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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATION BY SPLINE FUNCTIONS

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Abstract - **The purpose of this study is to find approximate solution of initial value problem concerning nonlinear differential equation of the nth-order by spline functions. Comparasion of spline solution with the exact one is also disscussed.**

I.INTRODUCTION

Consider the nonlinear differential equation of nth-order

$$y^{(n)} = f(x,y) \qquad (1)$$

with initial conditions

$$y(0) = y_0\ ,\ y'(0) = y'_0, \Lambda\ , y^{(n-1)}(0) = y_0^{(n-1)} \qquad (2)$$

where f: $[0,A]\times\Re \to \Re$ is a smooth enough function and satisfies Lipschitz condition

$$|f(x,y_1)-f(x,y_2)| \le L|y_1-y_2|\ ,$$
$$\forall\ (x,y_1),\ (x,y_2)\in[0,A]\times\Re \qquad (3)$$

Suppose that $y:[0,b]\to\Re$ be the unique solution of problem (1)-(2). In what follow we construct a polynomial spline of degree $m \ge n+1$, which we denote by S to approximate solution y on the interval $[0,b]$. For convenience we subdivide the interval $[0,b]$ into N equal subintervals and select the mesh points

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$$0 = x_0 < x_1 < x_2 < \mathrm{K} < x_N = b$$

where $x_i = ih$, h=b/N.

On the interval $[0,h]$, S(x) is defined as follows

$$S(x) = y(0) + y'(0)x + \Lambda\ + \frac{y^{(m-1)}(0)}{(m-1)!}x^{m-1} + \frac{a_0}{m!}x^m$$
$$x\in[0,h] \qquad (4)$$

where $y(0),\ y'(0),\mathrm{K}\ y^{(n-1)}(0)$ are known from problem (1)-(2) and the remaining $y^{(n+1)}(0),\mathrm{K}\ ,y^{(m-1)}(0)$ can be found by derivation of (1). The coeficient a_0 remains to be determined so that S(x) satisfy (1) for $x = h$, i.e.,

$$S^{(n)}(h) = f(h,S(h)).$$

On $[h,2h]$ S(x) is defined in the same manner by

$$S(x) = \sum_{j=0}^{m-1}\frac{S^{(j)}(h)}{j!}(x-h)^j + \frac{a_1}{m!}(x-h)^m\ ,$$
$$x\in[h,2h] \qquad (5)$$

and again a_1 is determined so that (1) is satisfied in x=2h, i.e.,

$$S^{(n)}(2h) = f(2h,S(2h)).$$

From this equation a_1 may be uniquely determined.
In general, on interval [kh, (k+1)h], $k = 0,\mathrm{K}\ ,N-1$ S is defined by

$$S(x)=\sum_{j=0}^{m-1}\frac{S^{(j)}(kh)}{j!}(x-kh)^j+\frac{a_k}{m!}(x-kh)^m$$

and a_k can be uniquely determined from

$$S^{(n)}[(k+1)h]=f[(k+1)h,S(k+1)h].$$

By construction we obtain splines of degree $m \geq n+1$, belonging to $S \in C_{[0,b]}^{m-1}$.

THEOREM: If h is such that

$$\frac{Lh^n}{m(m-1)\mathrm{K}\,(m-n+1)}<1$$

then the spline approximation S defined by above construction exists and is unique.

Proof: On interval $[kh,(k+1)h]$, $k=0,\mathrm{K}\,,n-1$, where S has the expression

$$S(x)=\sum_{j=0}^{m-1}\frac{S^{(j)}(kh)}{j!}(x-kh)^j+\frac{a_k}{m!}(x-kh)^m \equiv$$

$$\equiv A_k(x)+\frac{a_k}{m!}(x-kh)^m \qquad (6)$$

Thus we have to show that a_k can be uniquely determined from

$$S^{(n)}[(k+1)h]=f[(k+1)h,S(k+1)h]. \qquad (7)$$

Replacing S given by (6) in (7) we obtain

$$a_k=\frac{(m-n)!}{h^{m-n}}\left\{f\left[(k+1)h,A_k((k+1)h)+\frac{a_k}{m!}h^m\right]-\right.$$
$$\left.-A_k^{(n)}((k+1)h)\right\} \qquad (8)$$

For the sake of simplicity we write (8) in the fixed point iteration form, i.e.,

$$a_k=g_k(a_k) \qquad (9)$$

We now prove that under the assumption of theorem, the operator $g_k:\Re\rightarrow\Re$, $a_k\rightarrow g_k(a_k)$ is of construction, which implies the existence and uniqueness of the solution a_k of (8).

