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On the Bertrand Mate of Cubic Bézier Curve by Using Matrix Representation in E^3

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Abstract. In this study, we have examined Bertrand mate of a cubic Bezier curve based on the control points with matrix form in $\mathbf{E}^3$. Frenet vector fields and also curvatures of Bertrand mate of the cubic Bezier curve are examined based on the Frenet apparatus of the first cubic Bezier curve in $\mathbf{E}^3$.

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1. Introduction and Preliminaries

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. However, the study of these curves was first developed in 1959 by the mathematician Paul de Casteljau using de Casteljau's algorithm which is a numerically stable method to evaluate Bézier curves [11]. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics and in animations as a tool to control motion. To guarantee smoothness, the control point at which two curves meet must be on the line between those two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users define the desired path in Bézier curves, and the application creates the required frames for the object to move along that path. For 3D animations, Bézier curves are often used to define 3D paths as well as 2D curves for key-frame interpolation. We have been motivated by the following studies. First Bezier-curves with curvature and torsion continuity has been examined in [4]. Also in [12] and [14] Bezier curves and surfaces has been given. In [3], Bézier curves are designed for Computer-Aided Geometric. Recently, equivalence conditions of control points and application to planar Bezier curves have been examined in [5]. A new way of designing ruled surfaces by means of Bezier curves has been given in [13]. In [7], Frenet apparatus of the cubic Bézier curves has been examined in $\mathbf{E}^3$.We have already examined the cubic Bézier curves in [7], whereas the Mannheim and the involutes of Bezier curves have been contented in [8] and [9], respectively. Before, $5^{th}order$ Bézier curve and its, first, second, and third derivatives based on the control points of $5^{th}order$ Bézier in E^3 is examined [10], as well. Moreover, the Bertrand mate of Nurbs curves has been studied in [2] whereas for open Nurbs curves in [6]. In this paper, the Bertrand mate of a cubic Bezier curve based on the control points with matrix form has been examined by Frenet apparatus.

The set, whose elements are Frenet vector fields and the curvatures of a curve $\alpha(t) \subset IE^3$, is called Frenet apparatus of the curves.

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Let $\alpha(t)$ be the curve, with $\eta = \|\alpha'(t)\| \neq 1$ and Frenet apparatus are $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$. Frenet vector fields are given for a non arc-lengthed curve

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, N(t) = B(t) \Lambda T(t), B(t) = \frac{\alpha'(t) \Lambda \alpha''(t)}{\|\alpha'(t) \Lambda \alpha''(t)\|},$$

where curvature functions are defined by

$$\kappa(t) = \frac{\left\|\alpha'(t) \Lambda \alpha''(t)\right\|}{\|\alpha'(t)\|^3} \ and \ \ \tau(t) = \frac{\left\langle \alpha'(t) \Lambda \alpha''(t), \alpha'''(t)\right\rangle}{\|\alpha'(t) \Lambda \alpha''(t)\|^2}.$$

Also, Frenet formulae are well known as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\eta\kappa & 0 & \eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Generally, Bézier curves can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ with the parametrization

$$\mathbf{B(t)} = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} (t) [P_i].$$

In this study, we will define and work on cubic Bézier curves which are defined in $\mathbf{E}^3$. For more detail see [1].

Definition 1.1. A cubic Bézier curve is a special Bézier curve has only four points P_0, P_1, P_2 and P_3, with the parametrization

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3.$$

The matrix form of the cubic Bezier curve with control points P_0, P_1, P_2, P_3, is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T, N, B, \kappa, \tau\}$ have already been given as in the following theorems by using matrix representation. For more detail see in [7].

The first derivative of a cubic Bézier curve by using matrix representation is

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix},$$

where

$$\begin{aligned} Q_0 &= 3(P_1 - P_0) = (x_0, y_0, z_0), \\ Q_1 &= 3(P_2 - P_1) = (x_1, y_1, z_1), \\ Q_2 &= 3(P_3 - P_2) = (x_2, y_2, z_2) \end{aligned}$$

are control points.

