SAINT-VENANT TYPE ESTIMATE FOR THE WAVE EQUATION

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Özet- **Bu çalışmada hızı azalan bir dalga denklemi için Uzaysal Azalım Kestirimi elde edilmiştir. Yük bölgesinden uzaklaşıldıkça son etkilerin, en azından kısa zaman aralıkları için çok hızlı bir şekilde azaldığı görülmüştür.**

Anahtar Kelimeler- **Uzaysal azalım kestirimi, Saint-Venant türü kestirim, dalga denklemi.**

Abstract-**It is established Spatial decay estimates of Saint-Venant type for the damped wave equation of transient linear wave equation. It is shown that the end effects decay, at least for short times, very fast with the distance from the loaded end.**

Keywords- **Spatial decay estimate, Saint-Venant type estimate, wave equation**

I. INTRODUCTION

We shall show that the energy methods allow us to establish spatial decay results for the damped wave equation. Particularly, we show that the total energy (sum of kinetic and strain energy) stored in the region Ω_z over the time interval $[0,t]$, decays exponentionaly with z, for $z<t$ along the characteristic line, so that the decay rate is described by the factor $exp(-z/t)$; while for $z>t$, the energy is vanishing. Same type of estimates are given for the parabolic equation by [2] and [3]. Recent developments on the spatial estimates can be found in [6].

II. STATEMENT OF PROBLEM

Let Ω be closed, bounded, regular region in three-dimensional space whose boundary $\partial\Omega$ includes a plane portion S_0. Choose cartesian coordinates x_1, x_2, x_3 so that S_0 lies in the plane $x_3=0$, and suppose that Ω lies in the half space $x_3>0$. Indices after comma denotes the differentiation with respect to spa variables.

Let $u(\mathbf{x},t)=u(x_1,x_2,x_3)$ satisfy the wave equation

$$u_{tt}-u_{,jj}+\beta u_t=0 \quad \text{on } \Omega\text{x}(0,t_0) \qquad (1$$

with nonlinear boundary condition

$$u_t\frac{\partial u}{\partial n}+\alpha u u_t=0 \quad \text{on } (\partial\Omega/S_0)\text{x}(0,t_0) \qquad (2$$

and initial conditions

$$u(x,0)=0, \quad u_t(x,0)=0 \quad \text{for } x\in\Omega \qquad (3$$

where α and β is the given nonnegative constant a last term on the lefthand side is damping term whi reduces the velocity. $\partial u/\partial n$ is the normal derivativ

To the function $u(\mathbf{x},t)$, solution of the initial bounda value problem (1)-(3), we associate the followi nonnegative energy functional $E(z,t)$, which is sum kinetic and strain energies stored in the portion Ω_z of over the time interval $[0,t]$, defined on $[0,L]\text{x}[0,t_0)$ by

$$E(z,t)=\frac{1}{2}\int_0^t\int_{\Omega_z}\left(u_t^{\,2}+u_{,j}u_{,j}\right)dVds \qquad (4)$$

By differentiating (4) with respect to z we get

$$\frac{\partial}{\partial z}E(z,t)=-\frac{1}{2}\int_0^t\int_{S_z}\left(u_t^{\,2}+u_{,j}u_{,j}\right)dAds \qquad (5)$$

Now, we are state and prove theorem for the proble (1)-(3).

Theorem 1: Let $u(\mathbf{x},t)$ be a solution of the init boundary value problem defined by (1)-(3). Then

$$E(z,t)=0, \quad \text{for} \quad t<z\le L \tag{6}$$

$$E(z,t)\le E(0,t)e^{\frac{-z}{t}}, \quad \text{for} \quad 0\le z\le t \tag{7}$$

Proof: Let us multiply equation (1) by u_t and integrate over $\Omega_z x[0,t]$. Use integration by parts and boundary conditions (2) and initial conditions (3) to obtain

$$\frac{1}{2}\int_{\Omega_z}\left(u_t^{\,2}+u_{,j}u_{,j}\right)dV+\frac{\alpha}{2}\int_{\partial\Omega_z/S_z}u^2 dS+$$

$$+\beta\int_0^t\int_{\Omega_z}u_t^{\,2}d\,Vds=-\int_0^t\int_{S_z}u_t u_{,3}dAds \tag{8}$$

Integrate (8) over $[0,t]$

$$E(z,t)+\frac{\alpha}{2}\int_0^t\int_{\partial\Omega_z/S_z}u^2dVds+\beta\int_0^t\int_0^s\int_{\Omega_z}u_t^{\,2}dVdrds$$

$$=-\int_0^t\int_0^s\int_{S_z}u_t u_{,3}\,dAdrds \tag{9}$$

Since α and β is nonnegative, constant and by using the arithmetic-geometric mean inequality, we deduce from equation (9)

$$\frac{\partial}{\partial t}E(z,t)\le\frac{1}{2}\int_0^t\int_{S_z}\left(u_t^{\,2}+u_{,3}\,u_{,3}\right)dAds$$

From (5) we obtain

$$\frac{\partial}{\partial t}E(z,t)+\frac{\partial}{\partial z}E(z,t)\le 0 \tag{10}$$

By integrating (10) along the characteristic line $z=t$ in the (z,t) plane through $(0,0)$ we find that at $z=t\in[0,L]$ we have

$$E(t,t)\le E(0,0) \tag{11}$$

From (3), we observe that $E(0,0)=0$. Moreover, $E(z,t)$ is nonincreasing function of z, so we have

$$E(z,t)\le E(t,t) \quad \text{for} \quad z\ge t \tag{12}$$

From (11) and (12) we deduce the result (6).

Now suppose that $0\le z<t$. From (9) and young's inequality we get

$$E(z,t)\le\frac{1}{2}\int_0^t\int_0^s\int_{S_z}\left(u_t^{\,2}+u_{,3}\,u_{,3}\right)dAdrds \tag{13}$$

By integration by part we obtain

$$\frac{1}{2}\int_0^t\int_0^s\left(u_t^{\,2}+u_{,3}u_{,3}\right)drds$$

$$=\frac{1}{2}\int_0^t(t-s)\left(u_t^{\,2}+u_{,3}u_{,3}\right)ds$$

$$\le\frac{t}{2}\int_0^t\left(u_t^{\,2}+u_{,3}u_{,3}\right)ds \tag{14}$$

If we use (14) and (13) we get

$$E(z,t)\le\frac{t}{2}\int_0^t\left(u_t^{\,2}+u_{,3}\,u_{,3}\right)ds$$

Then

$$t\frac{\partial}{\partial z}E(z,t)+E(z,t)\le 0 \tag{15}$$

Multiplying (15) by $e^{\frac{z}{t}}$ and integrate over $(0,z)$ we get

$$E(z,t)\le E(0,t)e^{\frac{-z}{t}}, \quad \text{for} \quad 0\le z\le t$$

For $0<t<z$, integration of first order differetial inequality (15) leads to relation (7) and the proof is complete.

III. RESULT

We noted that for the short values of the time variable, the decay rate of the end effects in the wave equation is very fast. As a conclusion, for appropiately short values of the time variable, the spatial decay of end effects in the wave equation problem is faster than that for the transient heat conduction[3]. The above spatial decay estimate is dynamical. We do not know other decay estimates for the wave equation to compare it with the above one.

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