Let $a_k^*,a_k^{**}\in\Re$. The distance $d(a_k^*,a_k^{**})=\left|a_k^*-a_k^{**}\right|$.

Taking into account (3) we find

$$d(g_k(a_k^*),g_k(a_k^{**}))=\left|g_k(a_k^*),g_k(a_k^{**})\right|\leq$$
$$\leq\frac{Lh^n}{m(m-1)\mathrm{K}\,(m-n+1)}d(a_k^*,a_k^{**})$$

If

$$\frac{Lh^n}{m(m-1)\mathrm{K}\,(m-n+1)}<1$$

the operator g_k has a unique fixed point for every k, therefore (8) has a unique solution and the unique spline solution exists. The proof is complete. For n=1 and for n=2 the theorem reduces to the theorem of Loscalzo-Talbot [1] and the theorem 2 from [3] respectively. For a connection between the spline method and the multistep method see [2].

II. NUMERICAL RESULTS

In this section the method discussed above is tested on two problems.

Problem 1. Spline solution for $y'=1+y^2$ over $[0,0.5]$ with $y(0)=1$

Table 1. Numerical results for problem 1.

x_k	y_k	$y(x_k)$ (Exact)	Errors*
0.025	0.025005	0.025005	0.000000
0.050	0.050042	0.050042	0.000000
0.075	0.075142	0.075141	0.000001
0.100	0.100338	0.100335	0.000003
0.125	0.125662	0.125655	0.000007
0.150	0.151147	0.151135	0.000012
0.175	0.176827	0.176800	0.000019
0.200	0.202736	0.202710	0.000026
0.225	0.228908	0.228875	0.000033
0.250	0.255378	0.255342	0.000036
0.275	0.282181	0.282149	0.000033
0.300	0.309355	0.309336	0.000019
0.325	0.336941	0.336948	0.000006
0.350	0.364987	0.365028	0.000042
0.375	0.393541	0.393627	0.000085
0.400	0.422661	0.422793	0.000133
0.425	0.452404	0.452583	0.000179
0.450	0.482834	0.483055	0.000221
0.475	0.514021	0.514227	0.000251
0.500	0.546035	0.546302	0.000267

*Error= Exact solution-Numerical solution

Problem 2. Spline Solution for $y'' = 2y^3$ over $[1,3]$ with $y(1) = 1$, $y'(1) = -1$

Table 2. Numerical results for problem 2.

x_k	y_k	$y(x_k)$ (Exact)	Error*
1.1	0.909086	0.909091	0.000005
1.2	0.833324	0.833333	0.000009
1.3	0.679218	0.769231	0.000013
1.4	0.714269	0.714286	0.000017
1.5	0.666646	0.666667	0.000021
1.6	0.624975	0.625000	0.000025
1.7	0.588206	0.588235	0.000029
1.8	0.555521	0.555556	0.000035
1.9	0.526275	0.526316	0.000040
2.0	0.499953	0.500000	0.000047
2.1	0.476136	0.476190	0.000054
2.2	0.454484	0.454545	0.000062
2.3	0.434712	0.434783	0.000070
2.4	0.416587	0.416667	0.000080
2.5	0.399910	0.400000	0.000090
2.6	0.384514	0.384615	0.000101
2.7	0.370257	0.370370	0.000113
2.8	0.357017	0.357143	0.000126
2.9	0.344688	0.344828	0.000140
3.0	0.333179	0.333333	0.000155

*Error=Exact solution-Numerical solution

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