The second derivative of a cubic Bézier curve by using matrix representation is

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix},$$

where $R_0 = 6(P_2 - 2P_1 + P_0)$, $R_1 = 6(P_3 - 2P_2 + P_1)$ are control points.

The third derivative of a cubic Bézier curve is constant by using matrix representation is

$$\alpha'''(t) = [R_0 R_1]$$

with the control point $[R_0 R_1] = R_1 - R_0 = 2[Q_1 Q_2] - 2[Q_0 Q_1]$

1.1. **Frenet Apparatus of a Cubic Bezier Curve.** Frenet apparatus $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ of a cubic Bézier curve have already been given as in the following theorems by using the matrix representation. For more detail see in [8].

Tangent vector field of a cubic Bezier curve α with, $\|\alpha'\| = \eta$ has the following the matrix representation

$$T(t) = \frac{1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}.$$

Binormal vector field of a cubic Bezier curve by using the matrix representation is

$$B(t) = \frac{6}{m}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

where $\|\alpha' \Lambda \alpha''\| = m$,

$$\begin{aligned}
b_{11} &= (y_0z_1 - y_1z_0 - y_0z_2 + y_2z_0 + y_1z_2 - y_2z_1), \\
b_{12} &= (x_1z_0 - x_0z_1 + x_0z_2 - x_2z_0 - x_1z_2 + x_2z_1), \\
b_{13} &= (x_0y_1 - x_1y_0 - x_0y_2 + x_2y_0 + x_1y_2 - x_2y_1), \\
b_{21} &= (2y_1z_0 + y_0z_2 - 2y_0z_1 - y_2z_0), \\
b_{22} &= (2x_0z_1 - 2x_1z_0 - x_0z_2 + x_2z_0), \\
b_{23} &= (2x_1y_0 - 2x_0y_1 + x_0y_2 - x_2y_0), \\
b_{31} &= y_0z_1 - y_1z_0, \\
b_{32} &= x_1z_0 - x_0z_1, \\
b_{33} &= x_0y_1 - x_1y_0.
\end{aligned}$$

Normal vector field of a cubic Bezier curve is 4 th degree and has the matrix representation as in

$$N(t) = \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \end{bmatrix} = \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix},$$

where

$$\begin{aligned}
n_{11} &= b_{12}d_{13} - b_{13}d_{12}, \\
n_{21} &= b_{12}d_{23} - b_{13}d_{22} + b_{22}d_{13} - b_{23}d_{12}, \\
n_{31} &= b_{12}d_{33} - b_{13}d_{32} + b_{22}d_{23} - b_{23}d_{22} + b_{32}d_{13} - b_{33}d_{12}, \\
n_{41} &= b_{22}d_{33} - b_{23}d_{32} + b_{32}d_{23} - b_{33}d_{22}, \\
n_{51} &= b_{32}d_{33} - b_{33}d_{32},
\end{aligned}$$

$$
\begin{aligned}
n_{12} &= b_{11}d_{13} - b_{13}d_{11},\\
n_{22} &= -b_{11}d_{23} - b_{21}d_{13} + b_{13}d_{21} + b_{23}d_{11},\\
n_{32} &= b_{23}d_{21} + b_{33}d_{11} - b_{11}d_{33} - b_{21}d_{23} + b_{13}d_{31} - b_{31}d_{13},\\
n_{42} &= -b_{21}d_{33} - b_{31}d_{23} + b_{23}d_{31} + b_{33}d_{21},\\
n_{52} &= -b_{31}d_{33} + b_{33}d_{31},\\
n_{13} &= b_{11}d_{12} - b_{12}d_{11},\\
n_{23} &= b_{11}d_{22} - b_{12}d_{21} + b_{21}d_{12} - b_{22}d_{11},\\
n_{33} &= b_{11}d_{32} - b_{12}d_{31} + b_{21}d_{22} - b_{22}d_{21} + b_{31}d_{12} - b_{32}d_{11},\\
n_{43} &= b_{21}d_{32} - b_{22}d_{31} + b_{31}d_{22} - b_{32}d_{21},\\
n_{53} &= b_{31}d_{32} - b_{32}d_{31}.
\end{aligned}
$$

