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MATRIX TRANSFORMATIONS OF $c_0(p,s)$, $l_\infty(p,s)$ and $l(p,s)$ INTO Q

Mikail ET[1] and Metin BAŞARIR[2]

[1] *Department of Mathematics, Fırat University, 23169, Elazığ / TURKEY*
[2] *Department of Mathematics, Sakarya University, 54100, Adapazarı / TURKEY*

Abstract - Necessary and sufficient conditions have been established for an infinite matrix $A=(a_{nk})$ to transform $c_0(p,s)$, $\ell_\infty(p,s)$ and $\ell(p,s)$ into Q (semiperiodic sequences space), where $c_0(p,s)$ and $\ell_\infty(p,s)$ the set of all complex sequences $x=(x_k)$ such that $\lim_k (k^{-s}|x_k|^{p_k})=0$ and $\sup_k (k^{-s}|x_k|^{p_k})<\infty$, respectively, $p=(p_k)$ strictly positive numbers and $s\geq 0$ is a real number.

I- INTRODUCTION

The generalized sequence spaces $\ell(p)$, $\ell_\infty(p)$ and $c_0(p)$ introduced by I.J.Maddox[3]. Recently, Bulut and Çakar[1] defined the sequence space $\ell(p,s)$ that generalizes $\ell(p)$. In a similar way, Başarır[5] introduced the generalized sequence spaces $c_0(p,s)$, $c(p,s)$ and $\ell_\infty(p,s)$ that several known sequence spaces are obtained by taking special s and (p_n). Sirajudeen and Somasundaram[4] obtained conditions to characterize the matrix transformations of $c_0(p)$, $\ell_\infty(p)$ and $\ell(p)$ into Q (the semiperiodic sequences space). In this paper, we obtain conditions to characterize the matrix transformations of $c_0(p,s)$, $\ell_\infty(p,s)$ and $\ell(p,s)$ into Q .

In §II we deal with definitions and some known results as lemma which will be used in §III for establishing conditions to characterize the matrix transformations.

II. BASIC FACTS AND DEFINITIONS

Let X,Y be two nonempty subsets of the space S of all complex sequences and $A=(a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n,k=1,2,3,...)$. For every $x=(x_k)\in X$ and every integer n we write $A_n(x)=\sum_k a_{nk}x_k$. Here and afterwards the sum without limits is always taken from $k=1$ to $k=\infty$. The sequence $Ax=(A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say that $A\in(X,Y)$ if and only if $Ax\in Y$ whenever $x\in X$.

Throughout the paper, unless otherwise indicated, $p=(p_k)$ will denote a sequence of strictly positive real numbers (not necessarily bounded in general) and $s\geq 0$ is a real number. e_k represents the sequence $(0,0,...,0,1,0,...)$ the 1 in the kth place.

Now we define ([1],[4],[5],[6])

$$\ell_\infty(p,s)=\{x: \sup_k k^{-s}|x_k|^{p_k}<\infty\}$$

$$c_0(p,s)=\{x: \lim_k k^{-s}|x_k|^{p_k}=0\}$$

$$\ell(p,s)=\{x: \sum_k k^{-s}|x_k|^{p_k}<\infty,\}$$

$$Q=\{x: x \text{ is a semiperiodic sequence}\}$$

A sequence $x=(x_k)$ is said to be semiperiodic, if to each $\varepsilon>0$, there exists a positive integer i such that $|x_k-x_{k+ri}|<\varepsilon$ for all r and k . The space Q is seperable subspace of ℓ_∞ , the bounded sequences space. It is easy to see that the necessary and sufficient condition for $c_0(p,s)$, $\ell_\infty(p,s)$ and $\ell(p,s)$ spaces to be linear is $0 < p_k \le \sup p_k < \infty$. $\ell(p,s)$ is a linear paranormed sequence space by

