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Keller's Conjecture Revisited

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Abstract

In 1930, Keller conjectured that every tiling of $\mathbb{R}^n$ by unit cubes contains a pair of cubes sharing a complete $(n-1)$-dimensional face. Only 50 years later, Lagarias and Shor found a counterexample for all $n \geq 10$. In this note we show that neither a modification of Keller's conjecture to tiles of more complex shape is true.

Keywords: *Tiling, tiling by unit cubes, Keller's conjecture.*
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1. Introduction

All cubes considered here have their edges parallel to the axes; we identify such a cube with its center. It is trivial to tile $\mathbb{R}^n$ by unit cubes. One might be tempted to ask whether there exists a tiling of $\mathbb{R}^n$ by unit cubes so that there is no pair of twin cubes, i.e., cubes sharing a complete $(n-1)$-dimensional face. In other words, can we find a tiling of $\mathbb{R}^n$ by unit cubes that is completely "messed up?"

Minkowski in 1904 [8] first raised this question in connection to his work on the geometry of numbers, and conjectured there is no such lattice tiling. A tiling by cubes is lattice if the centers of cubes in this tiling form a lattice. In 1939, Hajós found a reformulation of Minkowski's conjecture in terms of abelian groups and answered it in the affirmative.

Theorem 1.1 (Hajós, 1939 [3]). *Every* lattice *tiling of $\mathbb{R}^n$ by unit cubes contains twin cubes.*

Meanwhile in 1930, when Minkowski's conjecture was still open, Keller suggested that the lattice condition was redundant, that the nature of the problem is purely geometric, and not algebraic as assumed by Minkowski.

Conjecture 1.1 (Keller, 1930 [4]). *Every tiling of $\mathbb{R}^n$ by unit cubes contains twin cubes.*

In a recent breakthrough, Brakensiek et al. [1] completed resolved the conjuecture, building upon the works of Perron [9], Lagarias–Shor [5], Mackey [7], and Debroni et al. [2].

Theorem 1.2 (Brakensiek et al., 2020 [1]). *Keller's conjecture is true for $n \leq 7$ and false for $n \geq 8$.*

2. Main result

The falsity of Keller's conjecture is an uncomfortable reminder of how misleading our geometric intuition can be, especially in large dimensions. But may the conjecture be salvaged if there are more "corners" in the tile than the unit cube? Define a cluster of cubes as a union of unit cubes centered at integral points. Can things still go wrong when we tile $\mathbb{R}^n$ by a cluster of cubes that is not a parallelepiped (an $a_1 \times \cdots \times a_n$ box)?

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Theorem 2.1. *Assume $n \geq 9$. There exists a non-parallelepipedal cluster $C \subseteq \mathbb{R}^n$ that is contractible as a topological space, such that there is a tiling of $\mathbb{R}^n$ by C with each pair of twin cubes belonging to the same tile.*

Remark 2.1. For $n \leq 6$, the above theorem is false by a result of Łysakowska and Przesławski [6] stating that every tiling of $\mathbb{R}^n$ by unit cubes contains a column of cubes. We do not know what happens for the dimensions $n = 7$ and 8.

Proof. We first focus on $n = 9$. By [7] there exists a tiling $\mathcal{T}$ of $\mathbb{R}^8$ by unit cubes with no pair of twin cubes, and each cube in $\mathcal{T}$ is centered in $\frac{1}{2}\mathbb{Z}^8$. We dissect each cube in $\mathcal{T}$ into 3^8 small congruent cubes, and then change the unit distance along each axis so that all small cubes become unit cubes. Then the new tiling $\mathcal{T}'$ can be seen as a tiling of $\mathbb{R}^8$ by a cluster comprising 3^8 unit cubes forming the cube of edge length 3, where, as all cubes in $\mathcal{T}$ are centered in $\frac{1}{2}\mathbb{Z}^8$, each pair of twin cubes belongs to the same tile.

Let $\mathcal{S}$ be a tiling of $\mathbb{R}^9$ by the cluster C cubical in shape with the edge length of 3, where each pair of twin cubes belongs to the same tile, and each cube in $\mathcal{S}$ has an integral last coordinate. Such a tiling can be obtained by "stacking" layers of tiling $\mathcal{T}'$ with suitable translations made along adjacent layers. We then shift each cube $(x_1, \ldots, x_9)$ in $\mathcal{S}$ to become a cube centered at $(x_1, \ldots, x_8 + x_9, x_9)$. The new set of cubes constitutes a tiling $\mathcal{S}'$ of $\mathbb{R}^9$ by a tile whose cross-section $(0, 0, ..., 0, x_8, x_9)$ is depicted in Figure 1 such that the only pairs of twins are those that belong to the same tile.

To obtain the statement for $n > 9$, it suffices to repeat the construction for $\mathcal{S}$, i.e., "stacking" layers of a counterexample in $\mathbb{R}^{n-1}$ with suitable translations made along adjacent layers. $\square$

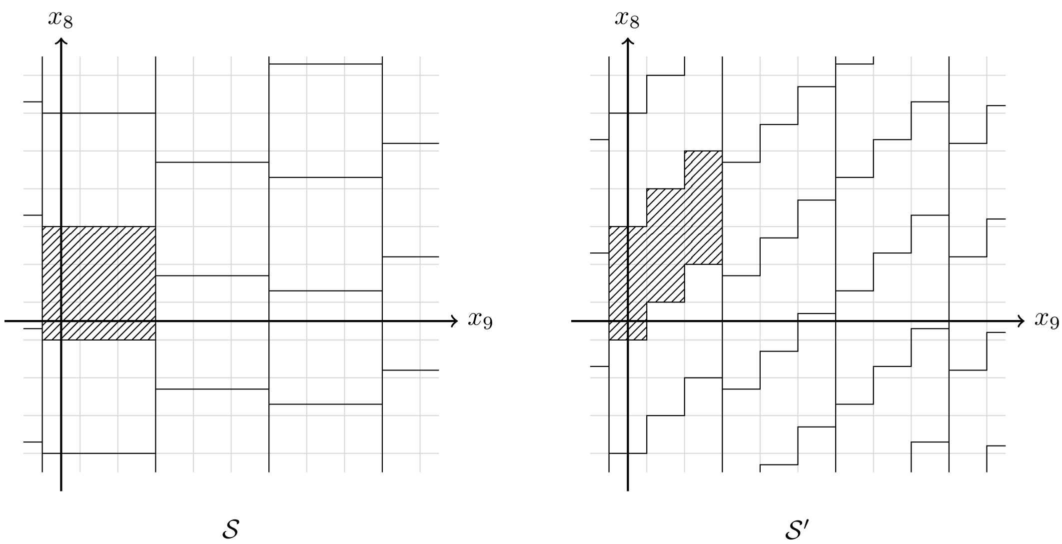

Figure 1. Cross-sections of the tiling $\mathcal{S}$ and $\mathcal{S}'$

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Author's contributions

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