First and second curvatures of a cubic Bezier curve by using the matrix representation are

$$
\kappa(t) = \frac{6}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix}
$$

and

$$
\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}.
$$

2. Bertrand Mate of a Cubic Bezier Curve

Definition 2.1. Five points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a 4*th order* Bézier curve with the following equation

$$
\alpha(t) = \sum_{i=0}^{4}\binom{4}{i}t^i(1-t)^{4-i}(t)[P_i] \ , \quad t \in [0,1].
$$

The matrix form of the 4*th order* Bézier curve based on the control points is

$$
\alpha(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.
$$

Definition 2.2. Let a curve α^* is Bertrand mate of α with Frenet-Serret apparatus $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, then they have common principal normal lines, i.e. $N = N^*$ where $\|\alpha'\| = \eta \neq 1$. Hence the equation of Bertrand mate α^* of the curve α has the following parametrization; $\alpha^*(t) = \alpha(t) + \mu N^*(t)$. Also it can be written as in the following parametrization, since μ *is* constant

$$
\alpha^*(t) = \alpha(t) + \mu N(t).
$$

Theorem 2.3. *The Bertrand mate of a cubic Bezier curve has the matrix form*

$$
\alpha^*(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6}{m\eta}N_0 \\ +\frac{6}{m\eta}N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6}{m\eta}N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6}{m\eta}N_3 + 3P_1 - 3P_0 \\ +\frac{6}{m\eta}N_4 + P_0 \end{bmatrix}.
$$

Proof. Let $\alpha^* = \alpha(t) + \mu N$, hence

$$\alpha^*(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} + \frac{6\mu}{\eta m} \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

$$= \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6\mu}{\eta m} N_0 \\ +\frac{6\mu}{\eta m} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{\eta m} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{\eta m} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{\eta m} N_4 + P_0 \end{bmatrix}.$$

□

Theorem 2.4. *Bertrand mate of a cubic Bezier curve can be written as the* 4*th order Bezier curve with the the control points* P_0, P_1, P_2 *and* P_3 *of any cubic Bezier curve with constant speed , as in the following waywhere* η, m *are constants*

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} P_0 \\ \frac{1}{4}P_0 + \frac{3}{4}P_1 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 \\ \frac{3}{4}P_2 + \frac{1}{4}P_3 \\ P_3 \end{bmatrix} + \begin{bmatrix} \frac{6\mu}{\eta m} N_4 \\ \frac{3\mu}{2m\eta} N_3 + \frac{6\mu}{\eta m} N_4 \\ \frac{\mu}{m\eta} N_2 + \frac{3\mu}{m\eta} N_3 + \frac{6\mu}{\eta m} N_4 \\ \frac{3\mu}{2m\eta} N_1 + \frac{3\mu}{m\eta} N_2 + \frac{9\mu}{2m\eta} N_3 + \frac{6\mu}{\eta m} N_4 \\ \frac{6\mu}{\eta m} N_0 + \frac{6\mu}{\eta m} N_1 + \frac{6\mu}{\eta m} N_2 + \frac{6\mu}{\eta m} N_3 + \frac{6\mu}{\eta m} N_4 \end{bmatrix}.$$

Proof. Let $P_0^*, P_1^*, P_2^*, P_3^*$ and P_4^* be control points of Bertrand mate of a cubic Bezier curve α^*, so we can write

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} \frac{6\mu}{m\eta} N_0 \\ +\frac{6\mu}{m\eta} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{m\eta} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{m\eta} N_4 + P_0 \end{bmatrix}.$$