$$h(x)=(\sum_k k^{-s} |x_k|^{p_k})^{1/M},$$

where $M=\max(1,\sup p_k)$. $c_o(p,s)$ is paranormed space by $g(x)=\sup_k (k^{-s/M} |x_k|^{p_k/M})$. Also $\ell_\infty(p,s)$ is paranormed by $g(x)$ if and only if $\inf p_k > 0$. All the spaces defined above are complete in their topologies. When $p_k = 1$ for all k, write $\ell_\infty(p,s)$, $c_o(p,s)$ and $\ell(p,s)$ as ℓ^∞_s, c^o_s and ℓ_s, respectively. When $p_k = 1/k$ for all k, $\ell_\infty(p,s)$ and $c_o(p,s)$ become, respectively, $\Gamma^*(s)$ and $\Gamma(s)$ spaces which generalizes the spaces introduced by V.G.Iyer[2]. When $p_k = p > 1$, $\ell(p,s)$ becomes ℓ_{ps} space.

It is well-known that, if (X,g) is a paranormed space, with the paranorm g, then we denote by X^* the continuous dual of X, i.e. the set of all continuous linear functionals on X. If E is a set of complex sequences $x = (x_k)$ then E^β will denote the generalized Köthe-Toeplitz dual of E:

$$E^\beta = \{ a : \sum_k a_k x_k \text{ converges, for all } x \in E \}$$

Now let us quote some required known results as follows.

Lemma A : [5] $c_o(p,s)^\beta = \bigcup_{N>1} \{ a=(a_k) : \sum_k |a_k| N^{-1/p_k} k^{s/p_k} < \infty \}$

Lemma B : [5] $\ell_\infty(p,s)^\beta = \bigcap_{N>1} \{ a=(a_k) : \sum_k |a_k| N^{1/p_k} k^{s/p_k} < \infty \}$

Lemma C : [1] i- Let $0<p_k\le 1$ for every k. Then $A \in (\ell(p,s), \ell_\infty)$ if and only if

$$\sup_{n,k} (k^s |a_{nk}|^{p_k}) < \infty.$$

ii- Let $1<p_k\le \sup p_k<\infty$ and $p_k^{-1}+q_k^{-1}=1$ for every k. Then $A \in (\ell(p,s), \ell_\infty)$ if and only if there exists an integer $M>1$ such that

$$\sup_n (\sum_k |a_{nk}|^{q_k} M^{-q_k} k^{s(q_k-1)}) < \infty.$$

Lemma D : [5] $A \in (\ell_\infty(p,s), \ell_\infty)$ if and only if $\sup_n \sum_k |a_{nk}| N^{1/p_k} k^{s/p_k} < \infty$ for every $N > 1$.

Lemma E : [5] Let $p \in \ell_\infty$. Then $A \in (c_o(p,s), \ell_\infty(p,s))$ if and only if there exists an absolute constant $B > 1$ such that

$$\sup_n n^{-s} \{ \sum_k |a_{nk}| B^{-r_k} k^{s r_k} \}^{p_n} < \infty,$$

where $r_k = p_k^{-1}$ and $s_k = q_k^{-1}$; $r_k + s_k = 1$.

III. MATRIX TRANSFORMATIONS

Theorem 1. Let $p \in \ell_\infty$. Then $A \in (c_o(p,s), Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q and

ii- There exists an absolute constant $M > 1$ such that $\sup_n \{ \sum_k |a_{nk}| M^{-1/p_k} k^{s/p_k} \} < \infty$.

Proof. Let $A \in (c_o(p,s), Q)$. Since $(e_k) \in c_o(p,s)$, the necessity of (i) is trivial. Since $Q \subset \ell_\infty$, the necessity of (ii) follows from lemma E.

Conversely, let (i) and (ii) hold and $(x_k) \in c_o(p,s)$. Then

$$\sum_k |a_{nk}| M^{-1/p_k} k^{s/p_k} \le L \text{ independent of } n \quad (1)$$

Since $p \in \ell_\infty$, we can take on $c_o(p,s)$, the paranorm

$$g(x) = \sup_k (k^{-s} |x_k|^{p_k})^{1/H},$$

where $H = \max(1, \sup p_k)$. Then

$$g(x - \sum_{k=1}^{n} x_k e_k) = \sup_{k\ge p+1} (k^{-s} |x_k|^{p_k})^{1/H} \to 0, \text{ as}$$

$p \to \infty$. So that $x = \sum_k x_k e_k$ with this topology on $c_o(p,s)$.

