rho_2')b_1 + (\rho_1 \rho_2 \rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) \quad = \quad -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1 k_2^2 + \rho_1)\delta_2 \\ -\rho_1 \delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \qquad \rho_2 = const.$$

$$\rho_1^2 + \rho_2^2 = \frac{\delta_1}{\delta_2} - 1,$$

$$\rho_2 \rho_3 = 0.$$

So, we get the proof. □

4. Conclusion

In this paper we charaecterized spacelike curves to be Sesqui-harmonic curves in LP-Sasakian manifolds. We gave four theorems about these curves. These theorems showed that if we change the vector fields of the Frenet frame $\{t,n,b_1,b_2\}$, then the equation of Sesqui-harmonic curves change. So, we introduced four different spacelike Sesqui-harmonic curves in this manner.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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