Using invers matrix we get the proof as in the following way,

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} P_0 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{4}P_0 + \frac{3}{4}P_1 + \frac{3\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{\mu}{m\eta} N_2 + \frac{3\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{3}{4}P_2 + \frac{1}{4}P_3 + \frac{3\mu}{2m\eta} N_1 + \frac{3\mu}{m\eta} N_2 + \frac{9\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ P_3 + \frac{6\mu}{m\eta} N_0 + \frac{6\mu}{m\eta} N_1 + \frac{6\mu}{m\eta} N_2 + \frac{6\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \end{bmatrix}.$$

□

2.1. Frenet Apparatus of Bertrand Mate of Any Cubic Bezier Curve in $\mathbf{E}^3$.

Theorem 2.5. *Let a curve* α^* *is Bertrand mate of* α *with Frenet-Serret apparatus* $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ *, then*

$$T^* = \frac{\gamma T + B}{\sqrt{\gamma^2 + 1}}, \quad N^* = N, \quad B^* = \frac{-T + \gamma B}{\sqrt{\gamma^2 + 1}},$$

$$\kappa^* = \frac{\gamma\kappa - \tau}{\mu\tau\sqrt{\gamma^2 + 1}}, \quad \mu\tau > 0$$

$$\tau^* = \frac{-\gamma\tau - \kappa}{\mu\tau\sqrt{\gamma^2 + 1}}, \quad \mu\tau > 0$$

Proof. It is trivial for a Bertrand curve α, that there are constants μ and γ such that $\frac{1}{\mu} = \kappa + \gamma\tau$, with Bertrand mate $\alpha^*(t) = \alpha + \mu N$. □

Theorem 2.6. *Tangent vector field of Bertrand mate of any cubic Bezier curve is*

$$T^*=\frac{\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} \frac{6}{m}b_{11}+\frac{\gamma}{\eta}x_0-2\frac{\gamma}{\eta}x_1+\frac{\gamma}{\eta}x_2 & \frac{6}{m}b_{12}+\frac{\gamma}{\eta}y_0-2\frac{\gamma}{\eta}y_1+\frac{\gamma}{\eta}y_2 & \frac{6}{m}b_{13}+\frac{\gamma}{\eta}z_0-2\frac{\gamma}{\eta}z_1+\frac{\gamma}{\eta}z_2\\ \frac{6}{m}b_{21}-2\frac{\gamma}{\eta}x_0+2\frac{\gamma}{\eta}x_1 & \frac{6}{m}b_{22}-2\frac{\gamma}{\eta}y_0+2\frac{\gamma}{\eta}y_1 & \frac{6}{m}b_{23}-2\frac{\gamma}{\eta}z_0+2\frac{\gamma}{\eta}z_1\\ \frac{6}{m}b_{31}+\frac{\gamma}{\eta}x_0 & \frac{6}{m}b_{32}+\frac{\gamma}{\eta}y_0 & \frac{6}{m}b_{33}+\frac{\gamma}{\eta}z_0\end{bmatrix}}{\sqrt{\gamma^2+1}}.$$

Proof. We have already known that $T^*=\frac{\gamma T+B}{\sqrt{\gamma^2+1}}$

$$T^*=\frac{\frac{\gamma}{\eta}\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} x_0 & y_0 & z_0\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{bmatrix}+\frac{6}{m}\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}}{\sqrt{\gamma^2+1}},$$

$$T^*=\frac{\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} \frac{6}{m}b_{11}+\frac{\gamma}{\eta}x_0-2\frac{\gamma}{\eta}x_1+\frac{\gamma}{\eta}x_2 & \frac{6}{m}b_{12}+\frac{\gamma}{\eta}y_0-2\frac{\gamma}{\eta}y_1+\frac{\gamma}{\eta}y_2 & \frac{6}{m}b_{13}+\frac{\gamma}{\eta}z_0-2\frac{\gamma}{\eta}z_1+\frac{\gamma}{\eta}z_2\\ \frac{6}{m}b_{21}-2\frac{\gamma}{\eta}x_0+2\frac{\gamma}{\eta}x_1 & \frac{6}{m}b_{22}-2\frac{\gamma}{\eta}y_0+2\frac{\gamma}{\eta}y_1 & \frac{6}{m}b_{23}-2\frac{\gamma}{\eta}z_0+2\frac{\gamma}{\eta}z_1\\ \frac{6}{m}b_{31}+\frac{\gamma}{\eta}x_0 & \frac{6}{m}b_{32}+\frac{\gamma}{\eta}y_0 & \frac{6}{m}b_{33}+\frac{\gamma}{\eta}z_0\end{bmatrix}}{\sqrt{\gamma^2+1}}.$$