Hence given $\varepsilon>0$, there exists $p\ge 1$ such that

$$k^{-s/H} |x_k|^{p_k/H} < \varepsilon / (4LM^{1/H}) \quad (2)$$

for $k\ge p$.

When p is fixed, since $(x_k) \in c_o(p,s)$, we have

$$|x_k| \le N^{-1/p_k} k^{s/p_k} \le R \quad (3)$$

where $R = \max \{ 1, N^{-1/p_k} k^{s/p_k} \}$

By (i), for $\varepsilon>0$ and for all n and r, there exists i_k; $k=1,2,\dots,p$ such that

$$|a_{nk} - a_{n+r i_k,k}| < \varepsilon/(2pR).$$

If i is the least common multiple of i_k; $k=1,2,...,p$ we have

$$\sum_{k=1}^{p} |a_{nk} - a_{n+ri,k}| < \varepsilon/(2R). \qquad (4)$$

Now we have

$$|y_n - y_{n+ri}| < S_1+S_2, \qquad (5)$$

where $S_1 = \sum_{k=1}^{p} |(a_{nk} - a_{n+ri,k}) x_k|$ and $S_2 = \sum_{k=p+1}^{\infty} |(a_{nk} - a_{n+ri,k}) x_k|$. We get $S_1 < \varepsilon/2$ using (3) and (4). Also, using (2) and (1)

$$\sum_{k=p+1}^{\infty} |a_{nk}||x_k| \le \sum_{k=p+1}^{\infty} |a_{nk}|(|x_k|^{p_k})^{1/p_k}$$

$$\le \varepsilon/(4L) \sum_{k=p+1}^{\infty} |a_{nk}| M^{1/p_k} k^{s/p_k} < \varepsilon/4.$$

Similarly, we get $\sum_{k=p+1}^{\infty} |a_{n+ri,k}||x_k| < \varepsilon/4$ so that $S_2 < \varepsilon/2$. Hence (5) gives

$$|y_n - y_{n+ri}| < \varepsilon$$

so that $(y_n) \in Q$. ◊

<u>Corollary 1.</u> $A \in (c^0_s, Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q

and

ii- $\sum_k |a_{nk}| k^s < M$ independent of n.

<u>Proof.</u> Take $p_k=1$ for all k.

<u>Corollary 2.</u> $A \in (\Gamma(s), Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q

and

ii- There exists an absolute constant $M > 1$ such that $\sup_n \{ \sum_k |a_{nk}| M^k k^{sk} \} < \infty$.

<u>Proof.</u> Take $p_k=1/k$ for all k.

<u>Theorem 2-</u> If for the set of all $p=(p_k)$, there exists an $N>1$ such that $\sum_k N^{-1/p_k} < \infty$ then $A \in (\ell_\infty(p,s), Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q

and

ii- $\sup_n \{ \sum_k |a_{nk}| M^{1/p_k} k^{s/p_k} \} < \infty$ for every integer $M>1$.

<u>Proof.</u> Let $A \in (\ell_\infty(p,s), Q)$. Since $(e_k) \in \ell_\infty(p,s)$, trivially (i) is necessary. Since $Q \subset \ell_\infty$, the necessity of (ii) follows from lemma D.

Conversely, let (i) and (ii) hold and $(x_k) \in \ell_\infty(p,s)$. Then

$$\sum_k |a_{nk}| M^{1/p_k} k^{s/p_k} \le L \quad \text{independent of } n.$$

(6)

Since for the set of all $p=(p_k)$, there exists an $N>1$ such that $\sum_k N^{-1/p_k} < \infty$, given an >0, there exists an $p \ge 1$ such that

$$\sum_{k=p+1}^{\infty} N^{-1/p_k} < \varepsilon/4L \qquad (7)$$

When p is fixed, since $(x_k) \in \ell_\infty(p,s)$, we have

$$|x_k| \le R^{1/p_k} k^{s/p_k} \le S \qquad (8)$$

where $S = \max(1, R^{1/p_k} k^{s/p_k})$; $k=1,2,...,p$.

By (i), for $\varepsilon>0$ and for n and r, there exists i_k; $k=1,2,...,p$ such that

$$|a_{nk} - a_{n+ri_k,k}| < \varepsilon/(2pS).$$

Then choosing i to be the least common multiple of i_k; $k=1,2,...,p$, we have

$$\sum_{k=1}^{p} |a_{nk} - a_{n+ri,k}| < \varepsilon/(2S). \qquad (9)$$

Now

$$|y_n - y_{n+ri}| < S_1+S_2, \qquad (10)$$

where $S_1 = \sum_{k=1}^{p} |(a_{nk} - a_{n+ri,k}) x_k|$ and $S_2 = \sum_{k=p+1}^{\infty} |(a_{nk} - a_{n+ri,k}) x_k|$. We get $S_1 < \varepsilon/2$ using (8) and (9). Further, using (6)

$$\sum_{k=p+1}^{\infty} |a_{nk}||x_k| \le$$

$$\sum_{k=p+1}^{\infty} L\ M^{-1/p_k}\ k^{-s/p_k}\ R^{1/p_k}\ k^{s/p_k}$$

$$< L \sum_{k=p+1}^{\infty} (R/M)^{1/p_k}.$$

Now choosing $M \geq NR$, we have $\sum_{k=p+1}^{\infty} |a_{nk}||x_k|$

$\leq L \sum_{k=p+1}^{\infty} N^{-1/p_k} < \varepsilon/4$, using (7).

Similarly, we get $\sum_{k=p+1}^{\infty} |a_{n+ri,k}||x_k| < \varepsilon/4$

so that $S_2 < \varepsilon/2$. Hence (10) gives

$$|y_n - y_{n+ri}| < \varepsilon$$

so that $(y_n) \in Q$. ◊

Corollary 3. $A \in (\Gamma^*(s), Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q

and

ii- $\sup_n \{ \Sigma_k |a_{nk}|\ M^k\ k^{sk} \} < \infty$ for every integer $M > 1$.

Proof. Take $p_k = 1/k$ for all k.

Remark. Theorem 2 is false in the general case even when we replace (i) by the stronger assumption that each column of the matrix $A=(a_{nk})$ is periodic as a counter examples in [4].

Theorem 3. $A \in (\ell(p,s), Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q

and

ii- $\sup_{n,k} \{ k^s |a_{nk}|^{p_k} \} < \infty$, when $0<p_k \leq 1$.

or

There exists an integer $M > 1$ such that

$$\sup_n \{ \Sigma_k |a_{nk}|^{q_k}\ M^{-q_k}\ k^{s(q_k-1)} \} < \infty,$$

when $1<p_k \leq \sup p_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$.

Proof. Let $A \in (\ell(p,s), Q)$. Since $(e_k) \in \ell(p,s)$, the necessity of (i) is obvious. Since $Q \subset \ell_\infty$, the necessity of (ii) follows from theorem 3 [1].

Conversely, let (i) and (ii) hold and $(x_k) \in \ell(p,s)$ and $H = \max(1, \sup p_k)$. From (ii), we have

$k^s |a_{nk}|^{p_k} \leq L$ independent of n, when $0<p_k \leq 1$,

$\Sigma_k |a_{nk}|^{q_k}\ M^{-q_k}\ k^{s(q_k-1)} \leq L$ independent of n for some integer $M>1$, when $1<p_k \leq \sup p_k < \infty$,

(11).

Since $(x_k) \in \ell(p,s)$, for a given $\varepsilon>0$, there exists a $p \geq 1$ such that

$\sum_{k=p+1}^{\infty} k^{-s} |x_k|^{p_k} < \varepsilon/4L$, when $0<p_k \leq 1$ and

$\left(\sum_{k=p+1}^{\infty} k^{-s} |x_k|^{p_k}\right)^{1/H} < \varepsilon/4M(L+1)$.

when $1<p_k \leq H < \infty$, (12)

When p is fixed, since $(x_k) \in \ell(p,s) \subset c_0(p,s)$, we have

$$|x_k| \leq N^{-1/p_k}\ k^{s/p_k} \leq R \qquad (13)$$

where $R = \max(1, N^{-1/p_k}\ k^{s/p_k})$; $k=1,2,\ldots,p$.