□

Corollary 2.7. *The control points S_0^*, S_1^*, S_2^* which are belong to tangent vector field of Bertrand mate which is a quadratic Bezier curve are*

$$\begin{bmatrix} S_0^*\\ S_1^*\\ S_2^*\end{bmatrix}=\frac{1}{\sqrt{\gamma^2+1}}\begin{bmatrix} \frac{6}{m}b_{31}+\frac{\gamma}{\eta}x_0 & \frac{6}{m}b_{32}+\frac{\gamma}{\eta}y_0 & \frac{6}{m}b_{33}+\frac{\gamma}{\eta}z_0\\ \frac{3}{m}b_{21}+\frac{6}{m}b_{31}+\frac{\gamma}{\eta}x_1 & \frac{3}{m}b_{22}+\frac{6}{m}b_{32}+\frac{\gamma}{\eta}y_1 & \frac{3}{m}b_{23}+\frac{6}{m}b_{33}+\frac{\gamma}{\eta}z_1\\ \frac{6b_{11}}{m}+\frac{6b_{21}}{m}+\frac{6b_{31}}{m}+\frac{\gamma x_2}{\eta} & \frac{6b_{12}}{m}+\frac{6b_{22}}{m}+\frac{6b_{32}}{m}+\frac{\gamma y_2}{\eta} & \frac{6b_{13}}{m}+\frac{6b_{23}}{m}+\frac{6b_{33}}{m}+\frac{\gamma z_2}{\eta}\end{bmatrix}.$$

Proof. Since η, m are constants and using inverse matrix we have the result. □

Theorem 2.8. *Normal vector field of Bertrand mate α^* of any cubic Bezier curve is*

$$N^*=N=\frac{6}{\eta m}\begin{bmatrix} t^4\\ t^3\\ t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} N_0\\ N_1\\ N_2\\ N_3\\ N_4\end{bmatrix}$$

with the same degree and control points of Bertrand curve.

Theorem 2.9. *Binormal vector field of Bertrand mate α^* of any cubic Bezier curve has the matrix representation as*

$$B^*=\frac{\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} \frac{2}{\eta}x_1-\frac{1}{\eta}x_0-\frac{1}{\eta}x_2+\frac{6\gamma}{m}b_{11} & \frac{2}{\eta}y_1-\frac{1}{\eta}y_0-\frac{1}{\eta}y_2+\frac{6\gamma}{m}b_{12} & \frac{2}{\eta}z_1-\frac{1}{\eta}z_0-\frac{1}{\eta}z_2+\frac{6\gamma}{m}b_{13}\\ \frac{2}{\eta}x_0-\frac{2}{\eta}x_1+\frac{6\gamma}{m}b_{21} & \frac{2}{\eta}y_0-\frac{2}{\eta}y_1+\frac{6\gamma}{m}b_{22} & \frac{2}{\eta}z_0-\frac{2}{\eta}z_1+\frac{6\gamma}{m}b_{23}\\ \frac{6\gamma}{m}b_{31}-\frac{1}{\eta}x_0 & \frac{6\gamma}{m}b_{32}-\frac{1}{\eta}y_0 & \frac{6\gamma}{m}b_{33}-\frac{1}{\eta}z_0\end{bmatrix}}{\sqrt{\gamma^2+1}}.$$

Proof. Since $B^* = \dfrac{-T+\gamma B}{\sqrt{\gamma^2+1}}$,

$$B^* = \frac{\frac{-1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{6\gamma}{m}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}{\sqrt{\gamma^2+1}},$$

$$B^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \frac{\frac{-1}{\eta}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{6\gamma}{m}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}{\sqrt{\gamma^2+1}},$$

which completes the proof. □

Corollary 2.10. *The control points K_0^*, K_1^*, K_2^* which are belong to Binormal vector field of Bertrand mate α^* which is a quadratic Bezier curve are*

$$\begin{bmatrix} K_0^* \\ K_1^* \\ K_2^* \end{bmatrix} = \frac{\begin{bmatrix} \frac{6\gamma b_{31}}{m} - \frac{x_0}{\eta} & \frac{6\gamma b_{32}}{m} - \frac{y_0}{\eta} & \frac{6\gamma b_{33}}{m} - \frac{z_0}{\eta} \\ \frac{3\gamma b_{21}}{m} - \frac{1}{\eta}x_1 + \frac{6\gamma b_{31}}{m} & \frac{3\gamma b_{22}}{m} - \frac{1}{\eta}y_1 + \frac{6\gamma b_{32}}{m} & \frac{3\gamma b_{23}}{m} - \frac{z_1}{\eta} + \frac{6\gamma b_{33}}{m} \\ \frac{6\gamma b_{11}}{m} - \frac{x_2}{\eta} + \frac{6\gamma b_{21}}{m} + \frac{6\gamma b_{31}}{m} & \frac{6\gamma b_{12}}{m} - \frac{y_2}{\eta} + \frac{6\gamma b_{22}}{m} + \frac{6\gamma b_{32}}{m} & \frac{6\gamma b_{13}}{m} - \frac{z_2}{\eta} + \frac{6\gamma b_{23}}{m} + \frac{6\gamma b_{33}}{m} \end{bmatrix}}{\sqrt{\gamma^2+1}}.$$

Proof. Since η, m are constants

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} K_0^* \\ K_1^* \\ K_2^* \end{bmatrix} = \begin{bmatrix} \frac{2x_1}{\eta} - \frac{x_0}{\eta} - \frac{x_2}{\eta} + \frac{6\gamma b_{11}}{m} & \frac{2}{\eta}y_1 - \frac{y_0}{\eta} - \frac{y_2}{\eta} + \frac{6\gamma b_{12}}{m} & \frac{2z_1}{\eta} - \frac{z_0}{\eta} - \frac{z_2}{\eta} + \frac{6\gamma b_{13}}{m} \\ \frac{2}{\eta}x_0 - \frac{2}{\eta}x_1 + \frac{6\gamma b_{21}}{m} & \frac{2}{\eta}y_0 - \frac{2}{\eta}y_1 + \frac{6\gamma b_{22}}{m} & \frac{2}{\eta}z_0 - \frac{2}{\eta}z_1 + \frac{6\gamma b_{23}}{m} \\ \frac{6\gamma b_{31}}{m} - \frac{1}{\eta}x_0 & \frac{6\gamma b_{32}}{m} - \frac{1}{\eta}y_0 & \frac{6\gamma b_{33}}{m} - \frac{1}{\eta}z_0 \end{bmatrix}.$$

By using inverse matrix, we have the result. □

Theorem 2.11. *The first and second curvature of Bertrand mate $\alpha^*$of any cubic Bezier curve in $\mathbf{E}^3$are*

$$\kappa^* = \frac{\frac{6\gamma}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix} - \tau}{\mu\tau\sqrt{\gamma^2+1}},$$

$$\tau^* = \frac{-\gamma\tau}{\mu\tau\sqrt{\gamma^2+1}} - \frac{6\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix}}{\mu\tau\eta^3\sqrt{\gamma^2+1}}.$$

Proof. Since Bertrand mate has $\kappa^* = \dfrac{\gamma\kappa - \tau}{\mu\tau\sqrt{\gamma^2+1}}$, curvatures $\tau^* = \dfrac{-\gamma\tau - \kappa}{\mu\tau\sqrt{\gamma^2+1}}$, $\mu\tau > 0$ it is trivial. □

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors have no conflicts of interest to declare.

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