By (i), for $\varepsilon>0$ and for n and r, there exists i_k; $k=1,2,\ldots,p$ such that

$$|a_{nk} - a_{n+ri_k,k}| < \varepsilon/(2pR).$$

If i is the least common multiple of i_k; $k=1,2,\ldots,p$ then

$$\sum_{k=1}^{p} |a_{nk} - a_{n+ri,k}| < \varepsilon/(2R). \qquad (14)$$

Now

$$|y_n - y_{n+ri}| < S_1 + S_2, \qquad (15)$$

where $S_1 = \sum_{k=1}^{p} |(a_{nk} - a_{n+ri,k})\, x_k|$ and $S_2 = \sum_{k=p+1}^{\infty} |(a_{nk} - a_{n+ri,k})\, x_k|$.

Case (a): When $0<p_k \leq 1$, since $(x_k) \in \ell(p,s)$, $\Sigma_k k^{-s} |x_k|^{p_k} < 1/L$. Where we can, without loss of generality, use the same L as in (11) so that

$$k^{-s/p_k} |x_k|\ L^{1/p_k} < 1.$$

Hence, using (11) and (12)

$$S_2 = \sum_{k=p+1}^{\infty} |(a_{nk} - a_{n+ri,k})\, x_k|$$

$$\leq \sum_{k=p+1}^{\infty} (|a_{nk}| |x_k| + |a_{n+ri,k}| |x_k|)$$

$$\leq 2 \sum_{k=p+1}^{\infty} k^{-s/p_k} L^{1/p_k} |x_k|$$

$$\leq 2 \sum_{k=p+1}^{\infty} L\, k^{-s} |x_k|^{p_k} < \varepsilon / 2$$

and $S_1 < \varepsilon / 2$, using (13) and (14). Then form (15), we have $|y_n - y_{n+ri}| < \varepsilon$. Hence $(y_n) \in Q$.

Case (b) : When $1 < p_k \leq H < \infty$, by the proof of Theorem 2[1] and the inequality

$$|ax| \leq B (|a|^q B^{-q} k^{s(q-1)} + k^{-s} |x|^p),$$

where $p^{-1} + q^{-1} = 1$, we have

$$\sum_{k=p+1}^{\infty} |a_{nk}| |x_k|$$

$$\leq M \{ (\sum_{k=p+1}^{\infty} |a_{nk}|^{q_k} M^{-q_k} k^{s(q_k-1)}) +$$

$$(\sum_{k=p+1}^{\infty} k^{-s} |x_k|^{p_k}) \}^{1/H}$$

$$\leq M (\sum_{k=p+1}^{\infty} |a_{nk}|^{q_k} M^{-q_k} k^{s(q_k-1)} + 1)$$

$$(\sum_{k=p+1}^{\infty} k^{-s} |x_k|^{p_k}) \}^{1/H} < \varepsilon / 4$$

using (11) and (12).

Similarly, we get $\sum_{k=p+1}^{\infty} |a_{n+ri,k}| |x_k| < \varepsilon / 4$ so that $S_2 < \varepsilon / 2$ and $S_1 < \varepsilon / 2$, using (13) and (14). Hence $(y_n) \in Q$ so that $A \in (\ell(p,s), Q)$. ◊

Corollary 4. $A \in (\ell_s , Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q and

ii- $k^s |a_{nk}| \leq M$ independent of n and k.

Proof. Take $p_k=1$ for all k.

Corollary 5. Let $p>1$ and $p^{-1} + q^{-1} = 1$. Then $A \in (\ell_{ps} , Q)$ if and only if

i- Each column of the matrix $A=(a_{nk})$ belongs to Q and

ii- $\sup_n \{ \sum_k k^{s(q-1)} |a_{nk}|^q \} < \infty$.

Proof. Take $p_k=p$ for all k so that $q_k=q$ for all k and $p_k^{-1} + q_k^{-1} = 1$ becomes $p^{-1} + q^{-1} = 1$.

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